Engineering Mathematics 2

Lecture 3

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Previously, we discussed

- curves
- parametric representations of curves
- arc length
- arc length as a parameter
- curvature
- torsion
- chain rule for functions of several variables

9.7 Gradient of a scalar field

- $grad\ f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

- differential operator $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

- Gradient of a scalar function is a vector function.

- If a vector function is obtained as the gradient of a scalar function, the scalar function is called the potential of the vector field.

- Directional derivative:

  Find the derivative of a function $f(x, y, z)$ in the direction of a vector $\vec{b}$

  (hint: set $\vec{r}(s) = x(s)\vec{\imath} + y(s)\vec{\jmath} + z(s)\vec{k} = \overrightarrow{p_0} + s\vec{b}$ and calculate df/ds).

$$f(x(s), y(s), z(s))$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]\cdot\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right]$$

$$= \nabla f \cdot \frac{d\vec{r}}{ds}$$

$$= \nabla f \cdot \vec{b}$$

- Surface normal vector

Find a vector that is normal to a level surface $f(x, y, z) = c$
(note that tangent vector of a point with its position vector $\vec{r}$ is $\vec{r}'$).

$$\vec{r} = [x(t), y(t), z(t)]$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

$$\nabla f \cdot \frac{d\vec{r}}{dt} = 0$$

tangent

- Gravitational field

Show that the force of attraction $\vec{p} = -\frac{c}{r^3}\vec{r}$ between around a fixed point (x_0, y_0, z_0) has the potential $\boxed{f(x, y, z) = \frac{c}{r}}$

in which $r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$.

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$\frac{\partial f}{\partial x} = \frac{df}{dr} \cdot \frac{\partial r}{\partial x} = -\frac{c}{r^2} \cdot \frac{\cancel{2}x}{\cancel{2}r} = -c\frac{1}{r^3}x$$

$$\nabla f = -\frac{c}{r^3}[x, y, z] = -\frac{c}{r^3}\vec{r} = \vec{p}$$

9.8 Divergence of a vector field

- For a vector $\vec{v} = [v_1, v_2, v_3]$, the divergence of the vector is

$$div\ \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

- The divergence of a vector function is a scalar function.

- If $\vec{v}$ is fluid velocity, its divergence roughly corresponds to 'net outflow'.

- For a scalar function $\rho = \rho(x, y, z)$, show that

$$\nabla \cdot (\rho \vec{v}) = \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v}$$

(hint: $\vec{v} = [v_1, v_2, v_3]$)

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\rho v_1, \rho v_2, \rho v_3\right]$$

$$= \frac{\partial}{\partial x}(\rho v_1) + \frac{\partial}{\partial y}(\rho v_2) + \frac{\partial}{\partial z}(\rho v_3)$$

$$= \left(\frac{\partial \rho}{\partial x} v_1 + \frac{\partial \rho}{\partial y} v_2 + \frac{\partial \rho}{\partial z} v_3\right) + \rho\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)$$

$$= \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v}$$

- Incompressible fluid does NOT mean constant density.

  For a fluid with the density field and the velocity field represented by $\rho = \rho(x, y, z, t)$ and $\vec{v} = [v_1(x, y, z, t), v_2(x, y, z, t), v_3(x, y, z, t)]$, the mass conservation equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0.$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho}_{=0} + \rho \nabla \cdot \vec{v} = 0$$

$$\frac{\partial}{\partial t} + \vec{v} \cdot \nabla = \frac{D}{Dt}$$

$$\frac{D\rho}{Dt} = 0 \rightarrow \nabla \cdot \vec{v} = 0$$

  Find the condition for $\nabla \cdot \vec{v} = 0.$

9.9 Curl of a vector field

- $curl\ \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$

- If $\vec{v}$ is fluid velocity, $curl\ \vec{v}$ is called the vorticity with its magnitude twice the rotational speed.

- Irrotational flow: $\nabla \times \vec{v} = 0$

Some important identities

- $curl\ (grad\ f) = \nabla \times (\nabla f) = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

- $div\ (curl\ \vec{v}) = \nabla \cdot (\nabla \times \vec{v}) = 0$

10.1 Line integrals

- A line integral of a vector function $\vec{F}(\vec{r})$ over a curve C: $\vec{r}(t)$

$$\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt} dt$$

- With $d\vec{r} = [dx, dy, dz]$,

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

$$= \int_a^b \underline{(F_1 x' + F_2 y' + F_3 z')dt}$$

- If the curve C is closed, the line integral is denoted as

$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r}$$

- Example 4

Suppose $\vec{F}(\vec{r})$ is a force with the parameter t now representing time. Then the first and second derivative or $\vec{r}(t)$ are velocity and acceleration vector, respectively.

Show that work done for $a \leq t \leq b$ is the same as the gain in kinetic energy during that period.

(Notice that $\vec{F} = m\vec{r}''$, in which m is the mass of the moving particle)

$$\int_a^b \vec{F} \cdot \underbrace{\frac{d\vec{r}}{dt}}^{=\vec{v}} dt = \int_a^b m \frac{d\vec{v}}{dt} \cdot \vec{v} \, dt$$

(From Newton's 2nd law: $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$)

$$= \int_a^b \frac{m}{2} \frac{d}{dt}(\vec{v} \cdot \vec{v}) \, dt$$

$$= \int_a^b \frac{m}{2} d(\vec{v} \cdot \vec{v})$$

$$= \frac{m}{2} |\vec{v}|^2 \Big|_{t=a}^{t=b}$$

10.2 Path independence of line integrals

- For SOME vector functions, the line integral in a <u>simply connected</u> domain depends only on the end points (say, A and B), NOT the path of integration.

- This is the case if $\vec{F} = grad\ f$, since

In 2D

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \int_C df$$

$$= f(B) - f(A)$$

In 3D

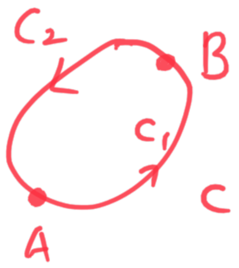
- Example 1

Calculate $\int_C (2x\,dx + 2y\,dy + 4z\,dz)$ from A: (0, 0, 0) to B: (2, 2, 2).

$\vec{F} = [2x, 2y, 4z] = \nabla f$, $f = x^2 + y^2 + 2z^2$

$$\int_A^B \nabla f \cdot d\vec{r} = \int_A^B df = f(B) - f(A)$$

$$= 16$$



$$\int_{C_1} df = -\int_{C_2} df$$

$$\left(\int_{C_1} + \int_{C_2}\right) df = 0 \, , \quad \oint_C df = 0$$

- Path independence implies that the line integral along a closed curve is zero (show it).

- If the line integral of the vector $\vec{F}$ is path-independent, then the vector field is called conservative.

- If not, it is called non-conservative or dissipative.

- If

$$\vec{F} \cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

then $(F_1 dx + F_2 dy + F_3 dz)$ is called exact.

- This is the case iff $\vec{F} = grad\ f$.

- From $curl\ (grad\ f) = 0$, we have test for exactness:

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$