445.204

Introduction to Mechanics of Materials

(재료역학개론)

Chapter 3: Strain

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- Displacement
- Definition of strain components: normal and shear strains
- Strain tensor
- Compatibility

Displacement field

- Displacement field, **u**(x,y,z) – vectors giving the displacement from each point **P**(x,y,z) to point **P**' (x,y,z) in the deformed geometry

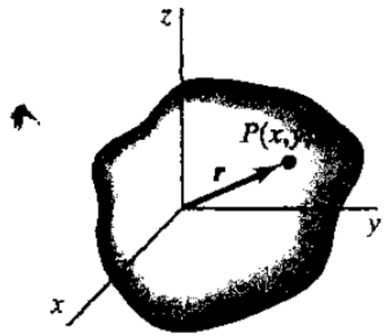


Figure 3.1. Position vector in a body.

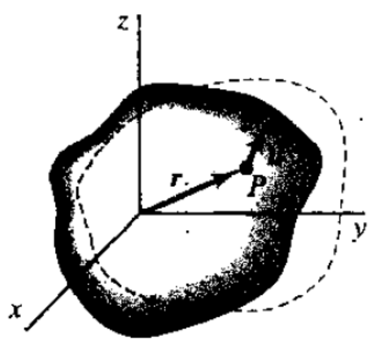


Figure 3.2. Displacement field vector.

How to measure deformation? - Strain

- Deformation of an infinitesimal element

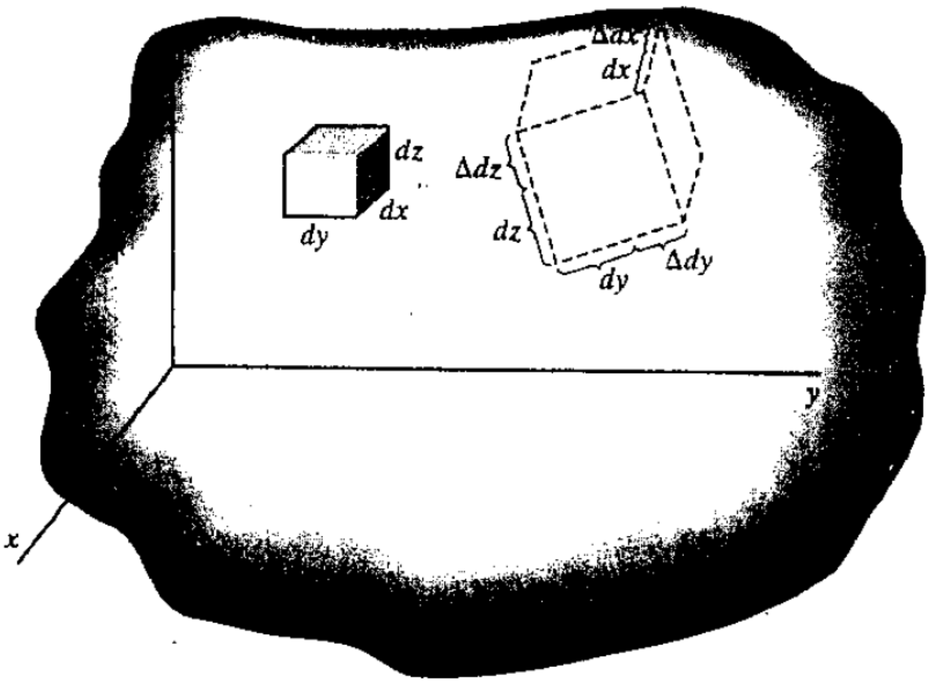


Figure 3.7. Deforming rectangular parallelepiped.

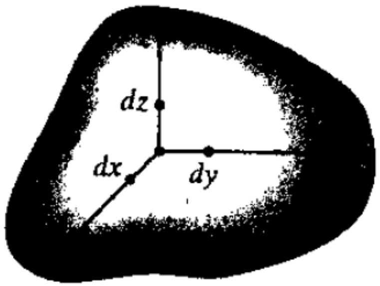


Figure 3.4. Four adjacent points.

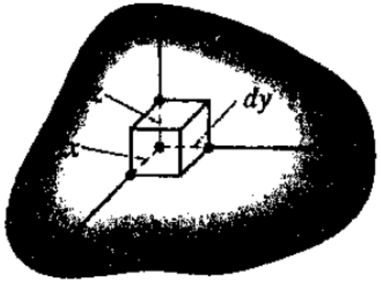
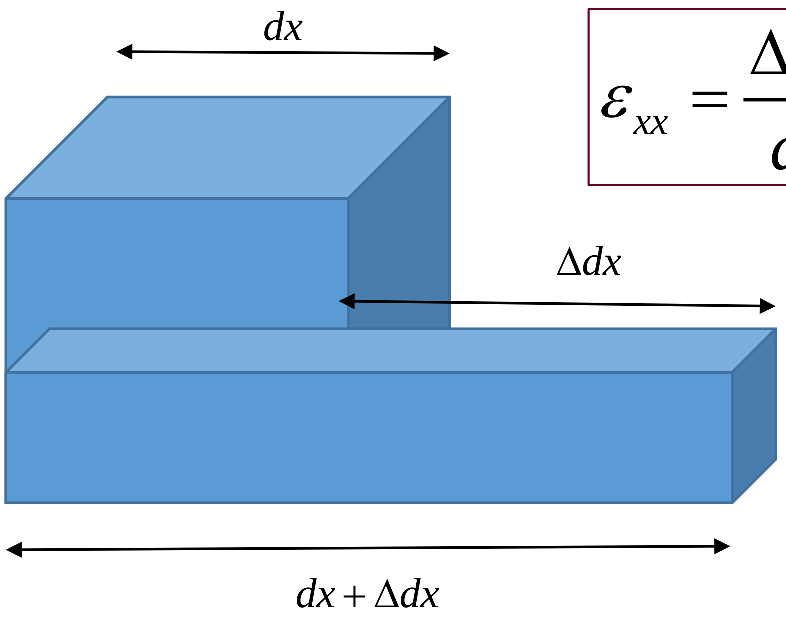


Figure 3.6. Adjacent points form a rectangular parallelepiped.

Strain – normal strain component

- Strain attributed to the “normal stress” components – length change
- Normal strain – change in length per unit original length



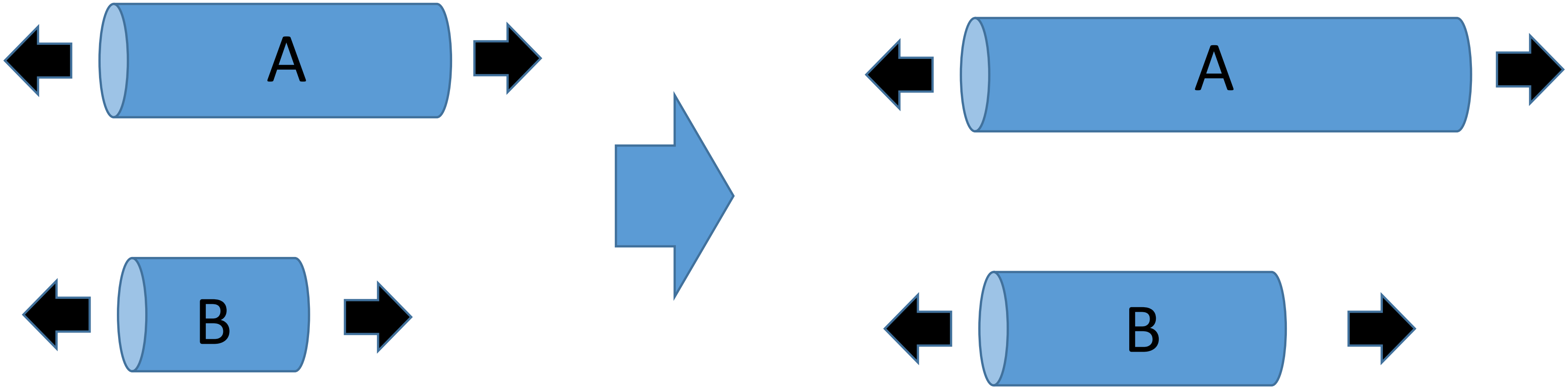
$$\varepsilon_{xx} = \frac{\Delta dx}{dx}$$

$$\varepsilon_{xx} = \frac{\Delta dx}{dx}$$

$$\varepsilon_{yy} = \frac{\Delta dy}{dy}$$

$$\varepsilon_{zz} = \frac{\Delta dz}{dz}$$

Why Strain instead of displacement



- For the same force, F, A is more elongated.
- Therefore, A is weaker or A is material with less strength??

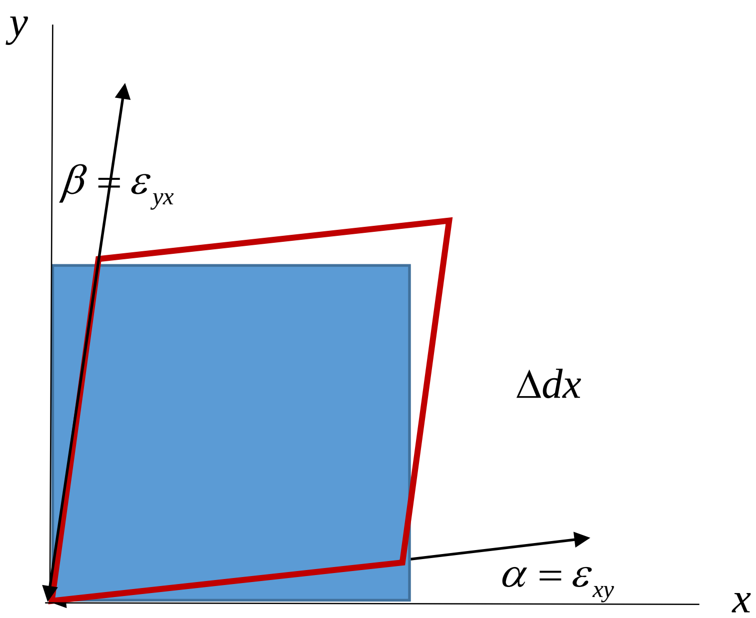
Strain – shear strain component

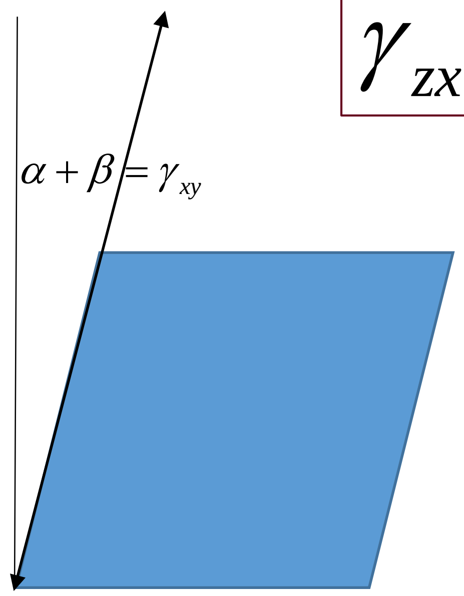
- Strain attributed to the “shear stress” components – angle (or orientation) of faces
- Shear strain or pure shear strain, or engineering shear strain

$$\gamma_{xy} = 2\varepsilon_{xy}$$

$$\gamma_{yz} = 2\varepsilon_{yz}$$

$$\gamma_{zx} = 2\varepsilon_{zx}$$





$$\gamma_{xy} = \alpha + \beta = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$$

Exercise – Axially loaded bar

- A bar with a constant square cross section is loaded by a force P.
- Q: If 2 in of total elongation is measured with no volume change (incompressible), what are the strains in x, y, z directions?

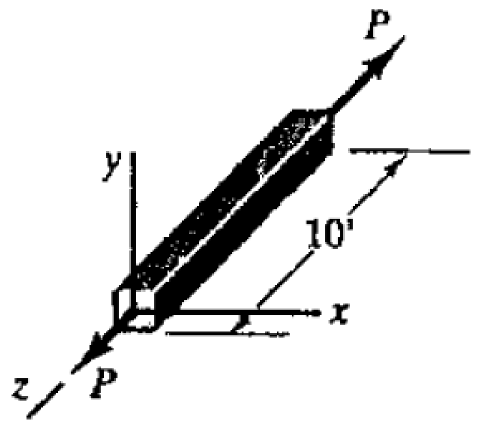
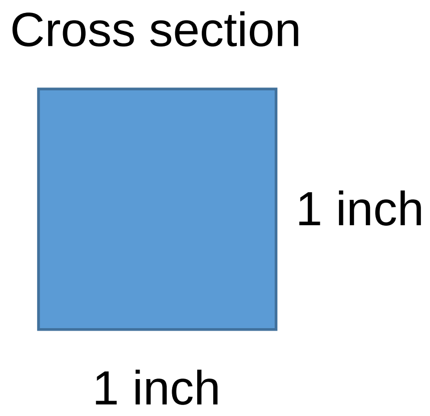


Figure 3.12. Axially loaded bar.



Exercise – Twist of thin-walled cylinder

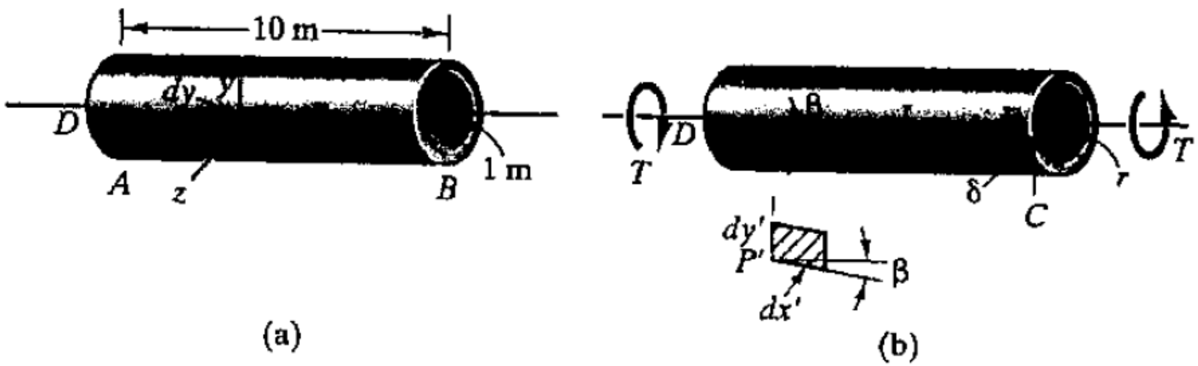


Figure 3.16. Thin-walled cylinder under torsion.

- A thin-walled cylinder is twisted by torques T, which leads to the twist of 15º relative to end A.
- Q: What is shear strain at point P on the surface of the cylinder?

Exercise – Circular rod hangs by its own weight

- A circular rod hangs by its own weight. If the normal strain in any direction is 1/E times the normal stress in that direction, what is the total deflection of the end A as a result of its weight? (Specific weight (=weight per unit volume, N/m^3) is a constant)

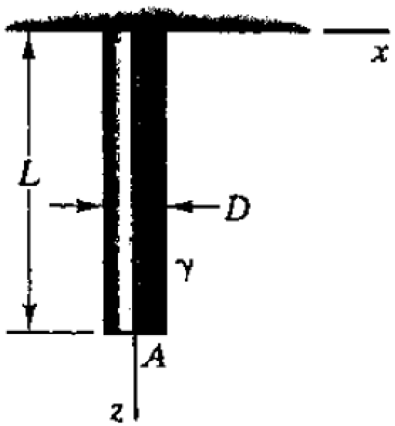
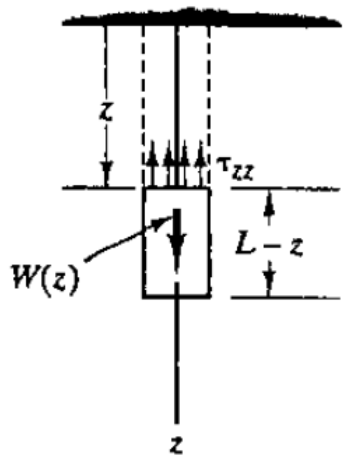


Figure 3.18. A rod suspended from above.

$$W(z) = \gamma \frac{\pi D^2}{4}(L - z)$$

Force equilibrium

$$\tau_{zz}\frac{\pi D^2}{4} = \gamma \frac{\pi D^2}{4}(L - z)$$



Constitutive equation (= relation between stress and strain)

$$\varepsilon_{zz} = \tau_{zz} / E$$

Kinematics (=displacement vs. strain)

$$d(dz) = \varepsilon_{zz} dz$$

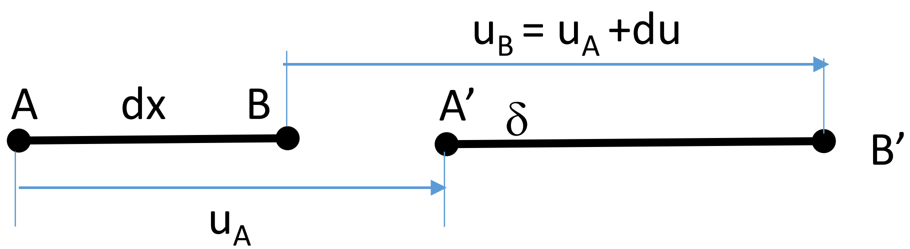
$$\boxed{\Delta = \int_0^L d(dz) = \int_0^L \varepsilon_{zz} dz}$$

Strain tensor

The kinematics: strain-displacement relationship which defines the strain (stretch or distortion) of material element

Displacement components: (u, v, w) which are functions of coordinate (x,y,z) or u=u(x,y,z) etc…

Consider two points A and B separated by a small distance dx. The material element experiences movement in the x direction. The displacement of A is u_A, while the displacement of B can be approximated by a Taylor's expansion of u(x) around the point x=A



$$u_B = u_A + du = u_A + \frac{\partial u}{\partial x} dx$$

Then, the differential motion (real stretch), δ $$\delta = u_B - u_A = \frac{\partial u}{\partial x} dx$$

By the definition of strain (displacement/initial length)

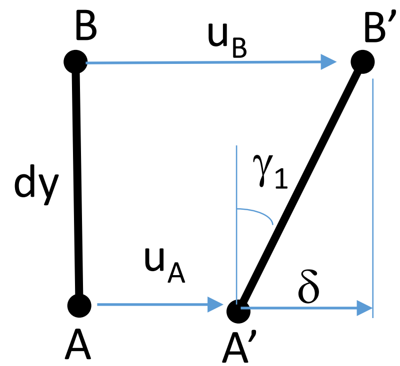
$$\varepsilon_{xx} = \frac{\delta}{dx} = \frac{\partial u}{\partial x}$$

Likewise, in the y- and z-direction

$$\varepsilon_{yy} = \frac{\delta}{dy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\delta}{dz} = \frac{\partial w}{\partial z}$$

Strain tensor

For the distortion (shearing) of the material, the following material element attached in the shearing material body can be considered



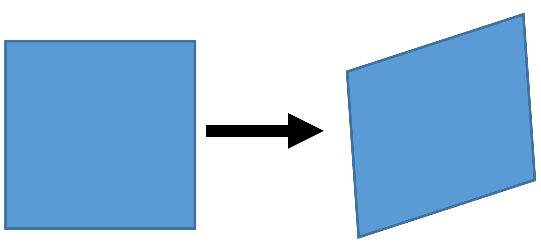
Then, the differential motion (real distortion), δ

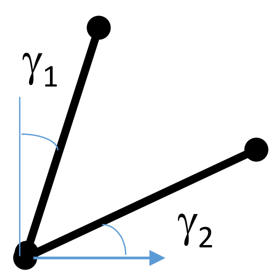
$$\delta = u_B - u_A = \left(u_A + \frac{\partial u}{\partial y} dy \right) - u_A = \frac{\partial u}{\partial y} dy$$

The change in angle is

$$\gamma_1 \cong \tan\gamma_1 = \frac{\delta}{dy} = \frac{\partial u}{\partial y}$$

Similarly, for 2D case





$$\boxed{\gamma_{xy} = \gamma_1 + \gamma_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}$$

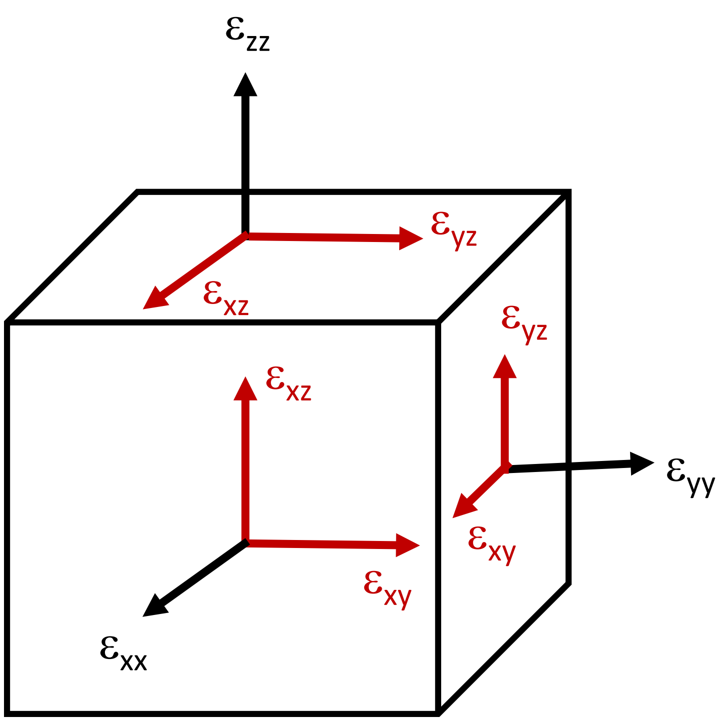
Strain tensor

Generalization in 3D: matrix notation

$$\varepsilon_{ij} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

Or

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$



- Strain tensor is symmetric, $\varepsilon_{ij}=\varepsilon_{ji}$
- Therefore, 6 independent components
- $\gamma_{ij}=2\varepsilon_{ji}$ (for example, $\gamma_{xy}=\varepsilon_{xy}+\varepsilon_{yx}=2\varepsilon_{xy}$)

Compatibility

- The displacements can be determined from the strains through integration.
- However, the strains are not independent but are cross related. For example, the 2-D strain-displacement relations are 3, while the displacements are only 2. In other words, when the strain components are known, the displacement cannot be uniquely determined (In contrast, if we know the two displacement, the strain can be determined)
- The relations between the strains are called “**compatibility conditions**”

For example,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y}, \quad \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y}\right) \quad \text{leads to} \quad \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

In general,

$$\boxed{\begin{aligned}
&\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}, \quad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \\
&\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \\
&\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z}\right)
\end{aligned}}$$

Questions ?