Engineering Mathematics 2

Lecture 2

Yong Sung Park

Previously, we discussed

- Vector vs. scalar
- position vector
- vector addition and scalar multiplication
- vector functions, scalar functions and their fields
- vector calculus, in particular, continuity, derivative and partial derivatives

9.5 Curves, arc length, curvature, torsion

- Parametric representation of a curve C,

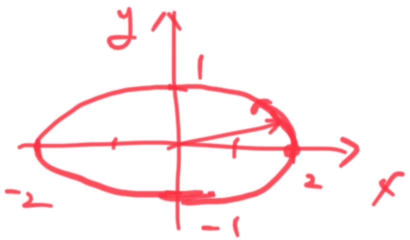
$$\vec{r}(t) = [x(t), y(t), z(t)\,]$$

  where x, y, z are coordinates of the position vector $\vec{r}$, which depends on the parameter t.

- Parametric representation is useful, since
  (i) coordinates play an equal role, and
  (ii) the parameter induces orientation of C (positive and negative senses).

(1) $x = 2\cos t$, $y = \sin t$

$$\left(\frac{x}{2}\right)^2 + (y)^2 = 1$$



(2)

- Examples

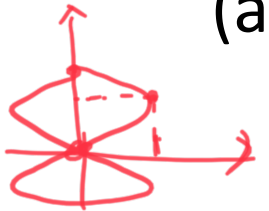
Sketch the curves represented by, respectively,

(1) $\vec{r}(t) = [2\cos t\,, \sin t\,, 0\,]$ (an ellipse)

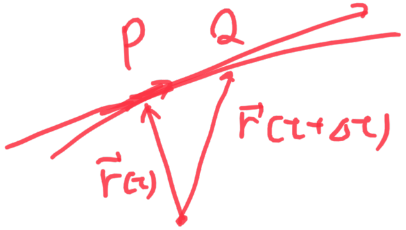
(2) $\vec{r}(t) = (3, 2) + t(1, 1)$ (a straight line)

(3) $\vec{r}(t) = [\cos t\,, \sin t\,, t\,]$ (a circular helix)

(4) $\vec{r}(t) = [\sin 2t\,, \cos t\,, 0\,]$ (a curve with a multiple point)



- Arc is a portion of a curve



$\vec{PQ} = \vec{r}(t+\Delta t) - \vec{r}(t)$

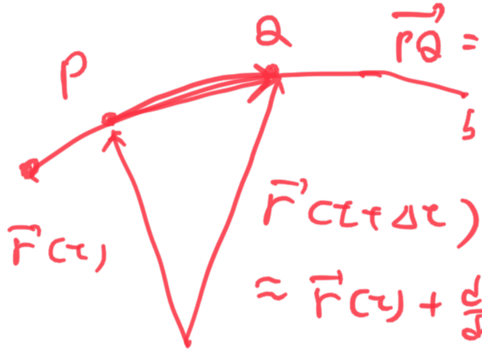
- Tangent to a simple curve C at a point P is obtained by taking the derivative of the position vector $\vec{r}(t)$ at the point, i.e. $\vec{r}'(t)$.

- The unit tangent vector

$$\boxed{\vec{u}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)\,|}}$$

- The tangent to C at P

$$\vec{q}(w) = \vec{r} + w\vec{r}\,'$$



$\vec{PQ} = \vec{r}(t+\Delta t) - \vec{r}(t)$

$\vec{r}(t+\Delta t) \approx \vec{r}(t) + \frac{d\vec{r}}{dt}\Delta t$

$\vec{PQ} \approx \frac{d\vec{r}}{dt}\Delta t + O(\Delta t^2)$

$|\vec{PQ}| = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}}\,\Delta t$

- Length of a curve C represented by $\vec{r}(t), a \leq t \leq b$:

$$l = \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{\Delta\vec{r}}{\Delta t}\cdot\frac{\Delta\vec{r}}{\Delta t}}\Delta t = \int_a^b \sqrt{\vec{r}'\cdot\vec{r}'}\,dt$$

where $\Delta\vec{r} = \vec{r}(t_k) - \vec{r}(t_{k-1})$ and $\Delta t = \frac{b-a}{n}$.

- Arc length s of a curve $\boxed{s = \int_a^t \sqrt{\vec{r}'\cdot\vec{r}'}\,d\tilde{t}}$.

$s = s(t)$

$\vec{r}' = \frac{d\vec{r}}{d\tilde{t}}$

$\frac{d\vec{r}(s)}{ds} = \vec{r}'(s) = \vec{u}(s)$

$1 = \frac{ds}{dt}\Big|_s = \sqrt{\vec{r}'\cdot\vec{r}'} = \left|\frac{d\vec{r}}{dt}\right|_s$

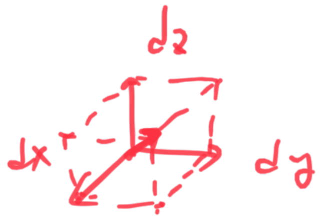
- Linear element ds :

$$\frac{ds}{dt} = \sqrt{\vec{r}' \cdot \vec{r}'} = \left|\frac{d\vec{r}}{dt}\right|$$

$$\left(\frac{ds}{dt}\right)^2 = \left|\frac{d\vec{r}}{dt}\right|^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

- denoting $\mathrm{d}\vec{r} = [dx, dy, dz]$,

$$ds^2 = dx^2 + dy^2 + dz^2$$



- Arc length is a natural parameter, which makes calculations simpler

e.g. the unit tangent vector $\vec{u}(s) = \dfrac{\vec{r}\,'(s)}{|\vec{r}\,'(s)\,|} = \vec{r}\,'(s)$.

Since t is arbitrary, by setting $t = s$, we get

$$\left|\frac{d\vec{r}}{ds}\right| = \frac{ds}{ds} = 1$$

- Show that a vector of a constant length is perpendicular to its derivative.

$$\left|\vec{r}(t)\right| = C$$

$$\left(\vec{r}(t) \cdot \vec{r}(t)\right)' = \left(C^2\right)'$$

$$\vec{r}\,' \cdot \vec{r} + \vec{r} \cdot \vec{r}\,' = 0$$

$$2\,\vec{r} \cdot \vec{r}\,' = 0$$

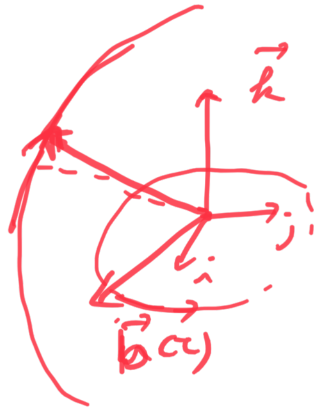
$$\underline{\vec{r} \cdot \vec{r}\,' = 0}$$

- For a curve C representing a path of moving body, $\vec{r}(t)$, time

  find velocity and acceleration vectors.

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$



- Example 8 Coriolis acceleration

  Calculate the acceleration vector of a projectile with its trajectory

$$\vec{r}(t) = R\cos\gamma t\,\vec{b}(t) + R\sin\gamma t\,\vec{k},$$

where

$$\vec{b}(t) = \cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}$$

is a unit vector normal to the axis of earth rotation in the direction $\vec{k}$.

$$\frac{d\vec{r}}{dt} = -R\gamma \sin\gamma t \, \vec{b}(t) + \underline{R\cos\gamma t \frac{d\vec{b}}{dt}} + \gamma R \cos\gamma t \, \vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = \boxed{-R\gamma^2 \cos\gamma t \, \vec{b}(t) - R\gamma^2 \sin\gamma t \, \vec{k}} - 2R\gamma \sin\gamma t \frac{d\vec{b}}{dt} + R\cos\gamma t \frac{d^2\vec{b}}{dt^2}$$

coriolis effect

$$= \underline{-\gamma^2 \vec{r}} \; \boxed{- 2R\gamma \sin\gamma t \frac{d\vec{b}}{dt}} + \underline{R\cos\gamma t} \frac{d^2\vec{b}}{dt^2}$$

$$\frac{d\vec{b}}{dt} = -\omega \sin\omega t \, \vec{i} + \omega \cos\omega t \, \vec{j}$$

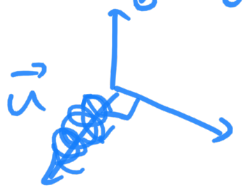
$$\frac{d^2\vec{b}}{dt^2} = -\omega^2 (\cos\omega t \, i + \sin\omega t \, \vec{j}) = \underline{-\omega^2 \vec{b}}$$

- curvature: measure of the rate of change of unit tangent vector

$$\kappa(s) = |\vec{u}'(s)| = |\vec{r}''(s)|$$

- tortion: measure of the rate of change of the osculating plane (spanned by $\vec{u}$ and $\vec{u}'$)

$\vec{b} = \vec{u} \times \vec{p}$



$\frac{\vec{u}'}{|\vec{u}'|} = \vec{p}$

$$\tau(s) = -\vec{p} \cdot \vec{b}'$$

$|\tau| = |\vec{b}'|$

$\vec{u} \cdot \vec{b} = 0,\quad \cancel{\vec{u}' \cdot \vec{b}} + \boxed{\vec{u} \cdot \vec{b}' = 0}$

9.6 Calculus review

- Chain rule:

  Given $w = f(x, y, z)$

  (1) For $x = x(u, v), y = y(u, v), z = z(u, v)$, find partial derivatives of w with respect to u and v .

  $f = f(u,v)$

  $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$

  (2) For $z = g(x, y)$, find partial derivatives of w w. r. t. x and y.

  $f(x,y,z) = f(x, y, g(x,y))$

  $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$