Dynamics (동역학)

Lecture 3: Particle Dynamics – Energy & Momentum

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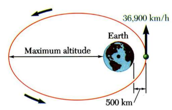
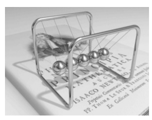
Introduction

- Previously, the particle dynamics solved by the fundamental equation of motion, $\vec{F} = m\vec{a}$. This chapter introduces two additional methods of analysis.
- ***Method of work and energy*****:** directly relates force, mass, velocity and displacement.
- ***Method of impulse and momentum*****:** directly relates force, mass, velocity, and time.





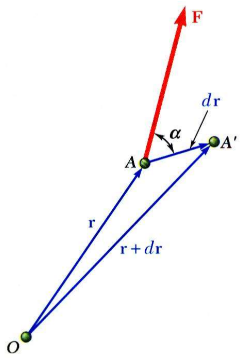
- can eliminate necessity of figuring out "internal" or "constraint" forces
- can solve for motion-related quantities without "integrating/solving" EoM
- typically only way to address "impact" with different restitutions

Forces and Accelerations → $\boxed{\sum \vec{F} = m\vec{a}_G}$ → **Newton's Second Law (last chapter)**

Velocities and Displacements → $\boxed{T_1 + U_{1\to 2} = T_2}$ → **Work-Energy**

Velocities and Time → $\boxed{m\vec{v}_1 + \int_{t_1}^{t_2} \vec{F}\, dt = m\vec{v}_2}$ → **Impulse-Momentum**

Work of a Force

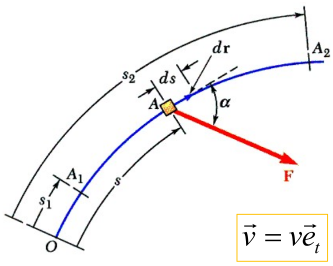


- Differential vector $d\vec{r}$ is the *particle displacement*.
- *Work of the force* is

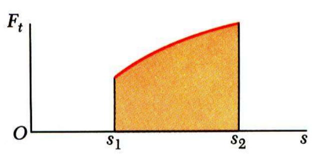
$$\begin{aligned} dU &= \vec{F} \bullet d\vec{r} \\ &= F\,ds\,\cos\alpha \\ &= F_x dx + F_y dy + F_z dz \end{aligned}$$

- Work is a scalar quantity, i.e., it has magnitude and sign but not direction.
- Work is zero when there is no displacement along the force even if force is non-zero (e.g., constraint force).
- Dimensions of work are $\text{length} \times \text{force}$. Units are $1\,\text{J}\,(joule) = (1\,\text{N})(1\,\text{m}) \qquad 1\,\text{ft}\cdot\text{lb} = 1.356\,\text{J}$

Work of a Force



$$\vec{v} = v\vec{e}_t$$

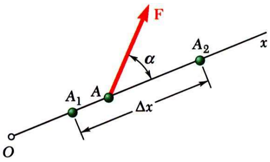


- Work of a force during a finite displacement,

$$\begin{aligned} U_{1\to 2} &= \int_{A_1}^{A_2} \vec{F} \bullet d\vec{r} \\ &= \int_{s_1}^{s_2} (F\cos\alpha)\,ds = \int_{s_1}^{s_2} F_t\ ds \\ &= \int_{A_1}^{A_2} \left(F_x dx + F_y dy + F_z dz\right) \end{aligned}$$

- No work is done by the force orthogonal to the motion (i.e., only tangential F_t works, not normal F_n)
- Work is represented by the area under the curve of F_t plotted against s.

Work by Gravity



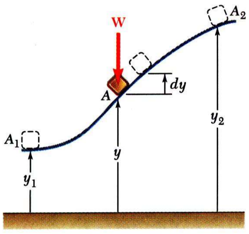
- Work of a constant force in rectilinear motion,

$$U_{1\to 2} = (F\cos\alpha)\Delta x$$

- Work done by the force of gravity,

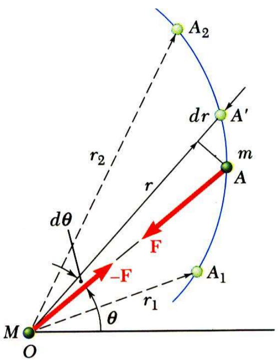
$$dU = F_x dx + F_y dy + F_z dz$$
$$= -W\,dy$$

$$\boxed{\begin{aligned} U_{1\to 2} &= -\int_{y_1}^{y_2} W\,dy \\ &= -W(y_2 - y_1) = -W\,\Delta y \end{aligned}}$$



- Work *by the gravity* is equal to product of weight W and vertical displacement Δy.
- Work *by the gravity* is positive when $\Delta y < 0$, (i.e., move down: speeding-up by pumping energy in); negative when $\Delta y > 0$ (i.e., move up: slow-down with extracting energy from)

Work by Gravitational Force



Work of a gravitational force (assume particle M is at the fixed position O while particle m follows path shown),

$$dU = -Fdr = -G\frac{Mm}{r^2}dr \qquad d\theta?$$

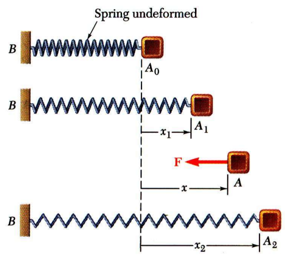
$$U_{1\to 2} = -\int_{r_1}^{r_2} G\frac{Mm}{r^2}dr = G\frac{Mm}{r_2} - G\frac{Mm}{r_1}$$

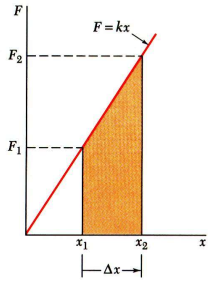
$if\ r_2 > r_1 \ \to U_{1\to 2}\ negative\ (energy\ extracted\ from\ m)$

$if\ r_2 < r_1 \ \to U_{1\to 2}\ positive\ (energy\ added\ on\ m)$

- Gravitation is the attractive force existing between any two objects that have mass.
- Gravity is the gravitational force that occurs between the earth and other bodies.

Work by Spring Force





- Magnitude of the force exerted by a linear spring is proportional to the deflection,

$$F = -kx$$
$$k = \text{spring constant}\,(\text{N/m or lb/in.})$$

- Work *done by the spring on A*,

$$dU = F\ dx = -kx\ dx$$
$$U_{1\to 2} = -\int_{x_1}^{x_2} kx\ dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$

- Work done by the spring to mass A is positive (i.e., speeding-up) when $x_2 < x_1$, i.e., spring returning to its un-deformed position while speeding-up A.

- Work done by the spring is equal to negative of area under curve of F plotted against x,

$$U_{1\to 2} = -\tfrac{1}{2}(F_1 + F_2)\Delta x = -\tfrac{1}{2}k(x_2^2 - x_1^2)$$

= negative of the spring potential increase.

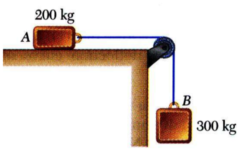
Work of a Force

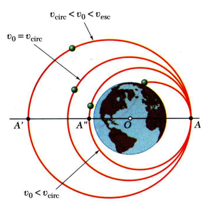
Forces which *do not* do work ($ds = 0$ or $\cos\alpha = 0$):

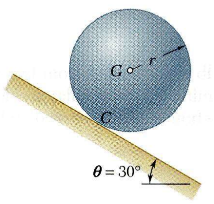
- *Work of the force* is

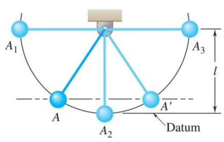
$$dU = \vec{F}\bullet d\vec{r} = F\,ds\,\cos\alpha = F_x dx + F_y dy + F_z dz$$

- reaction at frictionless surface when body in contact moves along surface,
- weight of a body when its center of gravity moves horizontally,
- reaction at frictionless pin supporting rotating body,
- reaction at a roller moving along its track with no-slip, etc.

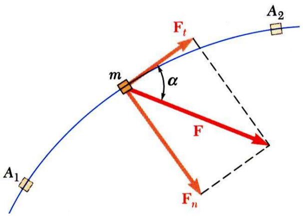








Principle of Work and Energy



$$\boxed{\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n}$$

- Consider a particle of mass m under the force $\vec{F}$

$$F_t = ma_t = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

$$\boxed{F_t\, ds = mv\, dv}$$ (no work done by F_n since $d\vec{r}_n = 0$)

- Integrating from A_1 to A_2 ,

$$\int_{s_1}^{s_2} F_t ds = m\int_{v_1}^{v_2} v\, dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

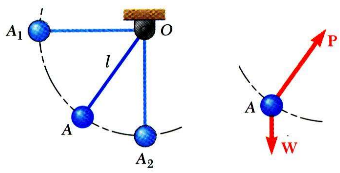
$$\boxed{\begin{aligned} U_{1\to 2} &= T_2 - T_1 \\ T_1 + U_{1\to 2} &= T_2 \end{aligned}} \qquad T = \tfrac{1}{2}mv^2 = kinetic\ energy$$

work done by all the (effective) forces acting on m

- *The work of the force $\vec{F}$ is equal to the change in kinetic energy of particle (no other energy storages).*

- Units of work and kinetic energy are the same: $T = \frac{1}{2}mv^2 = \mathrm{kg}\left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = \left(\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}\right)\mathrm{m} = \mathrm{N}\cdot\mathrm{m} = \mathrm{J}$

Applications of the Principle of Work and Energy



- Wish to determine velocity of pendulum bob at A_2. Consider work & kinetic energy.
- Cord tension $\vec{P}$ acts normal to the path and does no work.

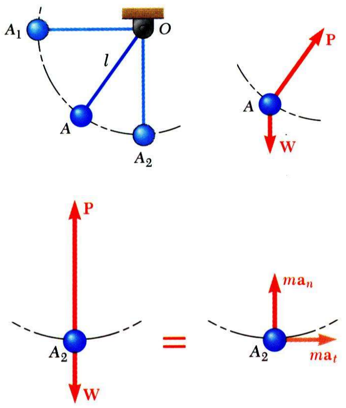
$$T_1 + U_{1\to 2} = T_2$$

$$0 + Wl = \frac{1}{2}\frac{W}{g}v_2^2$$

$$v_2 = \sqrt{2gl}$$

- Velocity found without determining expression for acceleration and integrating.
- All quantities are scalars and can be added directly.
- Forces which do no work are eliminated from the problem.

Applications of the Principle of Work and Energy



$$v_2 = \sqrt{2gl}$$

$$\boxed{\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n}$$

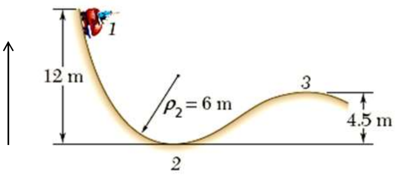
- Principle of work and energy cannot be applied to directly determine the acceleration vector of the pendulum bob.
- Calculating the tension in the cord requires supplementing the method of work and energy with an application of Newton's second law.
- As the bob passes through A_2 ,

$$\sum F_n = m a_n$$

$$P - mg = m\frac{v_2^2}{l}$$

$$P = mg + m\frac{2gl}{l} = 3W$$

Sample Problem 13.4



A 1000-kg car starts from rest at point 1 and moves without friction down the track shown.

Determine:

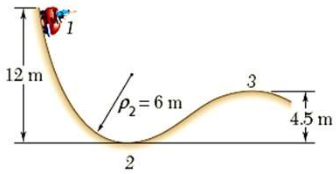
a) the force exerted by the track on the car at point 2, and

b) the minimum safe value of the radius of curvature (i.e., minimum-possible radius) at point 3.

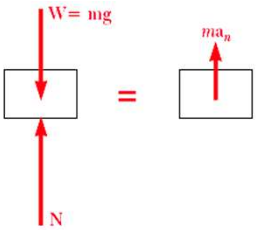
SOLUTION:

- Apply principle of work and energy to determine velocity at point 2.
- Apply Newton's second law to find normal force by the track at point 2.
- Apply principle of work and energy to determine velocity at point 3.
- Apply Newton's second law to find minimum-possible radius of curvature at point 3 with always downward force W and always upward track force N (maximum downward normal force limited by W with $N \geq 0$).

Sample Problem 13.4



$$\boxed{\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n}$$



SOLUTION:

- Apply principle of work and energy to determine velocity at point 2.

$$T_1 = 0 \qquad T_2 = \tfrac{1}{2} m v_2^2$$

$$U_{1\to 2} = +W(12\,\text{m}) = \text{mg}(12\text{m})$$

$$T_1 + U_{1\to 2} = T_2 : \qquad \boxed{0 + W(12\,\text{m}) = \frac{1}{2}\frac{W}{g} v_2^2}$$

$$v_2^2 = 2(12)g = 2(24\,\text{m})(9.81\,\text{m}/s^2) \qquad v_2 = 15.34\,\text{m/s}$$

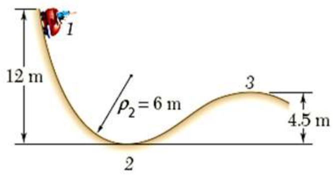
- Apply Newton's second law to find normal force by the track at point 2.

$$+\uparrow \sum F_n = m a_n :$$

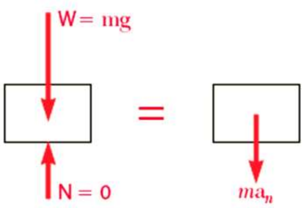
$$\boxed{-W + N = m a_n = \frac{m v_2^2}{\rho} = m\frac{24g}{6} = 4\text{mg} = 4W}$$

$$N = 5W \qquad \boxed{N = 49.05\,\text{kN}}$$

Sample Problem 13.4



$$\boxed{\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n}$$



- Apply principle of work and energy to determine velocity at point 3.

$$T_1 + U_{1\to 3} = T_3 \qquad \boxed{0 + mg(7.5\,\text{m}) = \frac{1}{2} m v_3^2}$$

$$v_3^2 = 15g = (15\,\text{m})(9.81\,\text{m/s}^2) \qquad v_3 = 12.13\,\text{m/s}$$

- Apply Newton's second law to find minimum-possible radius of curvature at point 3 such that a positive (upward) normal force is exerted by the track.

$$+\downarrow \sum F_n = m a_n :$$

$$mg - N = m\frac{v_3^2}{\rho}, \;\; N \ge 0$$

$$\boxed{\min \rho = \frac{v_3^2}{g} = \frac{15g}{g} \;\; \text{subj. to } mg - N = m\frac{v_3^2}{\rho}, N \ge 0}$$

$$\boxed{\rho = 15\,\text{m}}$$

Power and Efficiency

- *Power* = rate at which work is done.

$$= \frac{dU}{dt} = \frac{\vec{F} \bullet d\vec{r}}{dt}$$

$$= \vec{F} \bullet \vec{v}$$

- Dimensions of power are work/time or force × velocity. Units for power are

$$1\ \text{W (watt)} = 1\frac{\text{J}}{\text{s}} = 1\ \text{N} \cdot \frac{\text{m}}{\text{s}} \quad \text{or} \quad 1\ \text{hp} = 550\frac{\text{ft} \cdot \text{lb}}{\text{s}} = 746\ \text{W}$$

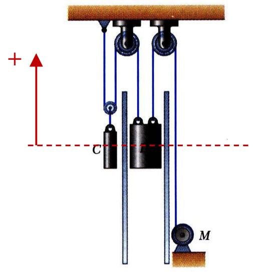
- η = efficiency

$$= \frac{\text{output work}}{\text{input work}}$$

$$= \frac{\text{power output}}{\text{power input}}$$

gear-shaft $\eta = \frac{T_o w_o}{T_i w_i}$

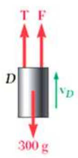
Sample Problem 13.5



The dumbwaiter D and its load have a combined weight of 300 kg, while the counterweight C weighs 400 kg.

Determine the power delivered by the electric motor M when the dumbwaiter *(a)* is moving up at a constant speed of 2.5 m/s and *(b)* has an instantaneous velocity of 2.5 m/s and an acceleration of 1 m/s^2, both directed upwards.

SOLUTION:



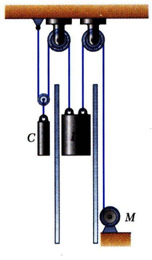
Force exerted by the motor cable has same direction as the dumbwaiter velocity. Power delivered by motor is equal to Fv_D, v_D = 2.5 m/s.

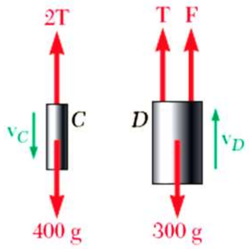
$$-v_M = v_D = -2v_C$$

$$-a_M = a_D = -2a_C$$

- In the first case, bodies are in uniform motion. Determine force exerted by motor cable from conditions for static equilibrium.
- In the second case, both bodies are accelerating. Apply Newton's second law (dynamic equilibrium) to each body to determine the required motor cable force.

Sample Problem 13.5





- In the first case, bodies are in uniform motion. Determine force exerted by motor cable from conditions for static equilibrium.

Free-body C:

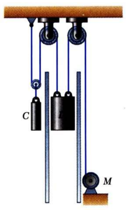
$$+\uparrow \sum F_y = 0: \quad 2T - 400\,\mathrm{g} = 0 \qquad T = 200\,\mathrm{g} \quad = 1962\,\mathrm{N}$$

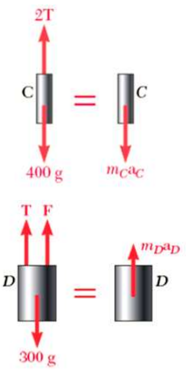
Free-body D:

$$+\uparrow \sum F_y = 0: \quad F + T - 300\,\mathrm{g} = 0$$
$$F = 300\,\mathrm{g} - T = 300\,\mathrm{g} - 200\,\mathrm{g} = 100\,\mathrm{g} = 981\,\mathrm{N}$$

$$Power = Fv_D = (981\,\mathrm{N})(2.5\,\mathrm{m/s})$$
$$= 2450\,\mathrm{W}$$

Sample Problem 13.5





- In the second case, both bodies are accelerating. Apply Newton's second law to each body to determine the required motor cable force.

$$a_D = 1\,\mathrm{m/s^2} \uparrow \qquad a_C = -\tfrac{1}{2} a_D = 0.5\,\mathrm{m/s^2} \downarrow$$

Free-body C: $400 - 2T = 400(0.5)$

$$+\downarrow \sum F_y = m_C a_C: \quad T = \frac{(400)(9.81) - 400(0.5)}{2} = 1862\,\mathrm{N}$$

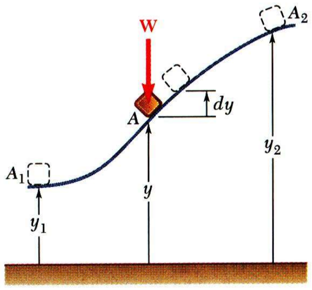
Free-body D:

$$+\uparrow \sum F_y = m_D a_D: \quad F + T - 300 = 300(1)$$
$$F + 1862 - 300(9.81) = 300 \qquad F = 1381\,\mathrm{N}$$

$$Power = Fv_D = (1381\,\mathrm{N})(2.5\,\mathrm{m/s}) = 3452\,\mathrm{W}$$

$$\boxed{Power = 3450\,\mathrm{W}}$$

Potential Energy



$$T_1 + U_{1\to 2} = T_2$$

- Work of the force of gravity $\vec{W}$,

$$U_{1\to 2} = mg\,y_1 - mg\,y_2$$

- Work is independent of path followed; depends only on the initial and final values of y.

$$V_g = Wy$$

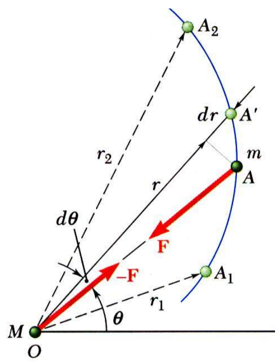
$=$ *potential energy* of the body with respect to *force of gravity*.

$$U_{1\to 2} = (V_g)_1 - (V_g)_2$$

- Choice of datum from which the elevation y is measured is arbitrary.
- Units of work and potential energy are the same:

$$V_g = Wy = \mathrm{N}\cdot\mathrm{m} = \mathrm{J}$$

Gravitation Potential Energy



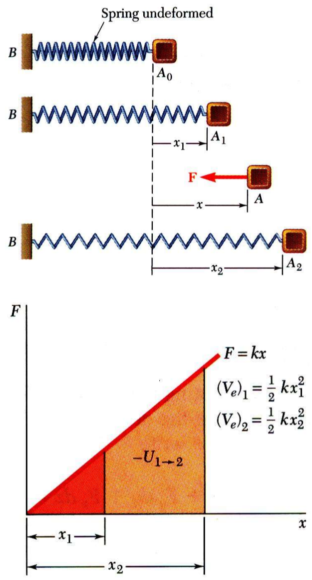
- Previous expression for potential energy of a body with respect to gravity is only valid when the weight of the body can be assumed constant.
- For a space vehicle, the variation of the force of gravity with distance from the center of the earth should be considered.
- Work of a gravitational force,

$$U_{1\to 2} = \frac{GMm}{r_2} - \frac{GMm}{r_1} = V_{g,1} - V_{g,2}$$

- Potential energy V_g when the variation in the force of gravity can not be neglected,

$$V_g = -\frac{GMm}{r} = -\frac{WR^2}{r}$$

Spring Potential Energy



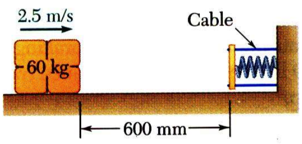
- Work of the force exerted by a spring depends only on the initial and final deflections of the spring,

$$U_{1\to2} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$

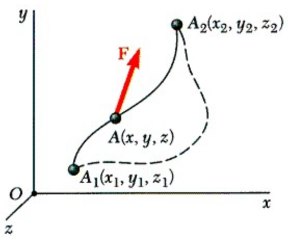
- The potential energy of the body with respect to the elastic force,

$$\boxed{\begin{aligned} V_e &= \tfrac{1}{2}kx^2 \\ U_{1\to2} &= (V_e)_1 - (V_e)_2 \end{aligned}}$$

- Note that the preceding expression for V_e is valid only if the deflection of the spring is measured from its undeformed position.



Conservative Forces



- Conservative force: total work done between two points independent of the path taken:

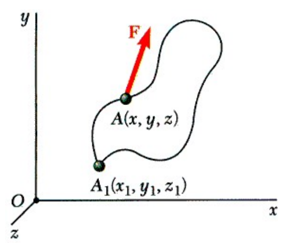
$$\boxed{U_{1\to2} = V(x_1, y_1, z_1) - V(x_2, y_2, z_2)}$$

*- Conservative force defines potential energy**
- Potential energy produces conservative force

** This is in fact the famous Poincare's lemma.*

- For any conservative force applied on a closed path,

$$\boxed{\oint \vec{F} \bullet d\vec{r} = 0}$$

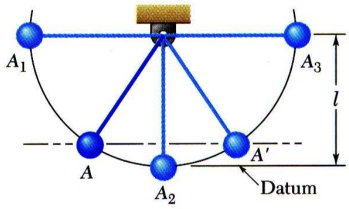


- Work corresponding to displacement between two neighboring points,

$$\begin{aligned} dU &= F_x dx + F_y dy + F_z dz \\ &= V(x, y, z) - V(x + dx, y + dy, z + dz) \\ &= -dV(x, y, z) = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) \end{aligned}$$

$$\boxed{\vec{F} = -\left(\frac{\partial V}{\partial x}; \frac{\partial V}{\partial y}; \frac{\partial V}{\partial z}\right) = -\mathbf{grad}\, V}$$

Energy Conservation



$T_1 = 0 \quad V_1 = W\ell$

$T_1 + V_1 = W\ell$

$T_2 = \frac{1}{2} m v_2^2 = \frac{1}{2}\frac{W}{g}(2g\ell) = W\ell \quad V_2 = 0$

$T_2 + V_2 = W\ell$

- Particle moving under a force:
  $\boxed{U_{1\to 2} = T_2 - T_1}$
- If this force is a conservative force:
  $\boxed{U_{1\to 2} = V_1 - V_2}$
- Combining these $\Rightarrow$ energy conservation
  $$\boxed{\begin{array}{l} T_1 + V_1 = T_2 + V_2 \\ E = T + V = \text{constant} \end{array}}$$
- When a particle moves under conservative forces, total mechanical energy is constant.
- Friction forces are not conservative. Total mechanical energy of a system involving friction decreases.
- Mechanical energy is dissipated by friction into thermal energy. Total energy is constant.

Sample Problem 13.1

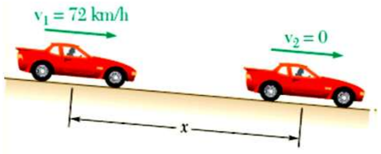
An automobile weighing 1000 kg is driven down a 5° incline at a speed of 72 km/h when the brakes are applied causing a constant total breaking force of 5000 N.

Determine the distance traveled by the automobile as it comes to a stop.

SOLUTION:

- Evaluate the change in kinetic energy.
- Determine the distance required for the work to equal the kinetic energy change.

Sample Problem 13.1



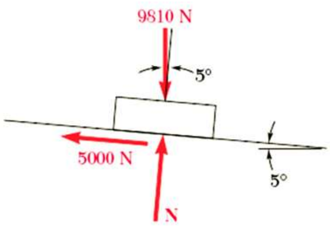
SOLUTION:

- Evaluate the change in kinetic energy.

$$v_1 = \left(72\frac{\text{km}}{\text{h}}\right)\left(\frac{1000\,\text{m}}{1\text{km}}\right)\left(\frac{1\text{h}}{3600\,\text{s}}\right) = 20\,\text{m/s}$$

$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(1000\text{kg})(20\text{m/s})^2 = 20{,}000\ \text{j}$$

$$v_2 = 0 \qquad T_2 = 0$$



- Determine the distance required for the work to equal the kinetic energy change.

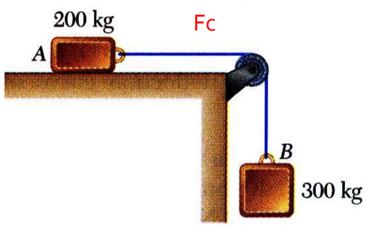
$$U_{1\to 2} = -5000x + (1000\,\text{kg})(9.81\text{m/s}^2)(\sin 5°)x$$
$$= -4145x$$

$$T_1 + U_{1\to 2} = T_2$$

$$200{,}000 - 4145x = 0$$

$$\boxed{x = 48.25\,\text{m}}$$

Sample Problem 13.2

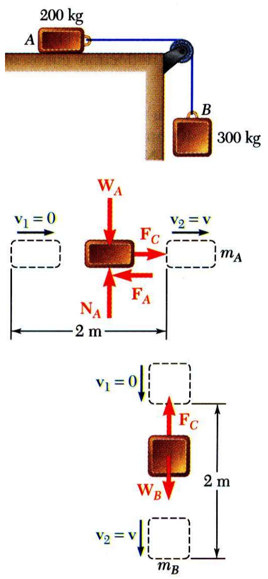


SOLUTION:

- Apply the principle of work and energy separately to blocks A and B.
- When the two relations are combined, the work of the cable forces cancel. Solve for the velocity.

Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block A after it has moved 2 m. Assume that the coefficient of friction between block A and the plane is $\mu_k = 0.25$ and that the pulley is weightless and frictionless.

Sample Problem 13.2



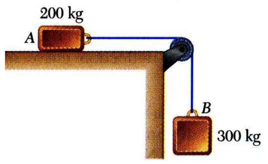
SOLUTION:

- Apply the principle of work and energy separately to blocks A and B.

$$W_A = (200\,\text{kg})(9.81\,\text{m/s}^2) = 1962\,\text{N}$$
$$F_A = \mu_k N_A = \mu_k W_A = 0.25(1962\,\text{N}) = 490\,\text{N}$$
$$T_1 + U_{1\to 2} = T_2:$$
$$0 + \boxed{F_C}(2\,\text{m}) - F_A(2\,\text{m}) = \tfrac{1}{2} m_A \boxed{v^2}$$
$$F_C(2\,\text{m}) - (490\,\text{N})(2\,\text{m}) = \tfrac{1}{2}(200\,\text{kg})v^2$$

$$W_B = (300\,\text{kg})(9.81\,\text{m/s}^2) = 2940\,\text{N}$$
$$T_1 + U_{1\to 2} = T_2:$$
$$0 - \boxed{F_c}(2\,\text{m}) + W_B(2\,\text{m}) = \tfrac{1}{2} m_B \boxed{v^2}$$
$$-F_c(2\,\text{m}) + (2940\,\text{N})(2\,\text{m}) = \tfrac{1}{2}(300\,\text{kg})v^2$$

Sample Problem 13.2



- When the two relations are combined, the work of the cable forces cancel. Solve for the velocity.

$$F_C(2\,\text{m}) - (490\,\text{N})(2\,\text{m}) = \tfrac{1}{2}(200\,\text{kg})v^2$$
$$-F_c(2\,\text{m}) + (2940\,\text{N})(2\,\text{m}) = \tfrac{1}{2}(300\,\text{kg})v^2$$

$$(2940\,\text{N})(2\,\text{m}) - (490\,\text{N})(2\,\text{m}) = \tfrac{1}{2}(200\,\text{kg} + 300\,\text{kg})v^2$$
$$4900\,\text{J} = \tfrac{1}{2}(500\,\text{kg})v^2$$

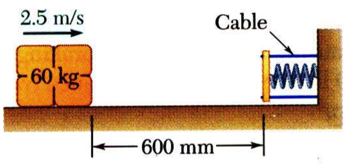
$$\boxed{v = 4.43\,\text{m/s}}$$

- Energy conservation:

$$0 + F_C(2\,\text{m}) - F_A(2\,\text{m}) = \tfrac{1}{2} m_A v^2$$
$$0 - F_c(2\,\text{m}) + m_B g(2\,\text{m}) = \tfrac{1}{2} m_B v^2$$

$$\boxed{\Rightarrow \tfrac{1}{2} m_A 0^2 + \tfrac{1}{2} m_B 0^2 + m_B g(2\,\text{m}) - F_A(2\,\text{m}) = \tfrac{1}{2} m_A v^2 + \tfrac{1}{2} m_B v^2}$$

Sample Problem 13.3



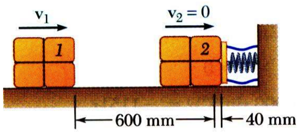
A spring is used to stop a 60 kg package which is sliding on a horizontal surface. The spring has a constant $k = 20$ kN/m and is held by cables so that it is initially compressed 120 mm. The package has a velocity of 2.5 m/s in the position shown and the maximum deflection of the spring is 40 mm.

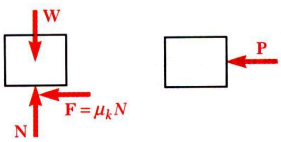
Determine *(a)* the coefficient of kinetic friction between the package and surface and *(b)* the velocity of the package as it passes again through the position shown.

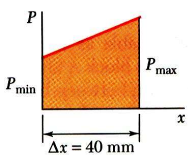
SOLUTION:

- Apply the principle of work and energy between the initial position and the point at which the spring is fully compressed and the velocity is zero. The only unknown in the relation is the friction coefficient.
- Apply the principle of work and energy for the rebound of the package. The only unknown in the relation is the velocity at the final position.

Sample Problem 13.3







SOLUTION:

- Apply principle of work and energy between initial position and the point at which spring is fully compressed.

$$T_1 = \tfrac{1}{2} m v_1^2 = \tfrac{1}{2}(60\,\text{kg})(2.5\,\text{m/s})^2 = 187.5\,\text{J} \qquad T_2 = 0$$

$$\left(U_{1\to2}\right)_f = -\mu_k W x$$
$$= -\mu_k (60\,\text{kg})\left(9.81\,\text{m/s}^2\right)(0.640\,\text{m}) = -(377\,\text{J})\mu_k$$

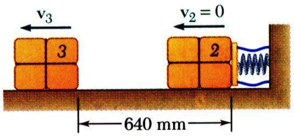
$$P_{\min} = k x_0 = (20\,\text{kN/m})(0.120\,\text{m}) = 2400\,\text{N}$$
$$P_{\max} = k(x_0 + \Delta x) = (20\,\text{kN/m})(0.160\,\text{m}) = 3200\,\text{N}$$
$$\left(U_{1\to2}\right)_e = -\tfrac{1}{2}(P_{\min} + P_{\max})\Delta x$$
$$= -\tfrac{1}{2}(2400\,\text{N} + 3200\,\text{N})(0.040\,\text{m}) = -112.0\,\text{J}$$

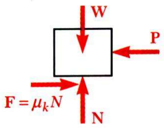
$$U_{1\to2} = \left(U_{1\to2}\right)_f + \left(U_{1\to2}\right)_e = -(377\,\text{J})\mu_k - 112\,\text{J}$$

$$\boxed{\begin{array}{l} T_1 + U_{1\to2} = T_2 : \\ \quad 187.5\,\text{J} - (377\,\text{J})\mu_k - 112\,\text{J} = 0 \end{array}}$$

$$\boxed{\mu_k = 0.20}$$

Sample Problem 13.3





- Apply the principle of work and energy for the rebound of the package.

$$T_2 = 0 \qquad T_3 = \tfrac{1}{2} m v_3^2 = \tfrac{1}{2}(60\,\mathrm{kg}) v_3^2$$

$$U_{2\to3} = \left(U_{2\to3}\right)_f + \left(U_{2\to3}\right)_e = -(377\,\mathrm{J})\mu_k + 112\,\mathrm{J}$$
$$= +36.5\,\mathrm{J}$$

$$\boxed{\begin{aligned} &T_2 + U_{2\to3} = T_3 : \\ &0 + 36.5\,\mathrm{J} = \tfrac{1}{2}(60\,\mathrm{kg}) v_3^2 \end{aligned}}$$

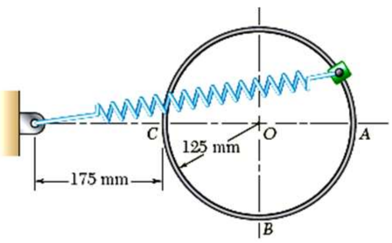
$$\boxed{v_3 = 1.103\,\mathrm{m/s}}$$

- Energy conservation:

$$\boxed{\begin{aligned} &E_{1\to2} : \tfrac{1}{2} m v_1^2 - \mu_k mg(0.6+0.04) + \tfrac{1}{2} K(0.12)^2 = \tfrac{1}{2} K(0.16)^2 \\ &E_{2\to3} : \tfrac{1}{2} K(0.16)^2 - \mu_k mg(0.6+0.04) = \tfrac{1}{2} m v_3^2 + \tfrac{1}{2} K(0.12)^2 \end{aligned}}$$

$$U_{1\to2} = \left(U_{1\to2}\right)_f + \left(U_{1\to2}\right)_e = -(377\,\mathrm{J})\mu_k - 112\,\mathrm{J}$$
$$U_{2\to3} = \left(U_{2\to3}\right)_f + \left(U_{2\to3}\right)_e = -(377\,\mathrm{J})\mu_k + 112\,\mathrm{J}$$

Sample Problem 13.6



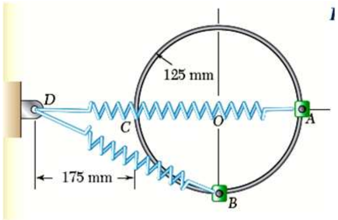
A 1.5-kg collar slides without friction along a circular rod in horizontal plane. The spring attached to the collar has an undeformed length of 150 mm and a constant of 400 N/m.

If the collar is released from equilibrium *A*, determine its velocity (a) as it passes through *B* and *C*.

SOLUTION:

- Apply the principle of conservation of energy between positions 1 and 2.
- The elastic and gravitational potential energies at 1 and 2 are evaluated from the given information. The initial kinetic energy is zero.
- Solve for the kinetic energy and velocity at 2.

Sample Problem 13.6



SOLUTION:

- Apply the principle of conservation of energy between positions 1 and 2.

Position 1: $\Delta L_{AD} - L_O = (175\text{mm} + 250\text{mm}) - (150\text{mm})$

$= 275\text{mm} = 0.275\text{mm}$

$V_A = \frac{1}{2}k(\Delta L_{AD})^2 = \frac{1}{2}(400\text{N/m})(0.275\text{m})^2$

$V_A = 15.125\text{J}$

Position 2: $T_B = \frac{1}{2}mv_B^2 = \left(\frac{1.5}{2}\text{kg}\right)(v_B^2) = (0.75)(v_B^2)$

$L_{BD} = (300^2\,\text{mm} + 125^2\,\text{mm})^{1/2} = 325\,\text{mm}$

$\Delta_{BD} = L_{BD} = (325\text{mm} + 150\text{m}) = 175\,\text{mm} = 0.175\,\text{m}$

$V_B \tfrac{1}{2} = k(\Delta_{BD})^2 = \frac{1}{2}(400\text{N/m})(0.175\text{m})^2$

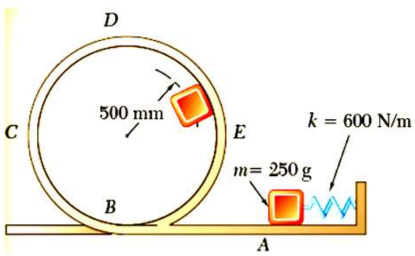
$= 6.125\text{j}$

Conservation of Energy: $T_A + V_A = T_B + V_B \Rightarrow 0 + 15.125 = 0.75v_B^2 + 6.125$

$v_B^2 = \left(\frac{(15.125 - 6.125)}{(0.75)}\right) = 12.00\text{m}^2/\text{s}^2$

$\boxed{v_B = 3.46\,\text{m/s}}$

Sample Problem 13.7

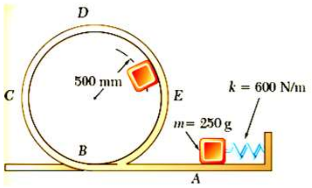


The 250-g pellet is pushed against the spring and released from rest at *A*. Neglecting friction, determine the smallest deflection of the spring for which the pellet will travel around the loop and remain in contact with the loop at all times.

SOLUTION:

- Since the pellet must remain in contact with the loop, the force exerted on the pellet must be greater than or equal to zero. Setting the force exerted by the loop to zero, solve for the minimum velocity at *D*.
- Apply the principle of conservation of energy between points *A* and *D*. Solve for the spring deflection required to produce the required velocity and kinetic energy at *D*.

Sample Problem 13.7

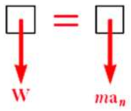


SOLUTION:

- Setting the force exerted by the loop to zero, solve for the minimum velocity at D.

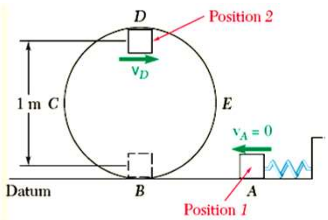
$+\downarrow \sum F_n = ma_n$:

$$W + N = ma_n \qquad mg + N = m v_D^2/r,\ N \ge 0$$
$$v_{D,\min}^2 = rg = (0.5\,\mathrm{m})(9.81\,\mathrm{m/s}) = 4.905\,\mathrm{m^2/s^2}$$



- Apply the principle of conservation of energy between points A and D.

$$V_1 = V_e + V_g = \tfrac{1}{2}kx^2 + 0 = \tfrac{1}{2}(600\,\mathrm{N/m})x^2 = 300x^2$$
$$T_1 = 0$$
$$V_2 = V_e + V_g = 0 + mgy = (0.25\,\mathrm{kg})(9.81\,\mathrm{m/s^2})(1\,\mathrm{m}) = 2.45\,\mathrm{j}$$
$$T_2 = \tfrac{1}{2}mv_D^2 = \frac{1}{2}\times(0.25\mathrm{kg})(4.905\,\mathrm{m^2/s^2}) = 0.613\,\mathrm{j}$$



$$T_1 + V_1 = T_2 + V_2$$
$$0 + 300x^2 = 2.45\mathrm{j} + 0.613\mathrm{j}$$

$$\boxed{x = 0.101\mathrm{m} = 101\,\mathrm{mm}}$$

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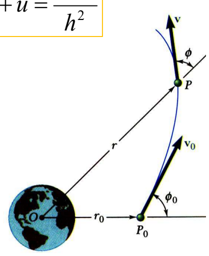
Motion Under a Conservative Central Force

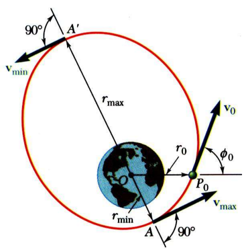
$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$



- When a particle moves under a **conservative and central** force, both angular momentum conservation and total energy conservation are satisfied:

$$r_0 m v_0 \sin\phi_0 = r m v \sin\phi$$

$$T_0 + V_0 = T + V$$
$$\tfrac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2}mv^2 - \frac{GMm}{r}$$



- Given r, we can directly/analytically obtain v and ϕ.

- At minimum and maximum r, $\phi = 90^o$. Given the launch conditions, we can also directly/analytically obtain (r_{min}, v_{max}), and (r_{max}, v_{min}).

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta$$

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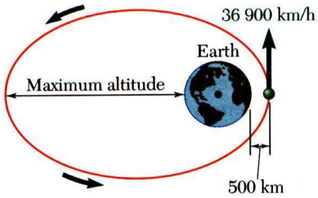
Sample Problem 13.9



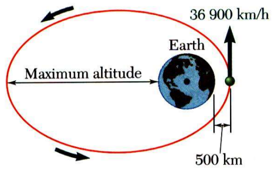
A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36900 km/h from an altitude of 500 km.

Determine *(a)* the maximum altitude reached by the satellite, and *(b)* the maximum allowable error in the direction of launching if the satellite is to come no closer than 200 km to the surface of the earth

SOLUTION:

- For motion under a conservative central force, the principles of conservation of energy and conservation of angular momentum may be applied simultaneously.
- Apply the principles to the points of minimum and maximum altitude to determine the maximum altitude.
- Apply the principles to the orbit insertion point and the point of minimum altitude to determine maximum allowable orbit insertion angle error.

Sample Problem 13.9



- Apply the principles of conservation of energy and conservation of angular momentum to the points of minimum and maximum altitude to determine the maximum altitude.

Conservation of energy:

$$T_A + V_A = T_{A'} + V_{A'} \qquad \tfrac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2} m v_1^2 - \frac{GMm}{r_1}$$

Conservation of angular momentum:

$$r_0 m v_0 = r_1 m v_1 \qquad v_1 = v_0 \frac{r_0}{r_1}$$

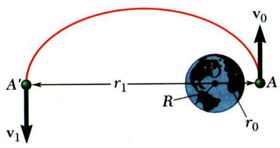
Combining,

$$\tfrac{1}{2} v_0^2 \left(1 - \frac{r_0^2}{r_1^2}\right) = \frac{GM}{r_0}\left(1 - \frac{r_0}{r_1}\right) \qquad 1 + \frac{r_0}{r_1} = \frac{2GM}{r_0 v_0^2}$$

$$r_0 = 6370\,\text{km} + 500\,\text{km} = 6870\,\text{km}$$

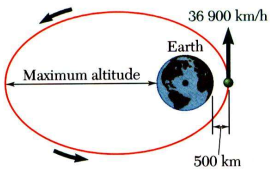
$$v_0 = 36900\,\text{km/h} = 10.25 \times 10^6\,\text{m/s}$$

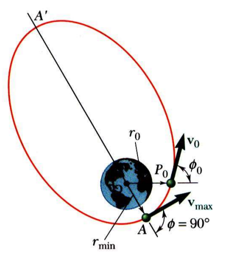
$$GM = gR^2 = \left(9.81\,\text{m/s}^2\right)\left(6.37 \times 10^6\,\text{m}\right)^2 = 398 \times 10^{12}\,\text{m}^3/\text{s}^2$$



$$\boxed{r_1 = 60.4 \times 10^6\,\text{m} = 60400\,\text{km}}$$

Sample Problem 13.9





- Apply the principles to the orbit insertion point and the point of minimum altitude to determine maximum allowable orbit insertion angle error.

Conservation of energy:

$$\boxed{T_0 + V_0 = T_A + V_A \qquad \tfrac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2}mv_{max}^2 - \frac{GMm}{r_{min}}}$$

Conservation of angular momentum:

$$\boxed{r_0 m v_0 \boxed{\sin\phi_0} = r_{min} m v_{max} \qquad v_{max} = v_0 \sin\phi_0 \frac{r_0}{r_{min}}}$$

Combining and solving for $\sin\varphi_0$,

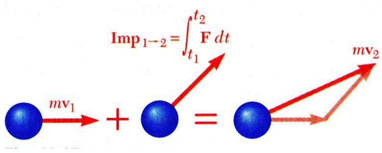
$$\sin\phi_0 = 0.9801$$

$$\varphi_0 = 90° \pm 11.5° \qquad \boxed{\text{allowable error} = \pm 11.5°}$$

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = \sum F_r = -F \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = \sum F_\theta = 0$$

$$\boxed{\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta}$$

Principle of Impulse and Momentum



- From Newton's second law,

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \qquad \boxed{m\vec{v} = \text{linear momentum}}$$

$$\vec{F}dt = d(m\vec{v})$$

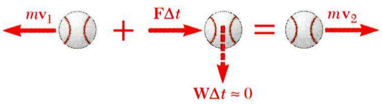
$$\int_{t_1}^{t_2} \vec{F}dt = m\vec{v}_2 - m\vec{v}_1$$

$$\int_{t_1}^{t_2} \vec{F}dt = \mathbf{Imp}_{1\to2} = \boxed{\text{impulse of the force } \vec{F}}$$

$$\boxed{m\vec{v}_1 + \mathbf{Imp}_{1\to2} = m\vec{v}_2}$$

- Dimensions of the impulse of a force are
  force × time.
- Units for the impulse of a force are
  $\mathrm{N \cdot s} = \left(\mathrm{kg \cdot m/s^2}\right)\cdot \mathrm{s} = \mathrm{kg \cdot m/s}$
- The final momentum of the particle can be obtained by adding vectorially its initial momentum and the impulse of the force during the time interval.

Impulsive Force



- Force acting on a particle during a very short time interval that is large enough to cause a significant change in momentum is called an ***impulsive force***.
- When impulsive forces act on a particle,

$$m\vec{v}_1 + \sum \vec{F}\,\Delta t = m\vec{v}_2$$

- When a baseball is struck by a bat, contact occurs over a short time interval but force is large enough to change sense of ball motion.
- ***Nonimpulsive forces*** are forces for which $\vec{F}\Delta t$ is small and therefore, may be neglected.

Sample Problem 13.10

An automobile weighing 1800 kg is driven down a 5° incline at a speed of 100 km/h when the brakes are applied, causing a constant total braking force of 7000 N.

Determine the time required for the automobile to come to a stop.

$$m\vec{v}_1 + \int_{t_1}^{t_2} \vec{F}dt = m\vec{v}_2$$

SOLUTION:

- Apply the principle of impulse and momentum. The impulse is equal to the product of the constant forces and the time interval.

An automobile weighing 1000 kg is driven down a 5° incline at a speed of 72 km/h when the brakes are applied causing a constant total breaking force of 5000 N.

Determine the distance traveled by the automobile as it comes to a stop.

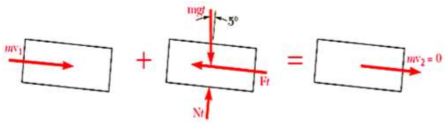
Sample Problem 13.10

SOLUTION:

- Apply the principle of impulse and momentum.

$$m\vec{v}_1 + \sum \mathbf{Imp}_{1\to 2} = m\vec{v}_2$$

Taking components parallel to the incline,



$$mv_1 + (mg\sin 5°)t - Ft = 0$$

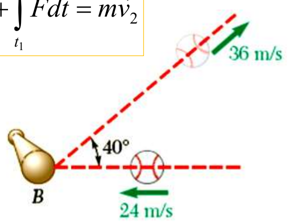
$$100\text{km/h} = 100 \times \frac{1000\text{m}}{1\text{km}} \times \frac{3600\text{s}}{1\text{h}} = 27.78\text{m}$$

$$(1800\text{kg})(27.78\text{ m/s}) + (17660\sin 5°\text{N})t - (7000\text{N})t = 0$$

$$\boxed{t = 9.16\,\text{s}}$$

Sample Problem 13.11

$$\boxed{m\vec{v}_1 + \int_{t_1}^{t_2} \vec{F}dt = m\vec{v}_2}$$

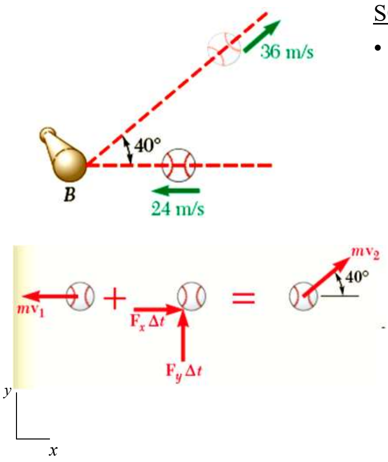


A 120-g baseball is pitched with a velocity of 24 m/s. After the ball is hit by the bat, it has a velocity of 36 m/s in the direction shown. If the bat and ball are in contact for 0.015 s, determine the average impulsive force exerted on the ball during the impact.

SOLUTION:

- Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.

Sample Problem 13.11



SOLUTION:

- Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.

$$m\vec{v}_1 + \mathbf{Imp}_{1\to 2} = m\vec{v}_2$$

x component equation:

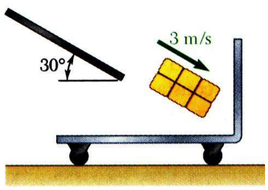
$$-mv_1 + F_x \Delta t = mv_2 \cos 40°$$
$$-(0.12\text{kg})(24\text{m/s}) + F_x(0.015\text{s}) = (0.12\text{kg})(36\text{m/s})\cos 40°$$
$$F_x = 412.6\,\text{N}$$

y component equation:

$$0 + F_y \Delta t = mv_2 \sin 40°$$
$$F_y(0.015) = (0.12\text{kg})(36\text{m/s})\sin 40°$$
$$F_y = 185.1\,\text{N}$$

$$\boxed{F = 452\text{ N}}$$

Sample Problem 13.12



A 10-kg package drops from a chute into a 25-kg cart with a velocity of 3 m/s. Knowing that the cart is initially at rest and can roll freely, determine *(a)* the final velocity of the cart, *(b)* the impulse exerted by the cart on the package, and *(c)* the fraction of the initial energy lost in the impact.

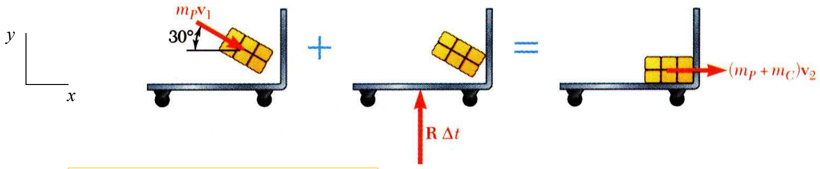
SOLUTION:

- Apply the principle of impulse and momentum to the package-cart system to determine the final velocity.
- Apply the same principle to the package alone to determine the impulse exerted on it from the change in its momentum.

Sample Problem 13.12

SOLUTION:

- Apply the principle of impulse and momentum to the package-cart system to determine the final velocity.



$$m_p\vec{v}_1 + \int_{t_1}^{t_2}\vec{F}_{\text{int}}dt = m_p\vec{v}_2$$

$$m_c\vec{v}_o + \int_{t_1}^{t_2}(-\vec{F}_{\text{int}} + \vec{R})dt = m_c\vec{v}_2$$

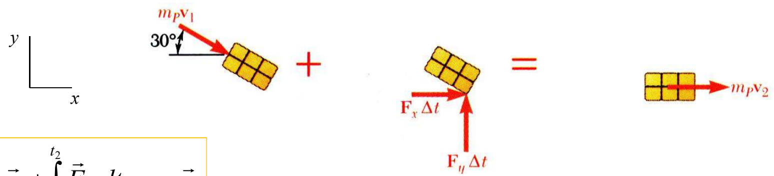
$$\Rightarrow \quad m_p\vec{v}_1 + \int_{t_1}^{t_2}\vec{R}dt = (m_p + m_c)\vec{v}_2$$

x components: $m_p v_1 \cos 30° + 0 = (m_p + m_c)v_2$

$(10\,\text{kg})(3\,\text{m/s})\cos 30° = (10\,\text{kg} + 25\,\text{kg})v_2$ $\quad \boxed{v_2 = 0.742\,\text{m/s}}$

Sample Problem 13.12

- Apply the same principle to the package alone to determine the impulse exerted on it from the change in its momentum.



$$m_p\vec{v}_1 + \int_{t_1}^{t_2}\vec{F}_{\text{int}}dt = m_p\vec{v}_2$$

x components: $m_p v_1 \cos 30° + F_x\Delta t = m_p v_2$

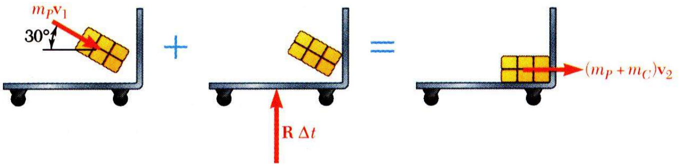
$(10\,\text{kg})(3\,\text{m/s})\cos 30° + F_x\Delta t = (10\,\text{kg})v_2$ $\quad F_x\Delta t = -18.56\,\text{N}\cdot\text{s}$

y components: $-m_p v_1 \sin 30° + F_y\Delta t = 0$

$-(10\,\text{kg})(3\,\text{m/s})\sin 30° + F_y\Delta t = 0$ $\quad F_y\Delta t = 15\,\text{N}\cdot\text{s}$

$$\boxed{\sum \mathbf{Imp}_{1\to 2} = \vec{F}\Delta t = (-18.56\,\text{N}\cdot\text{s})\vec{i} + (15\,\text{N}\cdot\text{s})\vec{j} \qquad F\Delta t = 23.9\,\text{N}\cdot\text{s}}$$

Sample Problem 13.12



To determine the fraction of energy lost,

$$T_1 = \tfrac{1}{2} m_p v_1^2 = \tfrac{1}{2}(10\,\text{kg})(3\,\text{m/s})^2 = 45\ \text{J}$$

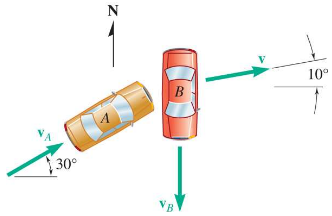
$$T_2 = \tfrac{1}{2}(m_p + m_c)v_2^2 = \tfrac{1}{2}(10\,\text{kg} + 25\,\text{kg})(0.742\,\text{m/s})^2 = 9.63\ \text{J}$$

$$\boxed{\frac{T_1 - T_2}{T_1} = \frac{45\ \text{J} - 9.63\ \text{J}}{45\ \text{J}} = 0.786}$$

- Internal force doesn't change total system momentum (only external force)
- Energy can be dissipated via impact (here, perfectly plastic impact assumed)

Concept Quiz

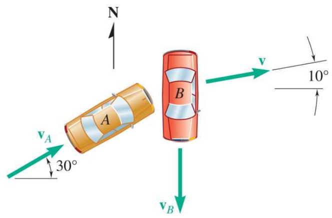
Car A and B crash into one another. Looking only at the impact, which of the following statements are true?



The total mechanical energy is the same before and after the impact	**True**	**False**
If car A weighs twice as much as car B, the force A exerts on car B is bigger than the force B exerts on car A.	**True**	**False**
The total linear momentum is the same immediately before and after the impact	**True**	**False**

Concept Quiz 2

Car A and B crash into one another. Looking only at the impact, which of the following statements are true?

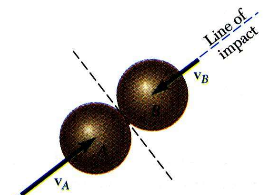


The total mechanical energy is the same before and after the impact — True / False

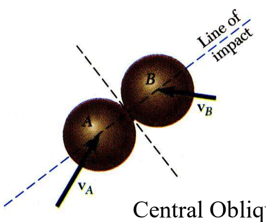
If car A weighs twice as much as car B, the force A exerts on car B is bigger than the force B exerts on car A. — True / False

The total linear momentum is the same immediately before and after the impact — True / False

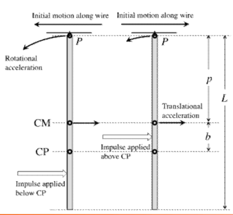
Impact



Central Direct Impact

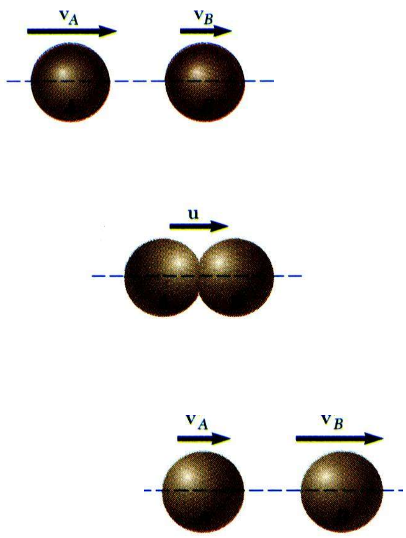


Central Oblique Impact



- *Impact:* Collision between two bodies which occurs during a small time interval and during which the bodies exert large forces on each other.
- ***Line of Impact:*** Common normal to the surfaces in contact during impact.
- ***Central Impact:*** Impact for which the mass centers of the two bodies lie on the line of impact; otherwise, it is an *eccentric impact.*
- *Direct Impact:* Impact for which the velocities of the two bodies are along the line of impact.
- *Oblique Impact:* Impact for which one or both of the bodies move along a line other than the line of impact (with no friction).
- *Central impact with no friction ⇒ rigid body with impact can be modeled by a particle with impact*

Central Direct Impact

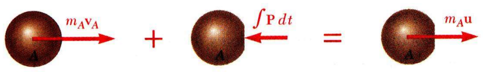


- Bodies moving in the same straight line, $v_A > v_B$.
- Upon impact the bodies undergo a ***period of deformation***, at the end of which, they are in contact and moving at a common velocity.
- A ***period of restitution*** follows during which the bodies either regain their original shape or remain permanently deformed.
- Wish to determine the final velocities of the two bodies. The total momentum of the two body system is preserved,

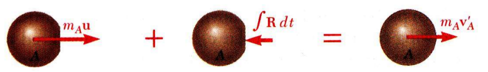
$$\boxed{m_A v_A + m_B v_B = m_B v'_B + m_B v'_B}$$

- Another relation between the final velocities is required.

Central Direct Impact



- Period of deformation: $m_A v_A - \int P dt = m_A u$



- Period of restitution: $m_A u - \int R dt = m_A v'_A$
- A similar analysis of particle B yields
- Adding these relations leads to the desired second relation between the final velocities.
- *Perfectly plastic impact,* $e = 0$: $v'_B = v'_A = v'$
- *Perfectly elastic impact,* $e = 1$: Total energy is also conserved (why?).

$e = coefficient\ of\ restitution$

$$= \frac{\int R dt}{\int P dt} = \frac{u - v'_A}{v_A - u}$$

$0 \le e \le 1$

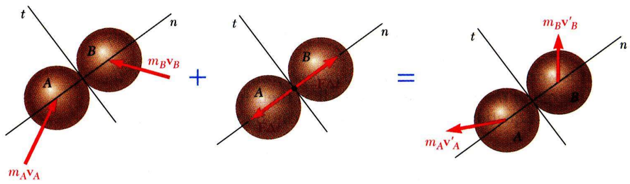
$$e = \frac{v'_B - u}{u - v_B}$$

$$\boxed{\begin{array}{l} v'_B - v'_A = e(v_A - v_B) \\ \text{not depends on } m_A, m_B! \end{array}}$$

$$m_A v_A + m_B v_B = (m_A + m_B) v'$$

$$v'_B - v'_A = v_A - v_B$$

Central Oblique Impact



- Final velocities are unknown in magnitude and direction. Four equations are required.

- No tangential impulse component (e.g., frictionless balls); tangential component of momentum for each particle is conserved.

$$(v_A)_t = (v'_A)_t \qquad (v_B)_t = (v'_B)_t$$

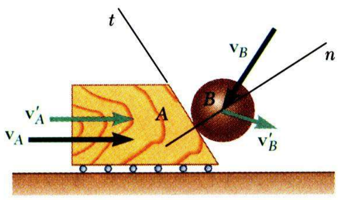
- Normal component of total momentum of the two particles is conserved.

$$m_A (v_A)_n + m_B (v_B)_n = m_A (v'_A)_n + m_B (v'_B)_n$$

- Normal components of relative velocities before and after impact are related by the coefficient of restitution.

$$(v'_B)_n - (v'_A)_n = e\left[(v_A)_n - (v_B)_n\right]$$

Central Oblique Impact

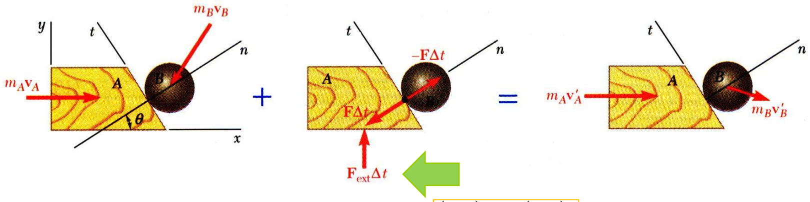


- Block constrained to move along horizontal surface.

- Impulses from internal forces $\vec{F}$ and $-\vec{F}$ along the n axis and from external force $\vec{F}_{ext}$ exerted by horizontal surface directed along the vertical to the surface.

- Final velocity of ball unknown in direction and magnitude and unknown final block velocity magnitude. Three equations required.

Central Oblique Impact

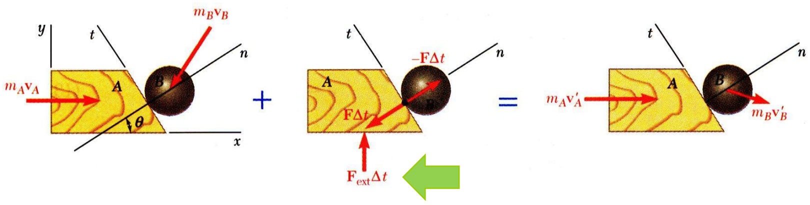


- Tangential momentum of ball is conserved (w/ no friction).
  $\boxed{(v_B)_t = (v'_B)_t}$
- Total horizontal momentum of block and ball conserved (w/ internal force).
  $\boxed{m_A(v_A) + m_B(v_B)_x = m_A(v'_A) + m_B(v'_B)_x}$
- Normal component of relative velocities of block and ball are related by coefficient of restitution.
  $\boxed{(v'_B)_n - (v'_A)_n = e\left[(v_A)_n - (v_B)_n\right]}$

- The last expression does not directly follow from the previous derivation of the restitution coefficient as the cart motion is now constrained (the same answer though).

$$\boxed{e = \frac{\int Rdt}{\int Pdt}} = \frac{\int Rdt\, cos\theta}{\int Pdt\, cos\theta}$$

Central Oblique Impact



block in horizontal direction

$$m_A v_A - \int Pdt\, cos\theta = m_A u$$
$$m_A u - \int Rdt\, cos\theta = m_A v'_A$$

ball in normal direction

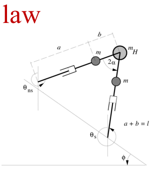
$$m_B(v_B)_n + \int Pdt = m_B u_n, \quad \boxed{u_n = u cos\theta}$$
$$m_B u_n + \int Rdt = m_B(v'_B)$$

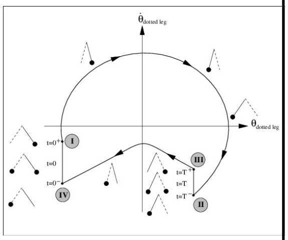
$$e = \frac{\int Rdt}{\int Pdt} = \frac{(v'_B)_n - u_n}{u_n - (v_B)_n} = \frac{\int Rdt\, cos\theta}{\int Pdt\, cos\theta} = \frac{u - v'_A}{v_A - u} = \frac{u_n - (v'_A)_n}{(v_A)\_n - u\_n}$$

$$\boxed{e = \frac{(v'_B)_n - (v'_A)_n}{(v_A)_n - (v_B)_n}}$$

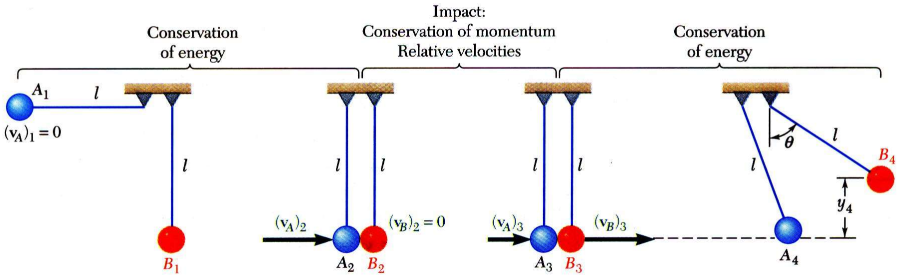
Problems Involving Energy and Momentum

- Three methods for the analysis of kinetics problems:
  - Direct application of Newton's second law
  - Method of work and energy
  - Method of impulse and momentum

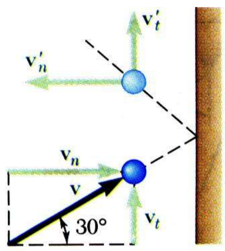




- Select the method best suited for the problem or part of a problem under consideration.



Sample Problem 13.14

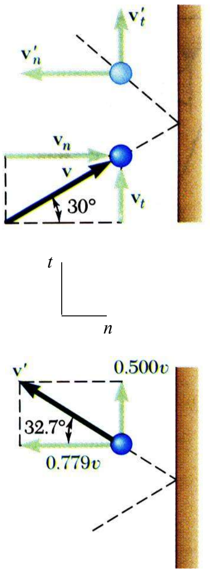


A ball is thrown against a frictionless, vertical wall. Immediately before the ball strikes the wall, its velocity has a magnitude v and forms angle of 30° with the horizontal. Knowing that $e = 0.90$, determine the magnitude and direction of the velocity of the ball as it rebounds from the wall. (unknowns: 2)

SOLUTION:

- Resolve ball velocity into components normal and tangential to wall.
- Impulse exerted by the wall is normal to the wall. Component of ball momentum tangential to wall is conserved.
- Assume that the wall has infinite mass so that wall velocity before and after impact is zero. Apply coefficient of restitution relation to find change in normal relative velocity between wall and ball, i.e., the normal ball velocity.

Sample Problem 13.14



SOLUTION:

- Resolve ball velocity into components parallel and perpendicular to wall.

$$v_n = v\cos 30° = 0.866v \qquad v_t = v\sin 30° = 0.500v$$

- Component of ball momentum tangential to wall is conserved.

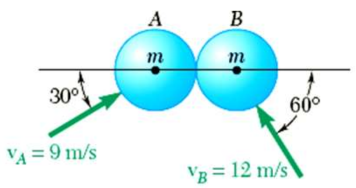
$$v'_t = v_t = 0.500v$$

- Apply coefficient of restitution relation with zero wall velocity.

$$0 - v'_n = e(v_n - 0)$$
$$v'_n = -0.9(0.866v) = -0.779v$$

$$\boxed{\vec{v}' = -0.779v\,\vec{\lambda}_n + 0.500v\,\vec{\lambda}_t \qquad v' = 0.926v \quad \tan^{-1}\left(\frac{0.779}{0.500}\right) = 32.7°}$$

Sample Problem 13.15

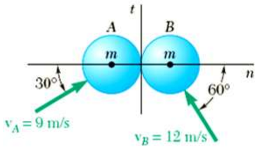


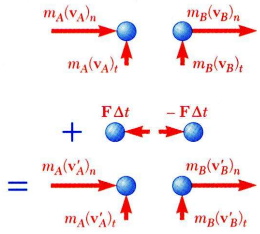
The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are as shown. Assuming e = 0.9, determine the magnitude and direction of the velocity of each ball after the impact. (unknowns: 4)

SOLUTION:

- Resolve the ball velocities into components normal and tangential to the contact plane.
- Tangential component of momentum for each ball is conserved (2 eqs).
- Total normal component of the momentum of the two ball system is conserved (1 eq).
- The normal relative velocities of the balls are related by the coefficient of restitution (1eq).
- Solve the last two equations simultaneously for the normal velocities of the balls after the impact.

Sample Problem 13.15





SOLUTION:

- Resolve the ball velocities into components normal and tangential to the contact plane.

$$(v_A)_n = v_A \cos 30° = 7.79\,\text{m/s} \qquad (v_A)_t = v_A \sin 30° = 4.5\,\text{m/s}$$
$$(v_B)_n = -v_B \cos 60° = -6\,\text{m/s} \qquad (v_B)_t = v_B \sin 60° = 10.39\,\text{m/s}$$

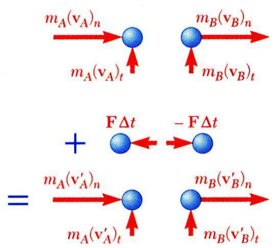
- Tangential component of momentum for each ball is conserved.

$$(v'_A)_t = (v_A)_t = 4.5\,\text{m/s} \qquad (v'_B)_t = (v_B)_t = 10.39\,\text{m/s}$$

- Total normal component of the momentum of the two ball system is conserved.

$$m_A(v_A)_n + m_B(v_B)_n = m_A(v'_A)_n + m_B(v'_B)_n$$
$$m(7.79) + m(-6) = m(v'_A)_n + m(v'_B)_n$$
$$(v'_A)_n + (v'_B)_n = 1.79$$

Sample Problem 13.15

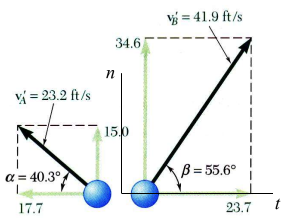


- The normal relative velocities of the balls are related by the coefficient of restitution.

$$(v'_A)_n - (v'_B)_n = e[(v_A)_n - (v_B)_n]$$
$$= 0.90[7.79 - (-6)] = 12.41$$

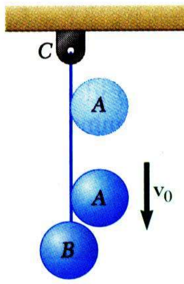
- Solve the last two equations simultaneously for the normal velocities of the balls after the impact.

$$(v'_A)_n = -5.31\,\text{m/s} \qquad (v'_B)_n = 7.1\,\text{m/s}$$



$$\vec{v}'_A = -5.31\vec{\lambda}_t + 4.5\vec{\lambda}_n$$
$$v'_A = 7.1\,\text{m/s} \quad \tan^{-1}\left(\frac{4.5}{5.31}\right) = 40.3°$$
$$\vec{v}'_B = 7.1\vec{\lambda}_t + 10.39\vec{\lambda}_n$$
$$v'_B = 12.58\,\text{m/s} \quad \tan^{-1}\left(\frac{10.39}{7.1}\right) = 55.6°$$

Sample Problem 13.16

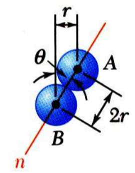


Ball B is hanging from an inextensible cord. An identical ball A is released from rest when it is just touching the cord and acquires a velocity v_0 before striking ball B. Assuming perfectly elastic impact ($e = 1$) and no friction, determine the velocity of each ball immediately after impact. (unknowns: 3)

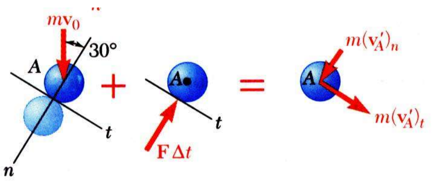
SOLUTION:

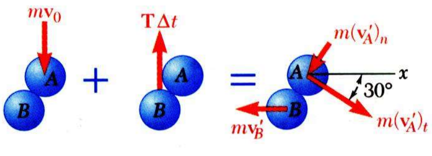
- Determine orientation of impact line of action.
- The momentum component of ball A tangential to the contact plane is conserved.
- The total horizontal momentum of the two ball system is conserved.
- The relative velocities along the line of action before and after the impact are related by the coefficient of restitution.
- Solve the last two expressions for the velocity of ball A along the line of action and the velocity of ball B which is horizontal.

Sample Problem 13.16



$$\sin\theta = \frac{r}{2r} = 0.5$$
$$\theta = 30°$$





SOLUTION:

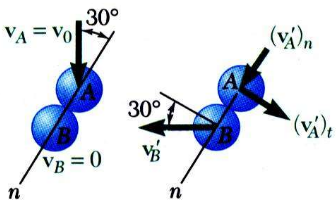
- Determine orientation of impact line of action.
- The momentum component of ball A tangential to the contact plane is conserved.

$$m\vec{v}_A + \vec{F}\Delta t = m\vec{v}'_A$$
$$mv_0 \sin 30° + 0 = m(v'_A)_t$$
$$(v'_A)_t = 0.5v_0$$

- The total horizontal (x component) momentum of the two ball system is conserved.

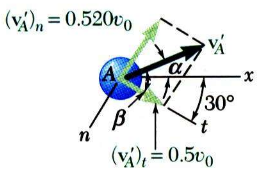
$$m\vec{v}_A + \vec{T}\Delta t = m\vec{v}'_A + m\vec{v}'_B$$
$$0 = m(v'_A)_t \cos 30° - m(v'_A)_n \sin 30° - mv'_B$$
$$0 = (0.5v_0)\cos 30° - (v'_A)_n \sin 30° - v'_B$$
$$0.5(v'_A)_n + v'_B = 0.433v_0$$

Sample Problem 13.16



- The relative velocities along the line of action before and after the impact are related by the coefficient of restitution.

$$(v'_B)_n - (v'_A)_n = e\left[(v_A)_n - (v_B)_n\right]$$
$$v'_B \sin 30° - (v'_A)_n = v_0 \cos 30° - 0$$
$$0.5v'_B - (v'_A)_n = 0.866v_0$$



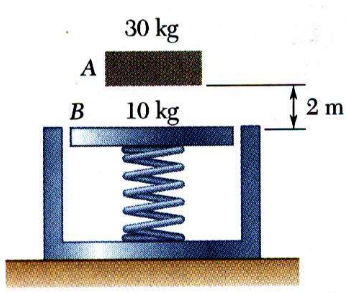
- Solve the last two expressions for the velocity of ball *A* along the line of action and the velocity of ball *B* which is horizontal.

$$(v'_A)_n = -0.520v_0 \qquad v'_B = 0.693v_0$$

$$\vec{v}'_A = 0.5v_0\vec{\lambda}_t - 0.520v_0\vec{\lambda}_n$$
$$v'_A = 0.721v_0 \qquad \beta = \tan^{-1}\left(\frac{0.52}{0.5}\right) = 46.1°$$
$$\alpha = 46.1° - 30° = 16.1°$$
$$v'_B = 0.693v_0 \leftarrow$$

- Energy conservation can also be used instead of coefficient of restitution, since the impact here is assumed to be perfectly elastic (no energy dissipation).

Sample Problem 13.17

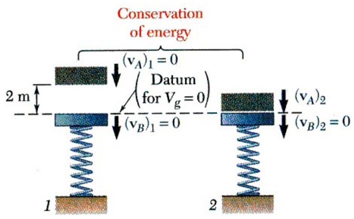


A 30-kg block is dropped from a height of 2 m onto the 10 kg-pan of a spring scale. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. The constant of the spring is $k = 20$ kN/m.

SOLUTION:

- Apply the principle of conservation of energy to determine the velocity of the block at the instant of impact.
- Since the impact is perfectly plastic, the block and pan move together at the same velocity after impact. Determine that velocity from the requirement that the total momentum of the block and pan is conserved.
- Apply the principle of conservation of energy to determine the maximum deflection of the spring.

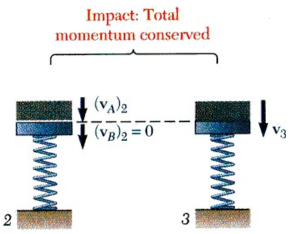
Sample Problem 13.17



SOLUTION:

- Apply principle of conservation of energy to determine velocity of the block at instant of impact.

$$T_1 = 0 \qquad V_1 = W_A y = (30)(9.81)(2) = 588\,\text{J}$$
$$T_2 = \tfrac{1}{2} m_A (v_A)_2^2 = \tfrac{1}{2}(30)(v_A)_2^2 \qquad V_2 = 0$$
$$T_1 + V_1 = T_2 + V_2$$
$$0 + 588\,\text{J} = \tfrac{1}{2}(30)(v_A)_2^2 + 0 \qquad (v_A)_2 = 6.26\,\text{m/s}$$



- Determine velocity after impact from requirement that total momentum of the block and pan is conserved.

$$m_A (v_A)_2 + m_B (v_B)_2 = (m_A + m_B) v_3$$
$$(30)(6.26) + 0 = (30 + 10) v_3 \qquad v_3 = 4.70\,\text{m/s}$$

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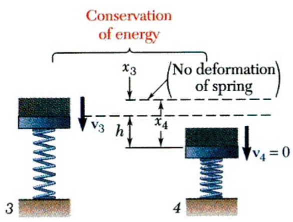
13 - 69

Sample Problem 13.17



Initial spring deflection due to pan weight:

$$x_3 = \frac{W_B}{k} = \frac{(10)(9.81)}{20 \times 10^3} = 4.91 \times 10^{-3}\,\text{m}$$

- Apply the principle of conservation of energy to determine the maximum deflection of the spring.

$$T_3 = \tfrac{1}{2}(m_A + m_B) v_3^2 = \tfrac{1}{2}(30 + 10)(4.7)^2 = 442\,\text{J}$$
$$V_3 = V_g + V_e = 0 + \tfrac{1}{2} k x_3^2 = \tfrac{1}{2}(20 \times 10^3)(4.91 \times 10^{-3})^2$$
$$T_4 = 0$$
$$V_4 = V_g + V_e = (W_A + W_B)(-h) + \tfrac{1}{2} k x_4^2$$

$$T_3 + V_3 = T_4 + V_4$$
$$442 + 0.241 = 0 - 392(x_4 - 4.91 \times 10^{-3}) + \tfrac{1}{2}(20 \times 10^3) x_4^2$$
$$x_4 = 0.230\,\text{m}$$
$$h = x_4 - x_3 = 0.230\,\text{m} - 4.91 \times 10^{-3}\,\text{m} \qquad \boxed{h = 0.225\,\text{m}}$$

may also directly start from the equilibrium while neglecting initial deformation of spring

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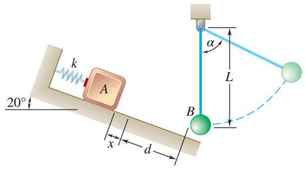
13 - 70

Group Problem Solving 13



Strategy:

- This is a multiple step problem. Formulate your overall approach.

A 2-kg block *A* is pushed up against a spring compressing it a distance *x*= 0.1 m. The block is then released from rest and slides down the 20º incline until it strikes a 1-kg sphere *B*, which is suspended from a 1 m inextensible rope. The spring constant is *k*=800 N/m, the coefficient of friction between *A* and the ground is 0.2, the distance *A* slides from the unstretched length of the spring d = 1.5 m, and the coefficient of restitution between *A* and *B* is 0.8. When α = 40º, find (*a*) the speed of *B* (*b*) the tension in the rope.

- Use work-energy to find the velocity of the block just before impact.
- Use conservation of momentum to determine the speed of ball B after the impact.
- Use work energy to find the velocity at **α.**
- Use Newton's 2nd Law to find tension in the rope.

$$\boxed{\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n}$$

Concept Question 1

Compare the following statement to the problem you just solved.

If the coefficient of restitution is smaller than the 0.8 in the problem, the tension T will be…

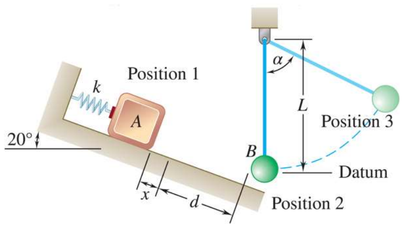
Smaller **Bigger**

If the rope length is smaller than the 1 m in the problem, the tension T will be…

Smaller **Bigger**

If the coefficient of friction is smaller than 0.2 given in the problem, the tension T will be…

Smaller **Bigger**



If the mass of A is smaller than the 2 kg given in the problem, the tension T will be…

Smaller **Bigger**

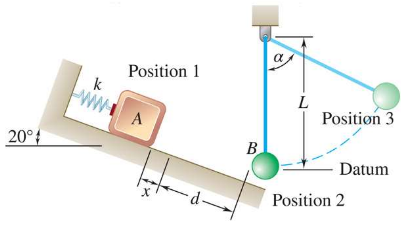
$$\boxed{\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n}$$

Concept Question 2

Compare the following statement to the problem you just solved.

If the coefficient of restitution is smaller than the 0.8 in the problem, the tension T will be…

Smaller **Bigger**



If the rope length is smaller than the 1 m in the problem, the tension T will be…

Smaller **Bigger**

If the mass of A is smaller than the 2 kg given in the problem, the tension T will be…

Smaller **Bigger**

If the coefficient of friction is smaller than 0.2 given in the problem, the tension T will be…

Smaller **Bigger**

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n$$