

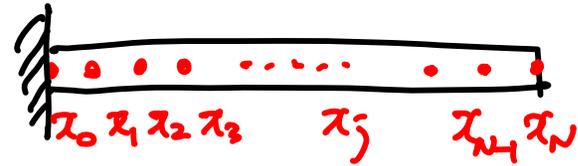
• Banded matrix

differential eq.  $\xrightarrow[\text{method}]{\text{numerical}}$  system of algebraic eqs

① ODE (ordinary diff'l eq.)

$$\frac{d^2 T}{dz^2} - a^2 T = Q(z)$$

↑ internal source



Introduce a discrete set of points  $x_j$ .  $j = 0, 1, 2, \dots, N$   
 ↳ grid points

Find  $T(x_j)$  or  $T_j$ .

• Finite difference method (FDM)

Taylor series expansion,  $x_{j+1} - x_j = h_j$  : grid spacing  
 $h$  (uniform grid spacing)

$$T(x_{j+1}) = T(x_j) + h \frac{dT}{dx} \Big|_j + \frac{1}{2} h^2 \frac{d^2T}{dx^2} \Big|_j + \frac{1}{6} h^3 \frac{d^3T}{dx^3} \Big|_j + \frac{h^4}{4!} \frac{d^4T}{dx^4} \Big|_j + \dots$$

$$- \left[ T(x_{j-1}) = \dots \right]$$

$$\Rightarrow \boxed{\frac{dT}{dx} \Big|_j = \frac{T(x_{j+1}) - T(x_{j-1}))}{2h}} - \frac{h^2}{6} \frac{d^3T}{dx^3} \Big|_j + \mathcal{O}(h^k)$$

Second-order FDM  $\curvearrowright$

leading error term  $\mathcal{O}(h^2)$

$$+ \left[ \dots \right] \text{ second-order FDM leading error term } \mathcal{O}(h^2)$$

$$\Rightarrow \boxed{\frac{d^2T}{dx^2} \Big|_j = \frac{T(x_{j+1}) - 2T(x_j) + T(x_{j-1}))}{h^2}} - \frac{1}{12} \frac{h^2}{h} \frac{d^4T}{dx^4} \Big|_j + \mathcal{O}(h^k)$$

→ Substitute this into diff'l eq.



$$\textcircled{2} \quad \left\{ \begin{array}{l} \frac{T_{j+1} - 2T_j + T_{j-1}}{h^2} - d^2 T_j = Q_j \quad j = 1, 2, \dots, N-1 \\ T_0 = 0 \\ T_N = S \end{array} \right. \quad \text{System of algebraic eqs.}$$

$$\rightarrow \left\{ \begin{array}{l} T_{j+1} - (2 + h^2 d^2) T_j + T_{j-1} = h^2 Q_j \\ T_0 = 0 \\ T_N = S \end{array} \right.$$

$$j=1 : T_2 - (2 + h^2 d^2) T_1 + \overset{0}{\textcircled{T_0}} = h^2 Q_1$$

$$j=2 : T_3 - (2 + h^2 d^2) T_2 + T_1 = h^2 Q_2$$

⋮

$$j=N-1 : \overset{S}{\textcircled{T_N}} - (2 + h^2 d^2) T_{N-1} + T_{N-2} = h^2 Q_{N-1}$$

$$\begin{bmatrix}
 -(2+h^2\alpha^2) & 1 & 0 & \dots & 0 \\
 1 & -(2+h^2\alpha^2) & 1 & 0 & \dots & 0 \\
 \dots & \dots & 1 & -(2+h^2\alpha^2) & 1 & 0 & \dots & 0 \\
 \vdots & & & & & & & \\
 0 & \dots & \dots & 0 & 1 & -(2+h^2\alpha^2)
 \end{bmatrix}
 \begin{bmatrix}
 T_1 \\
 T_2 \\
 \vdots \\
 T_j \\
 \vdots \\
 T_M
 \end{bmatrix}
 =
 \begin{bmatrix}
 h^2 Q_1 \\
 h^2 Q_2 \\
 \vdots \\
 h^2 Q_j \\
 \vdots \\
 h^2 Q_M - f
 \end{bmatrix}$$

tri-diagonal matrix  $Ax = b$

- banded matrix : non-zero elements only around the main diagonal.

↳ arises from FDM of diff'el eqs.

tridiagonal matrix :  $B [a_i, b_i, c_i]$

ex)  $a_i = 1, b_i = -(2+h^2\alpha^2), c_i = 1$

↔ Sparse matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

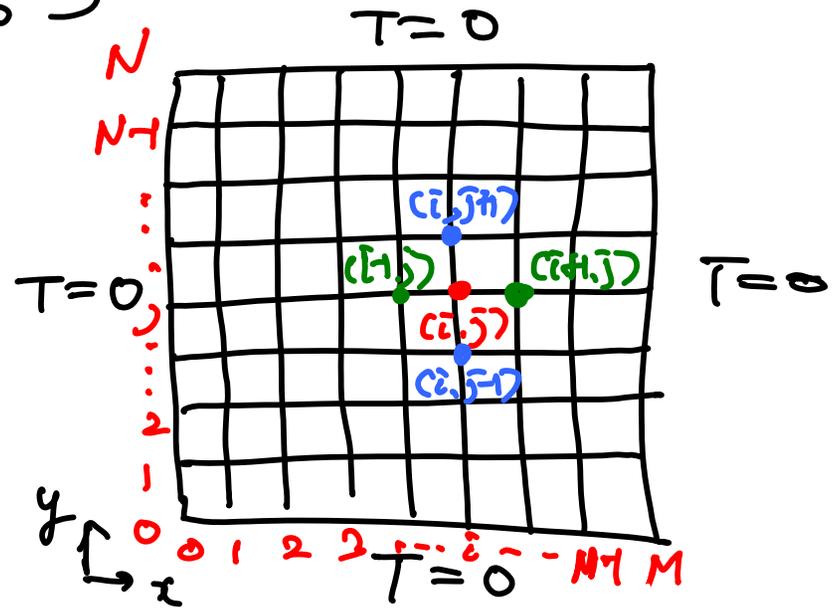
② PDE (partial diff'l eq.)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = Q(x, y)$$

$$\left. \begin{array}{l} x_{i+1} - x_i = h \\ y_{j+1} - y_j = h \end{array} \right\} \begin{array}{l} \text{uniform} \\ \text{grid spacings} \end{array}$$

2nd-order FDM

$$\rightarrow \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{h^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{h^2} = Q_{i,j} \quad \begin{array}{l} i=1, 2, \dots, M-1 \\ j=1, 2, \dots, N-1 \end{array}$$



$$\rightarrow T_{i+1,j} - 4T_{i,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} = h^2 Q_{i,j}$$

$$\bar{i}=1, \bar{j}=1 : T_{2,1} - 4T_{1,1} + T_{0,1} + T_{1,2} + T_{1,0} = h^2 Q_{1,1}$$

$$\bar{i}=2, \bar{j}=1 : T_{3,1} - 4T_{2,1} + T_{1,1} + T_{2,2} + T_{2,0} = h^2 Q_{2,1}$$

$$\vdots$$

$$\bar{i}=3, \bar{j}=3 : T_{4,3} - 4T_{3,3} + T_{2,3} + T_{3,4} + T_{3,2} = h^2 Q_{3,3}$$

$$M = 4, \\ N = 4$$

3x3

9 unknown

$$\Rightarrow \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{1,2} \\ T_{2,2} \\ T_{3,2} \\ T_{1,3} \\ T_{2,3} \\ T_{3,3} \end{bmatrix} = \begin{bmatrix} h^2 Q_{1,1} \\ h^2 Q_{2,1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ h^2 Q_{3,3} \end{bmatrix}$$

$$Ax = b$$
~~$$x = A^{-1}b$$~~

Block-tridiagonal matrix

$$\begin{pmatrix} B_1 & C_1 & 0 & \dots & 0 \\ A_2 & B_2 & C_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

◎ Solution technique

• Diagonal matrix

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

• Lower triangular matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

• Upper triangular matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

\* Gauss elimination (GE) : to make upper triangular matrix

$$\begin{cases} 4x_2 - x_3 = 5 \\ x_1 + x_2 + x_3 = 6 \\ 2x_1 - 2x_2 + x_3 = 1 \end{cases} \rightarrow \begin{pmatrix} 0 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$$

Interchange the first two rows.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

Subtract twice the first row from last row.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & -4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -11 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -6 \end{pmatrix} \quad \text{stop of GE}$$

Augmented matrix

$$\begin{pmatrix} 0 & 4 & -1 & 5 \\ 1 & 1 & 1 & 6 \\ 2 & -2 & 1 & 1 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 4 & -1 & 5 \\ 0 & 0 & -2 & -6 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 4 & 0 & 8 \\ 0 & 0 & -2 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$\rightarrow r_1=1, r_2=2, r_3=3$  "Gauss-Jordan elimination"

• General matrix system

$$Ax = b$$

$n \times n$

$$\xrightarrow{GE} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

0 ←

To eliminate  $a_{21}$ ,  $l_2 = a_{21}/a_{11}$

Multiply the first row by  $l_2$ , and subtract from 2nd row,

Continue until all elts. below  $a_{11}$  are eliminated.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22}' & \dots & a_{2n}' \\ \vdots & a_{m2}' & \dots & a_{mn}' \\ 0 & a_{n2}' & \dots & a_{nn}' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2' \\ \vdots \\ b_n' \end{pmatrix}$$

By same way (we drop prime(') for convenience)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \circ & & \dots & a_{nn} \\ \circ & & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

→

backward sweep

$$\begin{cases} x_n = b_n / a_{nn} \\ x_{n-1} = (b_{n-1} - a_{n-1n} x_n) / a_{n-1n-1} \\ x_j = (b_j - \sum_{k=j+1}^n a_{jk} x_k) / a_{jj}, \quad j = n-1, n-2, \dots, 1 \end{cases}$$