Engineering Mathematics 2

Lecture 21

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Concluding the semester

- 7 December: Q&A
- 9 December: Exam 3 (Refer to Course Announcement for specifics)

- Exam 3 covers Chapters 11 & 12.

Previously, we discussed

- Solution of 2D wave equation for rectangular membrane

  (1) method of separation of variables in two steps:

  $$u(x, y, t) = F(x, y)G(t),$$

  $$F(x, y) = H(x)Q(y),$$

  (2) then, double Fourier series (to satisfy the initial condition).

$$u(x, y, t) = \sum_{m} \sum_{n} (B_{mn} \cos \lambda_{mn} t + B_{mn}^{*} \sin \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

12.10 Circular membrane

- 2D wave equation in Cartesian coordinate:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- Want to express this in polar coordinate:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

- Exercise:

  Using

$$u_x = u_r r_x + u_\theta \theta_x$$

  express u_{xx} in terms of r and θ.

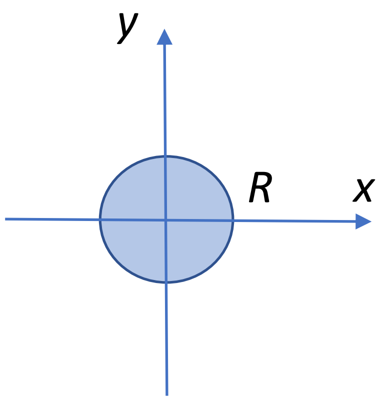
- Exercise:

  Show that

$$r_x = \frac{x}{r} \qquad \text{and} \qquad \theta_x = -\frac{y}{r^2}$$

- Wave equation for circular membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$



- Let us consider radially symmetric case:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

with BC: $u(R,t) = 0$ for all $t \geq 0$ and IC:

$$u(r,0) = f(r), \qquad u_t(r,0) = g(r)$$

- Exercise:

  Using the separation of variables

$$u(r,t) = W(r)G(t),$$

  obtain two ODEs :

$$\ddot{G} + \lambda^2 G = 0 \text{ and } W'' + \frac{1}{r}W' + k^2 W = 0 \text{ where } \lambda = ck.$$

- Exercise:

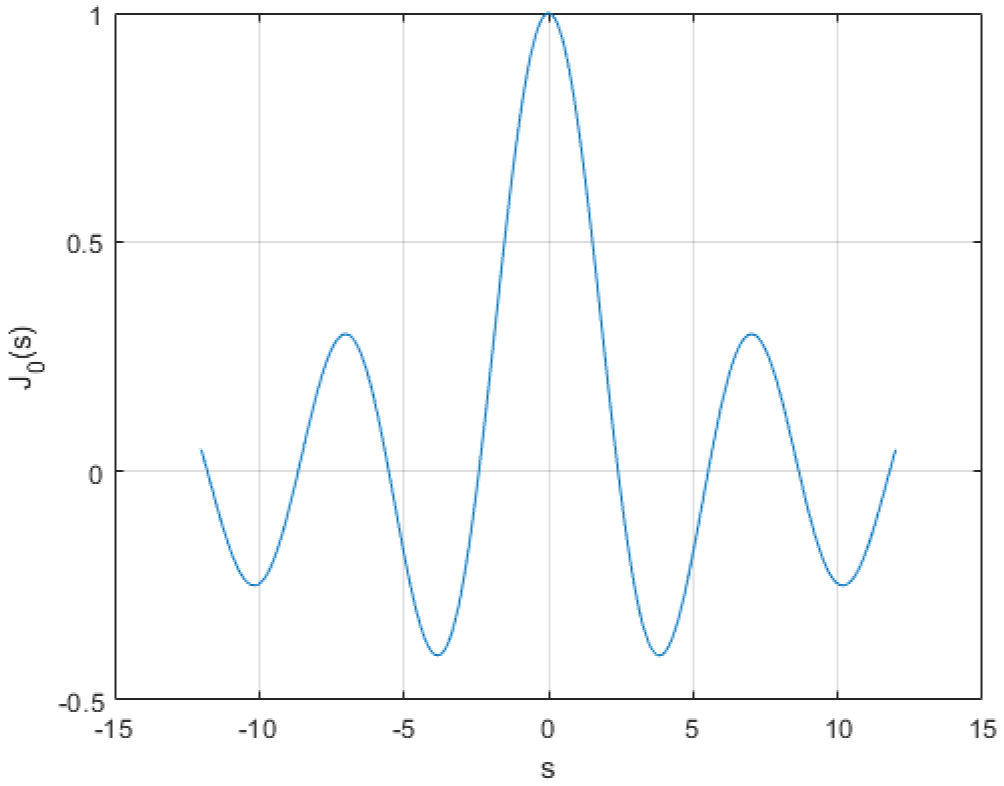
  By setting $s = kr$, show that the equation

$$W'' + \frac{1}{r}W' + k^2 W = 0$$

  can reduce to Bessel's equation:

$$\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W = 0$$

```
s = linspace(-12,12,301);
plot(s,besselj(0,s))
xlabel('s'), ylabel('J_{0}(s)'), grid on
```




```
s = linspace(-12,12,300);
plot(s,bessely(0,s))
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xlabel('s'), ylabel('Y_{0}(s)'), grid on, axis([0, 15 -1 1])
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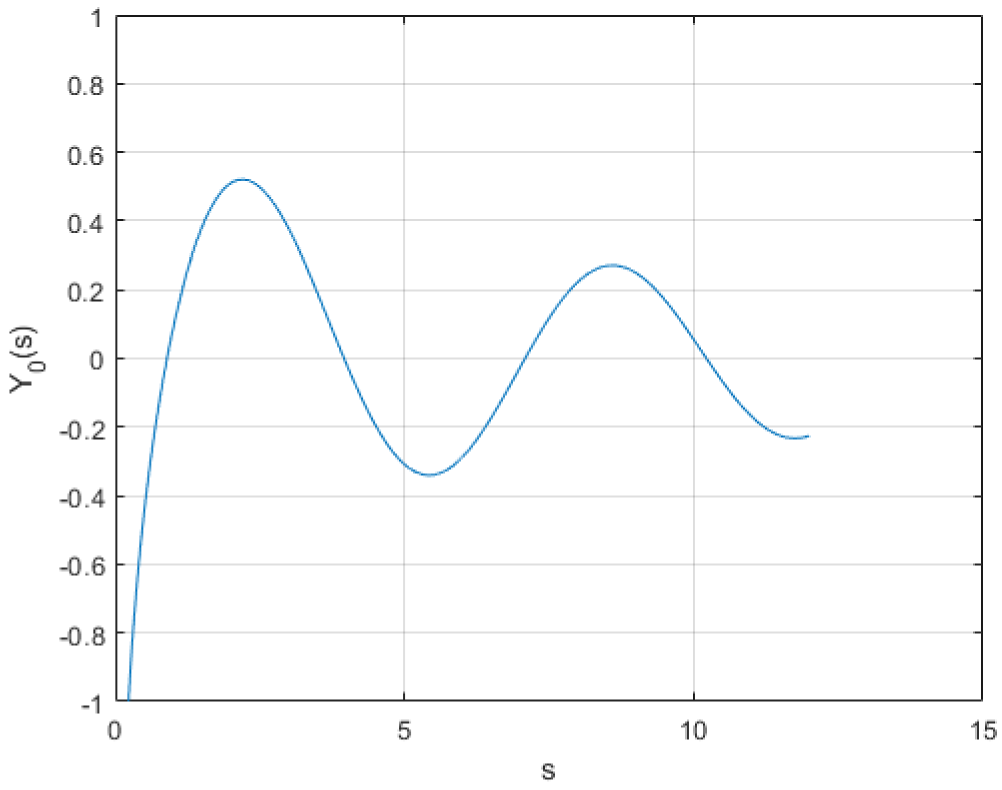

- Satisfying the boundary condition, $W_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{R} r\right)$

  Then, eigenfuctions for the wave equation for the circular membrane are

$$u_m(r,t) = (A_m \cos \lambda_m t + B_m \sin \lambda_m t) J_0\left(\frac{\alpha_m}{R} r\right)$$

- To satisfy the initial conditions

$$u(r,t) = \sum_{m=1}^{\infty} (A_m \cos \lambda_m t + B_m \sin \lambda_m t) J_0\left(\frac{\alpha_m}{R} r\right)$$

- Initial condition $u(r,0) = \sum_{m=1}^{\infty} A_m J_0\left(\frac{\alpha_m}{R} r\right) = f(r)$

- Fourier-Bessel series:

$$A_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r f(r) J_0\left(\frac{\alpha_m}{R} r\right) dr$$

- Exercise: find B_m