

⑥ ADI method using artificial time derivative term

노트 제목

2019-12-02

$$\nabla^2 \phi = b$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - b$$

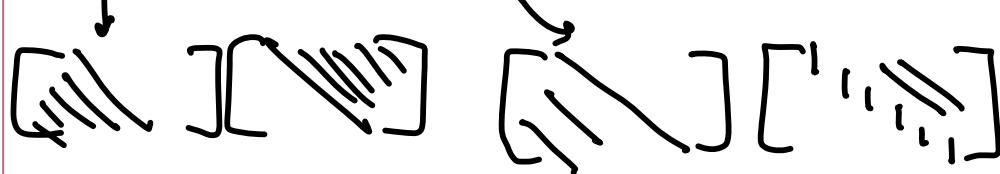
EE X
IE O

not very popular

⑦ strongly implicit procedure (SIP) stone (1968)

$$A\phi = b \rightarrow LU\phi^{k+1} = (LU-A)\phi^k - b$$

SIAM J. Num. Anal.
§, 530.



find L & U

s.t. $LU \approx A$

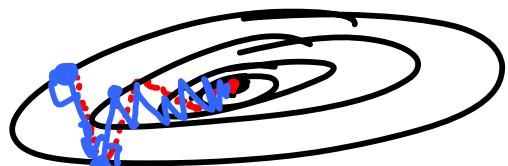
incomplete
LU decomp.
(ILU)

⑧

Conjugate Gradient Solver (CGS)

Hestenes & Stiefel (1952) Nat. Bur. Stand. J. Res. 49, 409.

Kershaw (1978) J. Comp. Phys. 26, 43.



— simple grad. method (steepest descent method)
 conjugate grad. method

for a symmetric matrix A , ($a_{ij} = a_{ji}$)

$$\phi = \frac{1}{2} x^T A x - x^T b = \frac{1}{2} \sum a_{ij} x_i x_j - \sum x_i b_i$$

$$\frac{\partial \phi}{\partial x_k} = \frac{1}{2} \sum a_{kj} x_j + \frac{1}{2} \sum a_{ik} x_i - b_k = 0 \text{ for } \min \phi$$

$$\rightarrow \sum a_{kj} x_j = b_k \quad (Ax = b) \rightarrow x = A^{-1} b$$

$$\begin{aligned}\Phi_{\min} &= \frac{1}{2} (A^{-1}b)^T A (A^{-1}b) - (A^{-1}b)^T b \\ &= \frac{1}{2} b^T A^{-1 T} A A^{-1} b - b^T A^{-1 T} b = -\frac{1}{2} b^T A^{-1 T} b\end{aligned}$$

\therefore minimizing ϕ & solving $Ax = b$ are equivalent probs.

One of the simplest strategies for minimizing ϕ is the method of steepest descent.

At a current point x_c , the ft ϕ decreases most rapidly in the direction of the negative gradient.

$$-\nabla \phi : -\frac{\partial \phi}{\partial x_i} = -\sum a_{ij} x_j + b_i = r_i^- \quad (\text{residual of } Ax = b)$$

If residual is non-zero, $\exists \alpha$ s.t. $\phi(x_i + \alpha r_i^-) < \phi(x_i)$.

$$\phi(x_i + \alpha r_i) = \frac{1}{2} \sum a_{ij} (x_i + \alpha r_i)(x_j + \alpha r_j) - \sum b_i (x_i + \alpha r_i)$$

$$\begin{aligned}\frac{\partial \phi}{\partial \alpha} &= \frac{1}{2} \sum a_{ij} r_i (x_j + \alpha r_j) + \frac{1}{2} \sum a_{ij} r_j (x_i + \alpha r_i) - \sum b_i r_i \\ &= \alpha \sum a_{ij} r_i r_j + \sum a_{ij} x_j r_i - \sum b_i r_i \quad (a_{ij} = a_{ji}) \\ &= \alpha \sum a_{ij} r_i r_j + \sum r_i (\underbrace{a_{ij} x_j - b_i}_{-r_i}) = 0\end{aligned}$$

$$\Rightarrow \alpha = \frac{\sum r_i r_i}{\sum a_{ij} r_i r_j} \quad k=0 : x_0 = 0, r_0 = b$$



$$\phi(x_i + \alpha p_i) < \phi(x_i) \quad p_i \text{ arbitrary}$$

$$q(\alpha) = \frac{1}{2} \sum a_{ij} (x_i + \alpha p_i)(x_j + \alpha p_j) - \sum b_i (x_i + \alpha p_i)$$

$$\frac{\partial \phi}{\partial \alpha} = \dots = \alpha \sum a_{ij} p_i p_j + \sum p_i (\underbrace{a_{ij} x_j - b_i}_{-r_i}) = 0$$

$$\Rightarrow \alpha = \frac{\sum p_i r_i}{\sum a_{ij} p_i p_j} = p^T r / p^T A p .$$

Now, pick two directions p_{1i}, p_{2i} .

$$x_i = x_i^0 + \alpha_1 p_{1i} + \alpha_2 p_{2i}$$

$$\phi(\alpha_1, \alpha_2) = \frac{1}{2} \sum a_{ij} (x_i^0 + \alpha_1 p_{1i} + \alpha_2 p_{2i})(x_j^0 + \alpha_1 p_{1j} + \alpha_2 p_{2j})$$

$$-\sum b_i^-(x_i^0 + \alpha_1 p_{1i}^- + \alpha_2 p_{2i}^-)$$

①

$$\frac{\partial \Phi}{\partial \alpha_1} = \sum a_{ij} p_{1j}^- x_i^0 + \alpha_1 \sum a_{ij} p_{1i}^- + \alpha_2 \sum a_{ij} p_{2i}^- p_{1j} - \sum b_i^- p_{1i}^- = 0$$

$$\frac{\partial \Phi}{\partial \alpha_2} = \sum a_{ij} p_{2j}^- x_i^0 + \alpha_1 \sum a_{ij} p_{1j}^- p_{2j}^- + \alpha_2 \sum a_{ij} p_{2i}^- p_{2j}^- - \sum b_i^- p_{2i}^- = 0$$

②

$$\text{Set } \sum a_{ij}^- p_{1i}^- p_{2j}^- = 0$$

$P_1^T A P_2 = 0 \rightarrow P_1 \text{ & } P_2 \text{ are conjugate.}$

Then, from ①

$$\sum a_{ij}^- p_{1j}^- x_i^0 + \alpha_1 \sum a_{ij}^- p_{1i}^- p_{1j}^- - \sum b_i^- p_{1i}^- = 0$$

$$\rightarrow \alpha_1 = - \frac{\sum p_{1i}^- (a_{ij}^- x_j^0 - b_i^-)}{\sum a_{ij}^- p_{1i}^- p_{1j}^-} = \frac{P_1^T r^0}{P_1^T A P_1}$$

from (2), - - - .

$$\rightarrow \alpha_2 = \frac{\sum P_{2i} r_i^T}{\sum a_{ij} P_{2i} P_{2j}} = \frac{P_2^T r^T}{P_2^T A P_2}$$

requiring $P_{k+1} A P_k = 0$, $P_k = r^{k+1} + \beta_k P_{k-1}$

$$\rightarrow \beta_k = -P_{k-1} A r^{k+1} / P_{k-1} A P_{k-1}$$

Algorithm : Guess x^0

$$r^k = -A x^k + b$$

if $k=0$, $P_k = r^k$ ← steepest descent

else, $\beta_k = -P_{k-1} A r^{k+1} / P_{k-1} A P_{k-1}$

$$P_k = r^{k+1} + \beta_k P_{k-1}$$

$$\left(\begin{array}{l} \alpha_k = r^{k+1} r^{k+1} / \|P_k A P_k\| \\ x^{k+1} = x^{k+1} + \alpha_k P_k \end{array} \right)$$



rate of convergence depends on condition number of A .

* Supplementary lecture : MW : 9 am - 10:45 am

(Thur)
* Final exam : Dec. 12

6:30 pm -

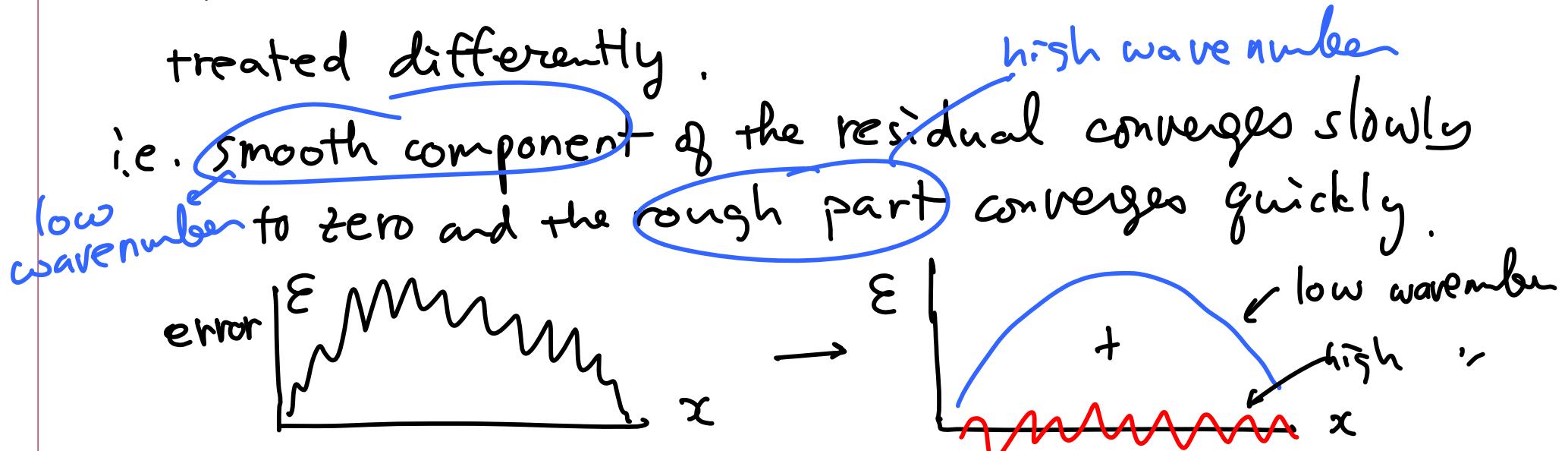
(ODE \rightarrow ")
Ch.5

⑨

Multigrid method (multigrid acceleration)

One of the most powerful acceleration schemes for the convergence of iterative methods in solving elliptic problems.

→ different components of the solution converge to the exact solution at different rates and thus should be treated differently.



$$A\phi = b$$

ψ : $\psi = \phi^k$ is an approx. to the sol. of after k iterations.

r : residual $r = b - A\psi$

ε : error $\varepsilon = \phi - \psi$

$$A\varepsilon = A(\phi - \psi) = b - A\psi = r \rightarrow$$

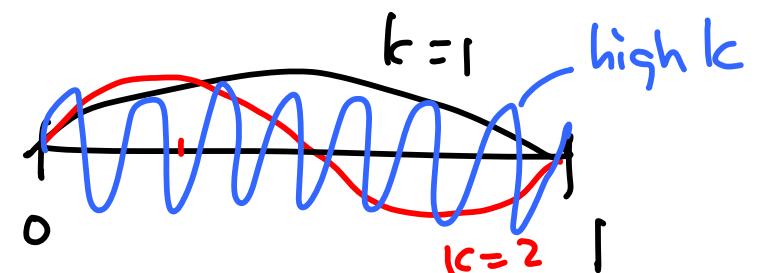
$$A\varepsilon = r$$
 residual eq.

ex) $\frac{d^2u}{dx^2} = \sin k\pi x \quad (0 \leq x \leq 1)$

$$u(0) = u(1) = 0$$

k : wavenumber

CD2: $\left[\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} \right] = \sin k\pi x_j, \quad j = 1, 2, \dots, N-1$



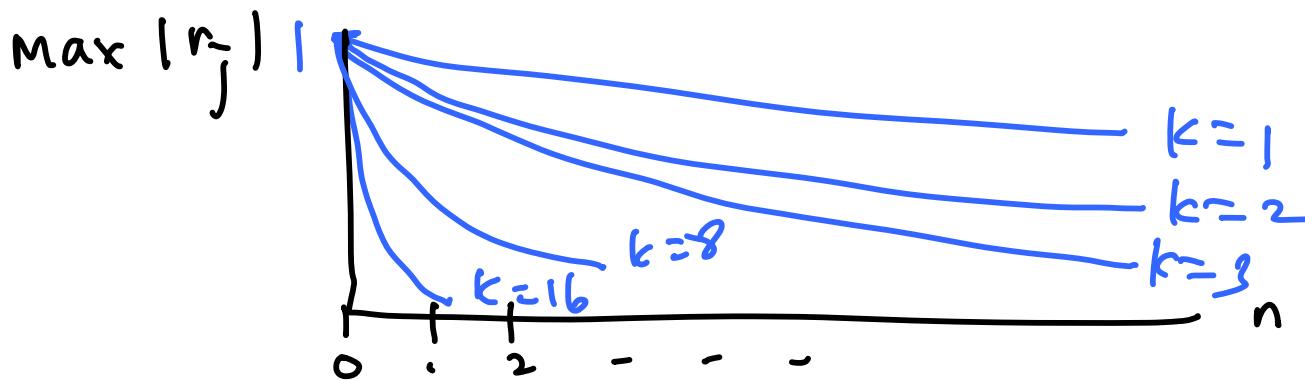
$$\{ u_0 = u_N = 0 \}$$

initial guess: $u^{(0)} = 0 \rightarrow r_j^{(0)} = \sin(k\pi x_j)$

$$GS: \frac{u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}}{\Delta x^2} = \sin k\pi x_j \quad n: \text{iteration index}$$

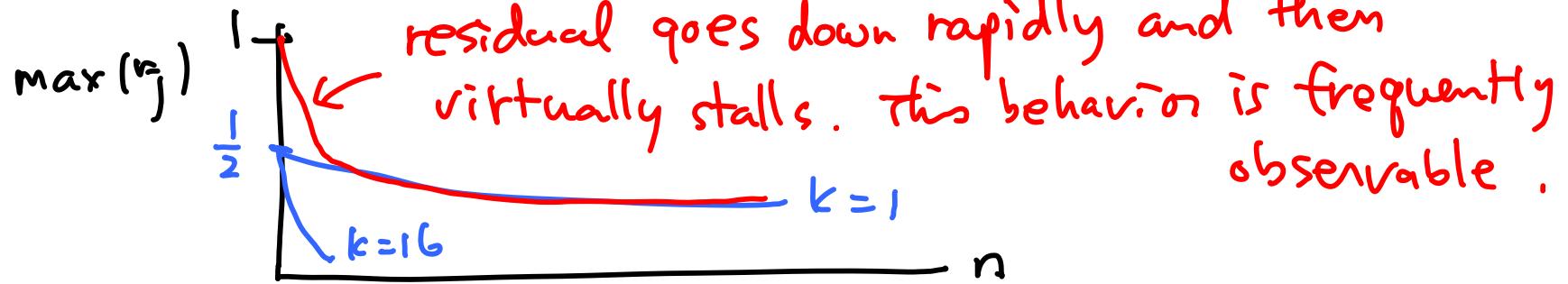
$$\rightarrow u_j^{(n+1)} = \frac{1}{2} (u_{j+1}^{(n)} + u_{j-1}^{(n)} - \Delta x^2 \sin k\pi x_j), \quad j=1, 2, \dots, N-1$$

$$r_j^{(n)} = \sin k\pi x_j - Au_j^{(n)}$$

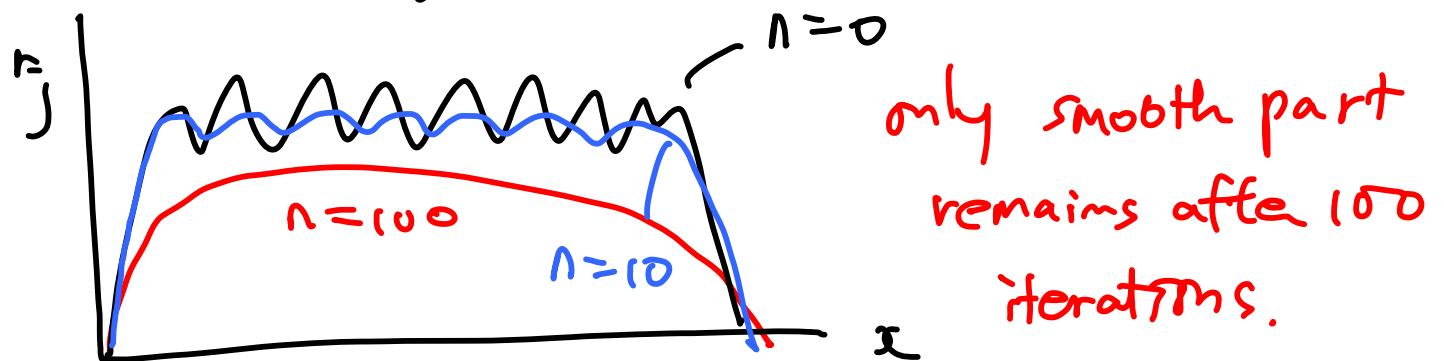


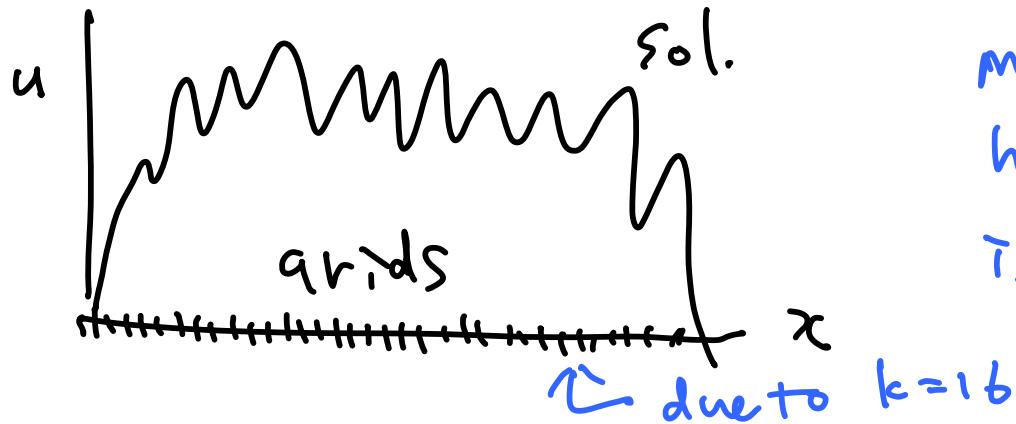
Convergence is faster for high values of k .

$$\text{ex) } \frac{d^2y}{dx^2} = \frac{1}{2} (\sin \pi x + \sin 16\pi x) \quad t=1 \text{ & } 16$$



The reason is that the rapidly varying part of the residual goes to zero quickly and the smooth part of it remains.





many grids are required for high k 's, but the convergence is fast for high k and slow for low k .

As $N \uparrow$, $|\lambda| \rightarrow 1 \rightarrow$ slow convergence
(due to high k)

t comes from low k

reduce N to $N/2 \rightarrow |\lambda|$ gets smaller.

for low k , $N/2$ is fine for resolution. \rightarrow faster convergence
 \rightarrow this is the basic idea of multigrid method
 A. Brandt. Math. Comput. 21. 233 (1977)