

6.1.6 Discrete Sine and cosine transforms

노트 제목

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$$\frac{2d}{2\pi} = 0$$

f is not periodic \rightarrow cannot use FT.

f is an even ft., $f(x) = f(-x)$: cosine transform

f is an odd ft., $f(x) = -f(-x)$: sine transform

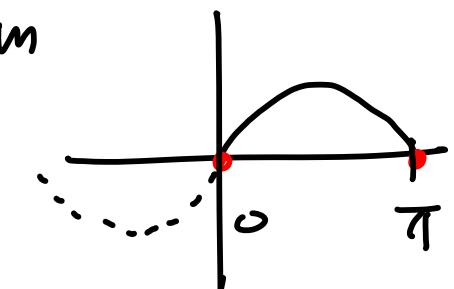
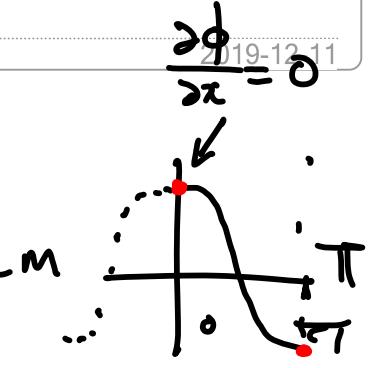
f_j : $N+1$ pts on $0 \leq x \leq \pi$, $x_j = h_j$, $j = \pi/N$

* cosine transform

$$\left\{ \begin{array}{l} f_j = \sum_{k=0}^N a_k \cos kx_j, \quad j = 0, 1, 2, \dots, N \\ a_k = \frac{2}{c_k N} \sum_{j=0}^N c_j f_j \cos kx_j, \quad k = 0, 1, 2, \dots, N \end{array} \right.$$

$$a_k = \frac{2}{c_k N} \sum_{j=0}^N c_j f_j \cos kx_j, \quad k = 0, 1, 2, \dots, N$$

where $c_j = \begin{cases} 2 & \text{if } j=0, N \\ 1 & \text{otherwise} \end{cases}$



$$\text{orthogonality } \sum_{j=0}^N \frac{1}{c_j} \cos kx_j \cos k' x_j = \begin{cases} 0 & \text{for } k \neq k' \\ \frac{1}{2} c_k N & \text{for } k = k' \end{cases}$$

* Sine transform

$$\left\{ \begin{array}{l} f_j = \sum_{k=0}^N b_k \sin k x_j, \quad j = 0, 1, 2, \dots, N \\ b_k = \frac{2}{N} \sum_{j=0}^N f_j \sin k x_j, \quad k = 0, 1, 2, \dots, N \end{array} \right.$$

6.2 Application of discrete Fourier Series

6.2.1 Direct sol. of finite difference elliptic egs.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = Q(x, y) \quad \text{with } \phi = 0 \text{ on boundaries}$$

CD 2

$\Delta x = \frac{\pi}{M}$
 $\Delta y = \frac{\pi}{N}$

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = \hat{Q}_{i,j}$$

$i=1, 2, \dots, M-1, \quad j=1, 2, \dots, N-1$

System of linear algebraic eqs. \rightarrow expensive to solve

Use Fourier sine series in x

Assume $\phi_{i,j} = \sum_{k=1}^{M-1} \hat{\phi}_{k,j} \sin kx_i, \quad x_i = h_x^{-} = \frac{\pi}{M} i$

$$\hat{Q}_{i,j} = \sum_{k=1}^{M-1} \hat{Q}_{k,j} \sin kx_i$$

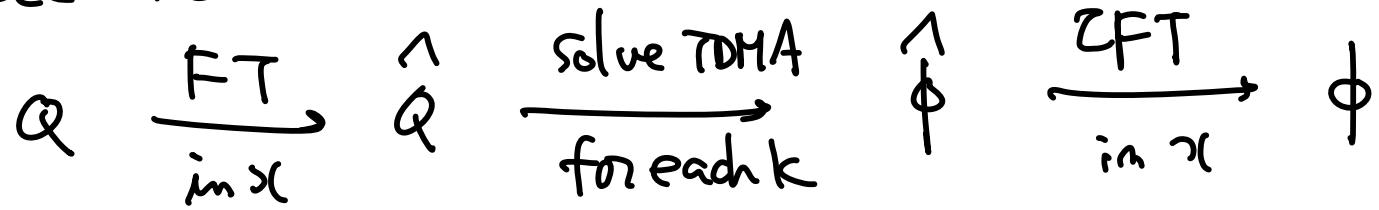
$$\begin{aligned}
 \rightarrow \textcircled{*}: & \sum_{k=1}^{M-1} \hat{\phi}_{k,j} \left(\underbrace{\sin kx_{i+1} - 2\sin kx_i + \sin kx_{i-1}}_{\sin kx_i} \right) / \partial x^2 \\
 & + \sum_{k=1}^{M-1} \left(\hat{\phi}_{k,j+1} - 2\hat{\phi}_{k,j} + \hat{\phi}_{k,j-1} \right) \cancel{\sin kx_i} / \partial y^2 \\
 & = \sum_{k=1}^{M-1} \hat{\phi}_{k,j} \sin kx_i \quad \sin kx_i \cdot \left(2\cos \frac{\pi k}{M} - 2 \right)
 \end{aligned}$$

Equating the coeff. of $\sin kx_i$ gives

$$\hat{\phi}_{k,j+1} + \left[\frac{\partial y^2}{\partial x^2} \left(2\cos \frac{\pi k}{M} - 2 \right) - 2 \right] \hat{\phi}_{k,j} + \hat{\phi}_{k,j-1} = \partial y^2 \hat{\phi}_{k,j} \quad k=1, 2, \dots, M-1$$

For each k , tri-diagonal sys. of eqs \rightarrow easy to solve

• procedure



$$N \Theta(M \log_2 M) \quad \text{MoCN}$$

$$N \Theta(M \log_2 M)$$

$\Rightarrow Q(MN \log_2 M)$ operations direct and low cost method

constraints : uniform grids in one direction

coeff. of PDE should not be a ft. of
transform direction

$$\frac{\partial \phi}{\partial x} = 0 \quad \downarrow$$

$$\frac{\partial}{\partial x} \left(\mu(\epsilon) \frac{\partial \phi}{\partial x} \right) \propto$$

Neumann b.c. \rightarrow use cosine transform

6.2.2 Differentiation of a periodic ft. using Fourier spectral method
 $f(x)$: periodic ft.

N equally spaced grid pts. $x_j = \sigma x j$, $j = 0, 1, 2, \dots, N-1$

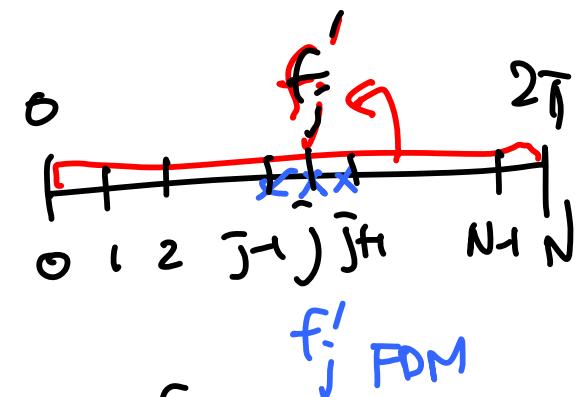
$$f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx_j}, \quad j = 0, 1, \dots, N-1$$

$$\frac{\partial f}{\partial x_j} = \sum_{k=-N/2}^{N/2-1} \hat{f}_k ik e^{ikx_j}$$

To get spectral derivative of f ,

$$f_j \xrightarrow{\text{FT}} \hat{f}_k \xrightarrow{i\sigma} i\hat{f}_k \xrightarrow{\text{IFT}} \frac{\partial f}{\partial x_j}$$

$$k = -\frac{N}{2}, -\frac{N}{2}+1, \dots, 0, \dots, \frac{N}{2}-1$$



$$\text{odd ball} \quad \hat{f}_{k=-\frac{N}{2}} \neq 0 \quad f' : ik\hat{f} \gg \epsilon \Rightarrow ik\hat{f}_{-\frac{N}{2}} = 0$$

\uparrow
 $-\frac{N}{2}$

before IFT

ensures that the derivative remains real in physical space.

This spectral derivative provides exact derivative of the harmonic ft. $f(x) = e^{ikx}$ at the grid pts. if $|k| \leq \frac{N}{2}-1$.

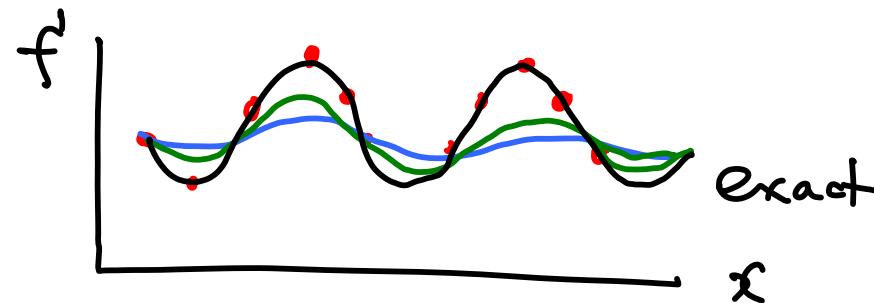
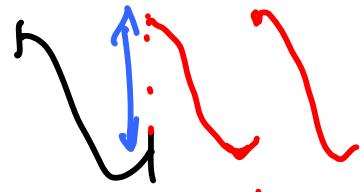
Spectral derivative is more accurate than any finite difference scheme for periodic ft., but cost is higher due to FFT.

$$f = \cos 3x$$

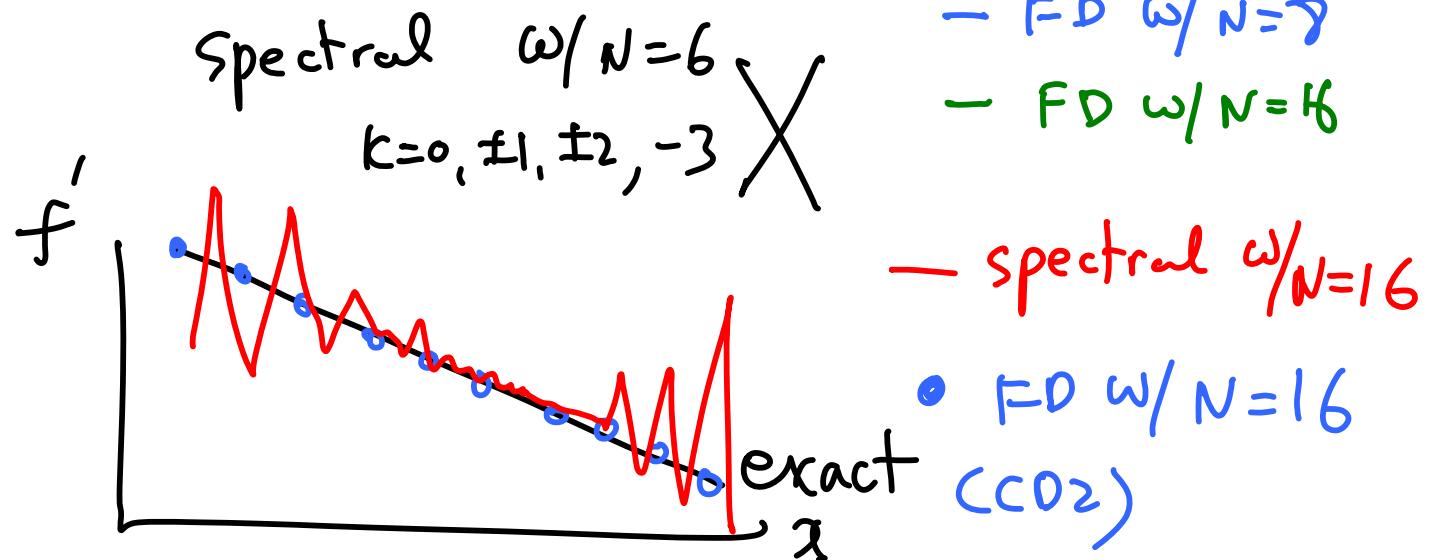
$$f' = -3 \sin 3x$$

$$f = 2\pi x - x^2$$

$$f' = 2\pi - 2x$$



- Spectral $\omega/N=8$
 $k=0, \pm 1, \pm 2, \pm 3, -4$
- FD $\omega/N=8$
- FD $\omega/N=16$



6.2.3 Numerical sol. of linear, const. coeff. diff'l. eq.

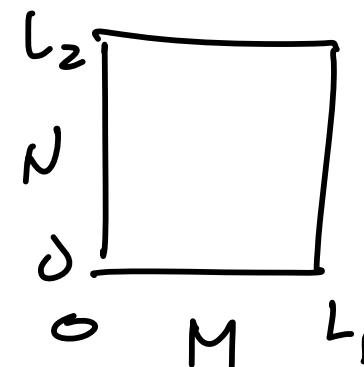
w/ periodic b.c.'s

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = Q(x, y)$$

$$P = \sum_{k_1} \sum_{k_2} \hat{P}(k_1, k_2) e^{ik_1 x} e^{ik_2 y}$$

$$Q = \sum_{k_1} \sum_{k_2} \hat{Q}(k_1, k_2) \quad " \quad "$$

$$-k_1^2 \hat{P}_{k_1, k_2} - k_2^2 \hat{P}_{k_1, k_2} = \hat{Q}_{k_1, k_2}$$



$$k_1 = \frac{2\pi}{L_1} n_1$$

$$k_2 = \frac{2\pi}{L_2} n_2$$

$$\hat{P}_{k_1, k_2} = -\frac{\hat{Q}_{k_1, k_2}}{k_1^2 + k_2^2}$$

for $k_1 = k_2 \neq 0$

$$\hat{P}_{k_1, k_2} = \frac{1}{M} \frac{1}{N} \sum_{l=0}^{M-1} \sum_{j=0}^{N-1} P_{l,j} e^{-ik_1 x_l} e^{-ik_2 y_j}$$

$$\rightarrow \hat{P}_{0,0} = \frac{1}{M} \frac{1}{N} \sum_{l=0}^{M-1} \sum_{j=0}^{N-1} P_{l,j} : \text{average of } p \text{ over the domain}$$

Sol. of Poisson eq. w/ periodic b.c.'s is indeterminant
to within an arbitrary constant.

Thus, set $\hat{P}_{0,0} = c$ (e.g., $c=0$)

$$Q_{l,j} \xrightarrow{\text{FT}} \hat{Q}_{k_1, k_2} \longrightarrow \frac{-\hat{Q}_{k_1, k_2}}{k_1^2 + k_2^2} = \hat{P}_{k_1, k_2} \xrightarrow{\text{IFT}} P_{l,j}$$

direct sol.

$$\iint \left[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = Q \right] dx dy \rightarrow \iint \left[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right] dx dy = 0$$



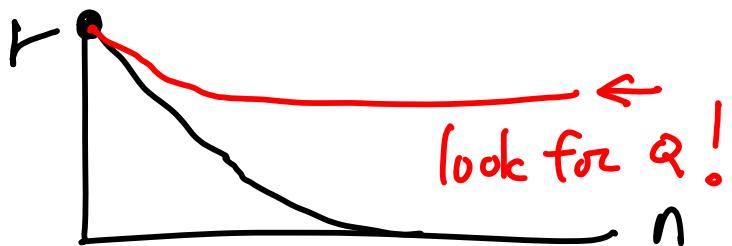
$$\iint Q dx dy = 0$$



$$\sum \sum Q = 0 \rightarrow Q_{0,0} = 0$$

↑
required for the well
posedness.

iterative method



$$\bullet \quad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f$$

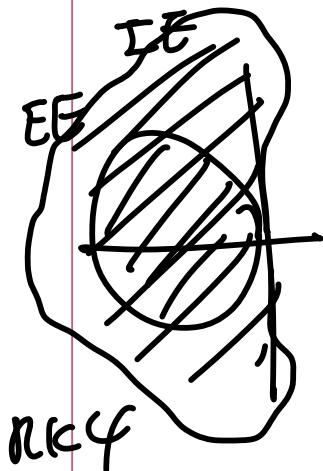
u : periodic in \mathbb{Z} \rightarrow FT

$$u = \sum \hat{u}_k e^{ikx}$$

$$\frac{d\hat{u}_k}{dt} + ik\hat{u}_k = -\nu k^2 \hat{u}_k + \hat{f}_k$$

$$\rightarrow \frac{d\hat{u}_k}{dt} = (-\nu k^2 - ik) \hat{u}_k + \hat{f}_k \quad \text{for } k = -\frac{N}{2}, \dots, 0, \dots, \frac{N}{2}-1$$

$\underbrace{\hspace{1cm}}_{\lambda}$



- apply numerical method for time integration
- do TFT to get u

6.4 Discrete Chebyshev transform and applications non-periodic ft. or non-uniform mesh

Chebyshev transform

$u(x)$ defined in $-1 \leq x \leq 1$

$$u(x) = \sum_{n=0}^N a_n T_n(x)$$

$$a \leq x \leq b$$

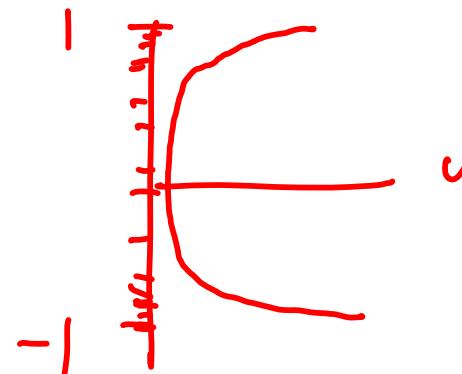
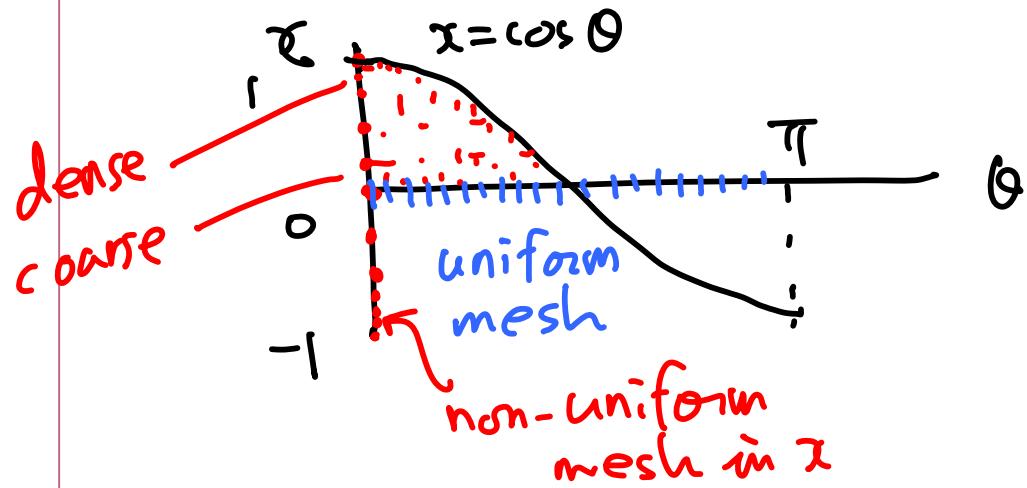
$$\hookrightarrow -1 \leq \xi \leq 1$$

$T_n(x)$: Chebyshev polynomials

$$\left(\frac{d}{dx} \left[\sqrt{1-x^2} \frac{dT_n}{dx} \right] + \frac{\lambda_n}{\sqrt{1-x^2}} T_n = 0, \quad \lambda_n = n^2 \right)$$

$$T_0 = 1, \quad T_1 = x, \quad T_2 = 2x^2 - 1, \quad T_3 = 4x^3 - 3x, \quad \dots$$

$$-1 \leq x \leq 1 \xrightarrow{x = \cos \theta} 0 \leq \theta \leq \pi \rightarrow T_n(\cos \theta) = \cos n \theta$$



recursive relation $T_{n+1}(x) + T_{n-1}(x) = 2x T_n(x) \quad (n \geq 1)$

mesh: cosine mesh

$$x_j = \cos \frac{\pi j}{N}$$

$j = 0, 1, \dots, N$

Discrete chebyshev transf.

$$u_j = \sum_{n=0}^N a_n T_n(x_j) = \sum_{n=0}^N a_n \cos \frac{n\pi}{N} j, \quad j = 0, 1, 2, \dots, N$$

$$a_n = \frac{2}{c_n N} \sum_{j=0}^N \frac{1}{c_j} u_j T_n(x_j) = \frac{2}{c_n N} \sum_{j=0}^N \frac{1}{c_j} u_j \cos \frac{n\pi}{N} j, \quad n = 0, 1, 2, \dots, N$$

The Chebyshev coeffs. for any ft. u in $-1 \leq x \leq 1$ are exactly the coeff. of the cosine transform obtained using the values of u at the cosine mesh, i.e. $u_j = u(\cos \frac{\pi j}{N})$.

6.4.1 Numerical differentiation using Chebyshev transform

$$T_n(x) = \cos n\theta, \quad x = \cos \theta$$

$$\rightarrow \frac{d T_n}{dx} = \frac{d \cos n\theta}{d\theta} \frac{d\theta}{dx} = \frac{n \sin n\theta}{\sin \theta}$$

$$\sin(n+1)\theta - \sin(n-1)\theta = 2 \sin \theta \cos n\theta$$

$$2 T_n(x) = \frac{1}{n+1} T_{n+1}' - \frac{1}{n-1} T_{n-1}' \quad (n > 1)$$

$$u(x) = \sum_{n=0}^N a_n T_n \longrightarrow u'(x) = \sum_{n=0}^N b_n T_n$$

\downarrow

$$u'(x) = \sum_{n=0}^N a_n T_n'$$

$$b_m = \frac{2}{C_m} \sum_{\substack{p=m+1 \\ p+m \text{ odd}}}^N P Q_p$$

$$u \xrightarrow{\text{CT}} a_n \longrightarrow b_n \xrightarrow{\text{ICT}} u'$$

6.4.2 Quadrature using Chebyshev transform

$$2T_n(x) = \frac{1}{n+1} T_{n+1}' - \frac{1}{n-1} T_{n-1}' \quad (n > 1)$$

Integrating both sides

$$\int T_n(x) dx = \begin{cases} T_0 + \alpha_0 & (n=0) \\ \frac{1}{4}(T_0 + T_2) + \alpha_1 & (n=1) \end{cases}$$

$$\left\{ \begin{array}{l} \frac{1}{2} \left[\frac{1}{n+1} T_{n+1} - \frac{1}{n-1} T_{n-1} \right] + a_n \text{ otherwise} \end{array} \right.$$

$$g(x) = \sum_{n=0}^{\infty} a_n \int T_n(x) dx = \sum_{n=0}^{N+1} d_n T_n$$

$$\left\{ \begin{array}{l} d_n = \frac{1}{2n} (c_{n+1} a_{n+1} - a_{n+1}) \\ \text{w/ } a_{N+1} = a_{N+2} = 0 \end{array} \right. \quad \begin{matrix} n=1, 2, \\ \dots, N+1 \end{matrix}$$

$$d_0 = d_1 - d_2 + d_3 - \dots + (-1)^{N+2} d_{N+1}$$

$$u \xrightarrow{\text{CT}} a_n \rightarrow d_n \xrightarrow{\text{ICT}} \int u dx$$

6.5 Method of weighted residuals (MWR)

$\mathcal{L}(u) = f(x, t)$ for $x \in D$ w/ $B(u) = g(x, t)$ on ∂D

\hat{u} : approx. sol

$$\hat{u} = \sum_{n=0}^N c_n(t) \phi_n(x)$$

\uparrow basis ft. or test ft.

$$\text{residual } R = \mathcal{L}(\hat{u}) - f$$

MWR aims to find the sol. \hat{u} which minimizes the residual R in the weighted integral sense

$$\int_D w_i R dx = 0 \quad i=0, 1, \dots, N$$

w_i : weight ft.

$$\rightarrow \int_{\Omega} w_i [L(\hat{u}) - f] dx = 0$$

$$\int_{\Omega} w_i [L(\sum_{n=0}^N c_n \phi_n) - f] dx = 0 : \text{weak form of original eq.}$$

$$L(u) = f \leftarrow FDM$$

$$w_i = 1 \leftarrow FVM$$

$$w_i = \phi_i \leftarrow \text{Galerkin method}$$

FEM

ODE