3/15/21 Gyro-motion and drift

Lorentz force for a single particle in $\vec{E}(\vec{x},t), \vec{B}(\vec{x},t)$

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) \qquad \vec{v} = \frac{d\vec{x}}{dt}$$

↑ Total derivative w.r.t t

1. Gyro motion ($\vec{E}=0$, $\vec{B}=$ const)

$$\left\{ m\frac{d\vec{v}}{dt} = q\,\vec{v}\times\vec{B} \right\}$$

$$\vec{v} = v_\parallel \hat{b} + \vec{v_\perp} \qquad \hat{b} = \vec{B}/B$$

parallel $\{\}\cdot\vec{B}$: $\quad m\frac{dv_\parallel}{dt} = 0 \qquad v_\parallel = \text{const.}$

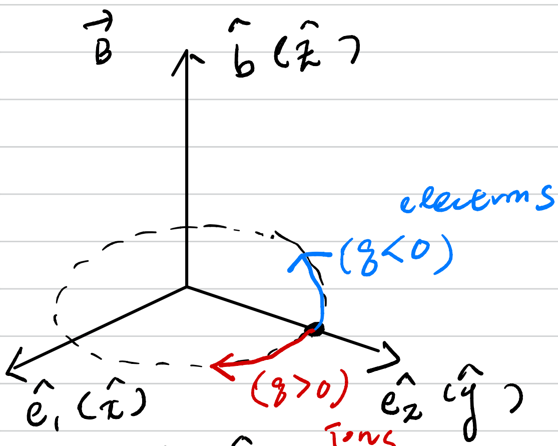
perp $\{\}\times\vec{B}$: $\quad m\frac{d}{dt}(\vec{v}\times\vec{B}) = q(\vec{v}\times\vec{B})\times\vec{B}$

$$= q\left(\vec{B}(\vec{v}\cdot\vec{B}) - \vec{v}B^2\right)$$

$$m\left(\frac{m}{q}\right)\frac{d^2\vec{v_\perp}}{dt^2} = -q B^2 \vec{v_\perp}$$

$$\ddot{\vec{v_\perp}} + \omega_c^2 \vec{v_\perp} = 0 \qquad \boxed{\omega_c \equiv \frac{|q|B}{m} \quad \text{gyro frequency (cyclotron frequency)}}$$

$$\vec{v_\perp} = \vec{v_{\perp 0}}\, e^{\pm i\omega_c t}$$



$(\hat{e}_1\times\hat{e}_2)\cdot\hat{b} = 1.$

Assign intial condition

$$\vec{v_x} = \hat{x}\, v_\perp \cos(\omega_c t) = v_x \hat{x}$$

$$m\frac{dv_x}{dt} = q v_y B$$

$$v_y = \frac{m}{qB} v_\perp (-\omega_c) \sin(\omega_c t)$$

$$= -\frac{q}{|q|} v_\perp \sin(\omega_c t)$$

$$\therefore \quad \vec{v_\perp} = v_\perp\left(\hat{e}_1 \cos(\omega_c t) - \frac{q}{|q|}\hat{e}_2 \sin(\omega_c t)\right)$$

$\vec{v}_{gyro}$

$$\vec{x_\perp} = \left(\frac{v_\perp}{\omega_c}\right)\left(\hat{e}_1 \sin(\omega_c t) + \frac{q}{|q|}\hat{e}_2 \cos(\omega_c t)\right)$$

$$\boxed{\rho \equiv \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|B} \quad \text{gyro radius (Larmor)}}$$

* Diamagnetism

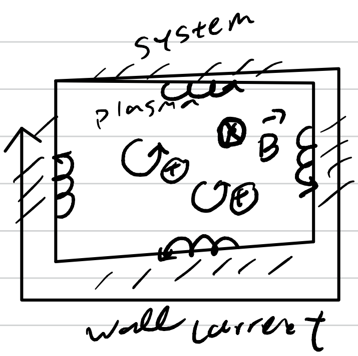
- gyromotion is diamagnetic
- plasma is diamagnetic.

* Feymann (paradox)

$$m\frac{d\vec{v}}{dt}\cdot\vec{v} = 0$$

since magnetic field can't change particle energy,
and total energy (particle + field) must be conserved.

plasma can't be (diamagnetic / paramagnetic) classically.



* Scale of magnetized plasma

let L: spatial scale of interest
τ: Temporal scale of interest

$\rho \gg L$, $\tau \ll \omega_c^{-1}$: Unmagnetized

$\rho \ll L$, $\tau \gg \omega_c^{-1}$: (strongly) magnetized
mag. fusion plasma

e.g) KSTAR $T \simeq 1\,\text{keV}$, $B_0 \simeq 1\,\text{T}$

$$\rho = \frac{m v_\perp}{qB} \sim \frac{\sqrt{mT}}{qB}$$

$\rho_{ion} \sim 0.01\,m$
$\rho_{ele} \sim 2\cdot 10^{-4}\,m$

$\rho_{ele} \ll L \ll \rho_{ion}$, $\omega_{ce}^{-1} \ll \tau \ll \omega_{ci}^{-1}$: geophysical system.

2. $\vec{E}\times\vec{B}$ drift ($\vec{E}, \vec{B} = \text{const} \neq 0$)

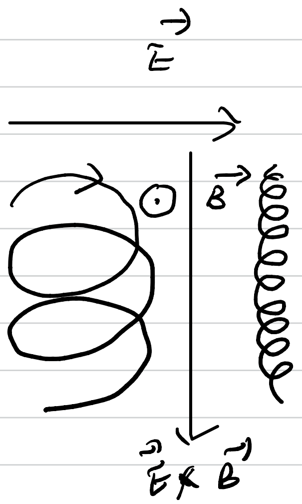
$$\left\{ m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) \right\}$$

$\{\}\cdot\vec{B}$ parallel : $\quad m\frac{dv_\parallel}{dt} = qE_\parallel \qquad v_\parallel = \frac{q}{m}E_\parallel t + \text{const}.$

$\{\}\times\vec{B}$ perp : $\quad m\left(\frac{m}{q}\right)\frac{d^2\vec{v}_\perp}{dt^2} = -qB^2\vec{v}_\perp + q\vec{E}\times\vec{B}$

Find a particular solution $\vec{V}_{\perp p} = \frac{\vec{E}\times\vec{B}}{B^2} \equiv \vec{V}_E$
due to $\vec{E}\times\vec{B}$
(inhomogeneous term)

Homogeneous solution same.



$$\vec{v}_\perp = \vec{v}_{gyro} + \vec{V}_E$$

$$\vec{v} = \vec{v}_{gyro} + v_\parallel\hat{b} + \vec{V}_E$$

$$\equiv \vec{v}_{gyro} + \vec{V}_{g.c}$$

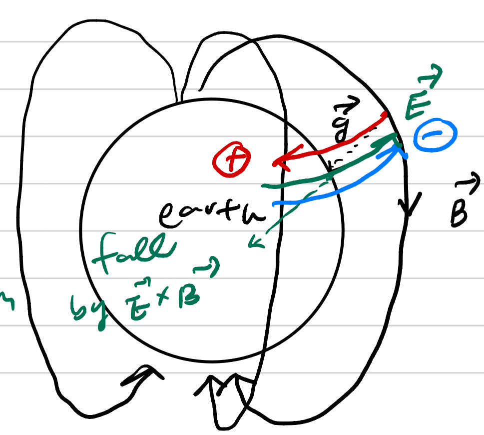
- independent of species
- The most important motion of particles in fusion plasma

3. $$m\frac{d\vec{v}}{dt} = \vec{F} + q(\vec{v}\times\vec{B})$$

$\vec{F}\times\vec{B}$ drift by a general (constant) force.

$$\vec{v}_{g.c} = v_\parallel\hat{b} + \frac{\vec{F}\times\vec{B}}{qB^2} = v_\parallel\hat{b} + \frac{\vec{E}\times\vec{B}}{B^2} + \frac{m\vec{g}\times\vec{B}}{qB^2}$$

$$\vec{F} = m\vec{g} + q\vec{E}$$

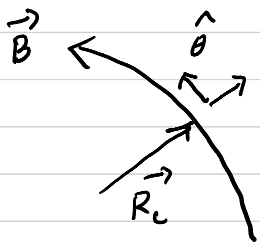


* $\vec{E}$ is time-varying
$\vec{E}$ is not as big as in vacuum
due to polarization current ✓

3/11/21 drift with inhomogeneous, time varying field

$* \quad \vec{V_{g.c}} = v_{\parallel}\hat{b} + \frac{\vec{E}\times\vec{B}}{qB^2}$ holds even for inhomogeneous, $\vec{E}$ or $\vec{B}$ if changes are slow compared to ρ, ω_c^{-1}

4. curvature drift

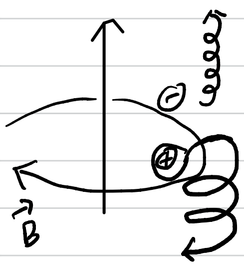


centrifugal force

$$\vec{F_{cf}} = \frac{mv_{\parallel}^2}{R_c}\hat{r} = mv_{\parallel}^2\frac{\vec{R_c}}{R_c^2}$$

$$\vec{V_{cf}} = \frac{mv_{\parallel}^2}{qB^2}\frac{\vec{R_c}\times\vec{B}}{R_c^2} = \frac{2W_{\parallel}}{qB^2}\frac{\vec{R_c}\times\vec{B}}{R_c^2}$$

In general $\vec{F_{cf}} = -mv_{\parallel}^2\vec{K} = \vec{F_{curv}}$



where, $\vec{K} = (\hat{b}\cdot\vec{\nabla})\hat{b} = \frac{1}{r}\frac{\partial\hat{\theta}}{\partial\theta}\Big|_{r=R_c} = -\frac{\vec{R_c}}{R_c^2}$

$$\vec{V_{curv}} = \frac{2W_{\parallel}}{qB^2}(\vec{B}\times\vec{K}) = \frac{2W_{\parallel}}{qB^2}\vec{B}\times(\hat{b}\cdot\vec{\nabla})\hat{b}$$

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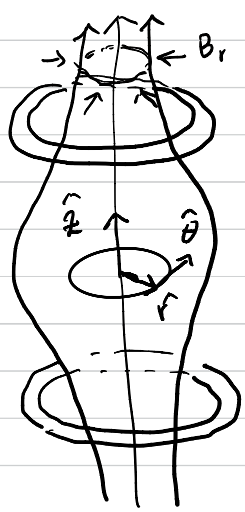
5. $\vec{\nabla}B$ - drift

preposition $\vec{F_{\nabla B}} = -\mu\vec{\nabla}B$ where $\mu \equiv \frac{\frac{1}{2}mv_{\perp}^2}{B}$ magnetic moment



$$\vec{\nabla}B = \frac{\partial B_z}{\partial z}\hat{z}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

$$B_r \approx -\frac{1}{2}r\left(\frac{dB_z}{\partial z}\right)$$

For a gyrating particle along $\vec{B}$ with $v_{\perp}$

$$B_r \approx -\frac{1}{2}\rho\frac{dB_z}{\partial z} \qquad \rho = \frac{mv_{\perp}}{|q|B}$$

Lorentz force

$$\vec{F_{\nabla B}} = -qv_{\theta}B_r\hat{z} = |q|v_{\perp}B_r\hat{z} = -\frac{\frac{1}{2}mv_{\perp}^2}{B}\frac{dB_z}{dz}\hat{z}$$

$$= -\mu\vec{\nabla}B$$

$$\vec{V}_{\nabla B} = \frac{\vec{F}_{\nabla B}\times\vec{B}}{qB^2} = \frac{\mu\vec{B}\times\vec{\nabla}B}{qB^2} = \frac{W_\perp}{qB^2}(\hat{b}\times\vec{\nabla}B)$$

For $\vec{\nabla}\times\vec{B}=0$ $\quad \hat{b}\times\vec{\nabla}B = \vec{B}\times(\hat{b}\cdot\vec{\nabla})\hat{b}$

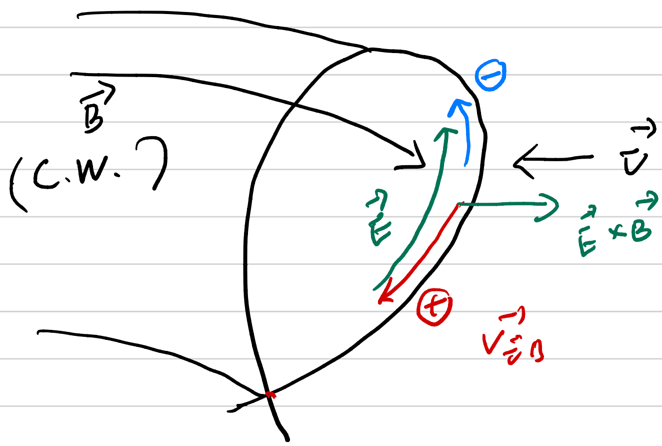
$$\vec{B}\times(\vec{\nabla}\times\vec{B}) = 0 = \frac{1}{2}\vec{\nabla}B^2 - (\vec{B}\cdot\vec{\nabla})\vec{B}$$

$$\hat{b}\times\left\{ \cancel{B}\nabla B = B(\hat{b}\cdot\vec{\nabla})(B\hat{b}) \right.$$
$$\left. = \hat{b}\cdot\cancel{(\hat{b}\cdot\vec{\nabla})}B + B(\hat{b}\cdot\vec{\nabla})\hat{b} \right\}$$

$$\hat{b}\times\vec{\nabla}B = \vec{B}\times(\hat{b}\cdot\vec{\nabla})\hat{b}$$

$$\vec{V}_{curv} + \vec{V}_{\nabla B} = \frac{2W_\parallel + W_\perp}{qB^2}(\hat{b}\times\vec{\nabla}B)$$

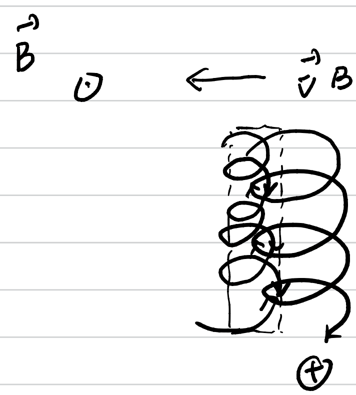
Maxwellian $\quad \int(\vec{V}_{curv} + \vec{V}_{\nabla B})f_M\, d^3v = \frac{T_\parallel + T_\perp}{qB^2}(\hat{b}\times\vec{\nabla}B)$



need rotational transform to mix $\vec{E}\times\vec{B}$ $\quad \propto \frac{1}{q}$

B_T is mostly C.W for Lower single null divertor configuration

favorable direction for L-H transition power ↓



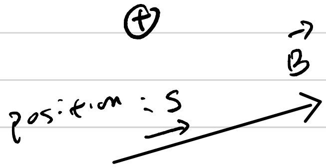
can you measure $\vec{\nabla}B$ drift?

(X)

* μ-invariant.

$$m\frac{dv_\parallel}{dt} = -\mu\nabla_\parallel B = -\mu\frac{dB}{ds} \times \left(\frac{ds}{dt} = v_\parallel\right)$$

$$mv_\parallel\frac{dv_\parallel}{dt} = -\mu\frac{dB}{dt}$$



$$\rightarrow \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) + \mu\frac{dB}{dt} = 0$$

energy conservation $\rightarrow \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) + \frac{d}{dt}(\mu B) = 0$

$$\frac{d\mu}{dt} = 0$$

6. polarization drift, time-varying $\vec{E}_\perp$ ($\vec{B}$ = const)

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$m\frac{d^2\vec{v}}{dt^2} = q\frac{d\vec{E}}{dt} + q\left(\frac{d\vec{v}}{dt}\right)\times\vec{B}$$

$$m\frac{d^2\vec{v}_\perp}{dt^2} = q\frac{d\vec{E}_\perp}{dt} + \left(\frac{q^2}{m}\right)(\vec{E}_\perp + \vec{v}_\perp\times\vec{B})\times\vec{B}$$

$$\because\ m\frac{dv_\parallel}{dt} = qE_\parallel \simeq \text{const.}$$

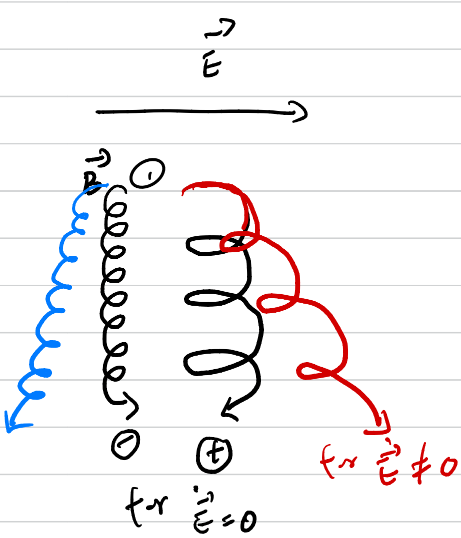
$$\frac{d^2\vec{v}_\perp}{dt^2} = -\omega_c^2\vec{v}_\perp + \left(\frac{q^2}{m^2}\right)\vec{E}\times\vec{B} + \left(\frac{q}{m}\right)\frac{d\vec{E}}{dt}$$

$$\vec{v}_\perp = \vec{v}_{gyro} + \vec{v}_{g.c}$$

$$0 \approx -\omega_c^2\vec{v}_{g.c} + \left(\frac{q^2}{m^2}\right)\vec{E}_\perp\times\vec{B} + \left(\frac{q}{m}\right)\frac{d\vec{E}_\perp}{dt}$$

$$\vec{v}_{g.c} \simeq \frac{\vec{E}\times\vec{B}}{B^2} + \frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt}$$

(To be exact. $\frac{d\vec{E}_\perp}{dt}$ = const, $E_\parallel$ = const, $\vec{B}$ = const)



$$\vec{v}_{pol} = \frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt}$$

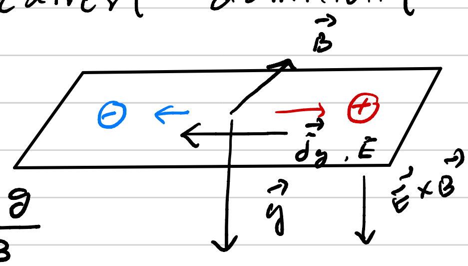
→ generate currents

$$\vec{j}_{pol} = \sum_s n_s q_s \vec{v}_{pol} = \frac{\sum_s m_s n_s}{B^2}\dot{\vec{E}}_\perp = \frac{\rho_m}{B^2}\dot{\vec{E}}_\perp$$

$$\epsilon_\perp = \epsilon_0 + \rho_m/B^2 \gg \epsilon_0$$

Ion polarization current dominant

* Goldston problem



$$j_g = \sum n_s q_s \frac{m}{q_s B^2} gB = \frac{\rho_m g}{B}$$

$$\dot{E}_\perp = \frac{j_g}{\epsilon_\perp} = \frac{\rho_m g/B}{\epsilon_0 + \rho_m/B^2}$$

$$\dot{v}_E = \dot{E}_\perp/B = g\frac{1}{1+\epsilon_0 B^2/\rho_m} \sim g$$