3/22/21 Asymptotic expansion for single particle motion

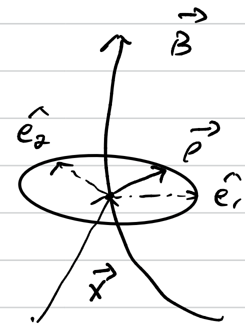
Again, Lorentz force $\frac{m}{q}\frac{d\vec{v}}{dt} = \vec{E} + \vec{v}\times\vec{B}$

$m/q \to 0$ Small "$\frac{m}{q}$" expansion

$\frac{m}{q} = \epsilon$ for expansion

$\frac{m}{q} \to \epsilon\frac{m}{q}$ and $\epsilon = 1$ after expansion

$$\epsilon\frac{m}{q}\frac{d\vec{v}}{dt} = \vec{E} + \vec{v}\times\vec{B}$$



<u>ordering</u> $\vec{x} = \vec{X} + \epsilon\vec{\rho}$

guiding gyro

$$\vec{B}(\vec{X}+\vec{\rho}) = \vec{B}(\vec{X}) + \epsilon(\vec{\rho}\cdot\nabla)\vec{B} + O(\epsilon^2)$$

$$\vec{E}_\perp(\vec{X}+\vec{\rho}) = \vec{E}_\perp(\vec{X}) + \epsilon(\vec{\rho}\cdot\nabla)\vec{E}_\perp + O(\epsilon^2)$$

$$\vec{E}_\parallel(\vec{X}+\vec{\rho}) = \epsilon\vec{E}_\parallel(\vec{X}) + O(\epsilon^2)$$

$\vec{\rho} = \rho(\hat{e}_1\sin\omega t + \hat{e}_2\cos\omega t)$

Assume $\rho = \frac{mv_\perp}{qB}$ $\omega = \frac{qB}{m}$

$\vec{\rho}(\gamma,t) = \vec{\rho} = \rho(\hat{e}_1\sin\gamma + \hat{e}_2\cos\gamma)$

$\gamma(t)$: gyro phase

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{d\vec{X}}{dt} + \frac{d\vec{\rho}}{dt}$$

$$= \vec{v}_0 + \omega\frac{\partial\vec{\rho}}{\partial\gamma} + \epsilon\vec{v}_1 + \epsilon\frac{\partial\vec{\rho}}{\partial t}$$

exp. $\frac{d\vec{\rho}}{dt} = \frac{\partial\vec{\rho}}{\partial t} + \dot{\gamma}\frac{\partial\vec{\rho}}{\partial\gamma}$ $(\dot{\gamma} = \omega)$

$= \epsilon\frac{\partial\vec{\rho}}{\partial t} + \cancel{\epsilon^{-1}}\omega\cancel{\epsilon}\frac{\partial\vec{\rho}}{\partial\gamma}$

$\frac{d\vec{X}}{dt} = \vec{v}_0 + \epsilon\vec{v}_1 + O(\epsilon^2)$

represents spatial dependency

(In fact, $\omega = \omega_0 + \epsilon\omega_1 + \cdots$

$\frac{\partial\vec{\rho}}{\partial t} = \frac{\partial\vec{\rho}_0}{\partial t} + \epsilon\frac{\partial\vec{\rho}_1}{\partial t} + \cdots$

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{v}_0 + \epsilon\vec{v}_1 + \cdots) + \underbrace{\left(\epsilon^{-1}\omega\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial t} + \vec{v}_0\cdot\nabla\right)}_{\frac{d}{dt}}\underbrace{\left(\epsilon^{-1}\omega\frac{\partial}{\partial\gamma} + \frac{\partial}{\partial t}\right)\epsilon\vec{\rho}}_{\frac{d\vec{\rho}}{dt}}$$

$$= \epsilon^{-1}\omega^2\frac{\partial^2\vec{\rho}}{\partial\gamma^2} + \frac{d}{dt}\vec{v}_0 + 2\omega\frac{\partial^2\vec{\rho}}{\partial t\,\partial\gamma} + \frac{d\omega}{dt}\frac{\partial\vec{\rho}}{\partial\gamma} + \omega(\vec{v}_0\cdot\nabla)\frac{\partial\vec{\rho}}{\partial\gamma} + O(\epsilon)$$

ϵ^0 : $\omega^2\frac{\partial^2\vec{\rho}}{\partial\gamma^2} - \frac{q}{m}\omega\frac{\partial\vec{\rho}}{\partial\gamma}\times\vec{B} = \vec{E}_\perp + \vec{v}_0\times\vec{B}$

$\omega = \frac{qB}{m}$. $\vec{\rho} = \rho(\hat{e}_1\sin\gamma + \hat{e}_2\cos\gamma)$ $\omega^2\frac{\partial^2\vec{\rho}}{\partial\gamma^2} - \frac{q}{m}\omega\frac{\partial\vec{\rho}}{\partial\gamma}\times\vec{B} = 0$

$$\vec{v}_0 = \frac{\vec{E}_\perp \times \vec{B}}{B^2}$$

$$\epsilon' : \quad 2\omega \frac{\partial^2 \vec{\rho}}{\partial t\, \partial \gamma} + \frac{d\omega}{dt}\frac{\partial \vec{\rho}}{\partial \gamma} + \omega (\vec{v}_0 \cdot \vec{\nabla}) \frac{\partial \vec{\rho}}{\partial \gamma} + \frac{d\vec{v}_0}{dt}$$

$$= \frac{q}{m}\Big[(\vec{\rho}\cdot\vec{\nabla})\vec{E}_\perp + \vec{E}_\parallel + \vec{v}_1 \times \vec{B} + \frac{\partial \vec{\rho}}{\partial t} \times \vec{B} + \vec{v}_0 \times (\vec{\rho}\cdot\vec{\nabla})\vec{B} + \omega \frac{\partial \vec{\rho}}{\partial \gamma} \times (\vec{\rho}\cdot\vec{\nabla})\vec{B} \Big]$$

(A) Take gyrophase average $\langle A \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} A\, d\gamma$

$$\frac{d\vec{v}_0}{dt} = \frac{q}{m}\left[\vec{E}_\parallel + \vec{v}_1 \times \vec{B} \right] + \frac{q}{m}\omega \left\langle \frac{\partial \vec{\rho}}{\partial \gamma} \times (\vec{\rho}\cdot\vec{\nabla})\vec{B} \right\rangle$$

$\langle \; \rangle : \quad \frac{\partial \vec{\rho}}{\partial \gamma} = \rho(\hat{x}\cos\gamma - \hat{y}\sin\gamma)$ $\qquad \langle \cos^2\gamma \rangle = \frac{1}{2}$

$$\times (\vec{\rho}\cdot\vec{\nabla})\vec{B} = \rho\left(\sin\gamma \frac{\partial \vec{B}}{\partial x} + \cos\gamma \frac{\partial \vec{B}}{\partial y}\right)$$

$$= \frac{1}{2}\rho^2 \Big(\underbrace{\frac{\partial B_y}{\partial y}\hat{z} + \frac{\partial B_x}{\partial x}\hat{z}}_{\vec{\nabla}\cdot\vec{B} = 0} - \frac{\partial B_z}{\partial y}\hat{y} - \frac{\partial B_z}{\partial x}\hat{x} \Big)$$

$$= -\frac{1}{2}\rho^2 \Big(\frac{\partial B_z}{\partial z}\hat{z} + \frac{\partial B_z}{\partial y}\hat{y} + \frac{\partial B_z}{\partial x}\hat{x} \Big)$$

$$B_z = B$$

$$= -\frac{1}{2}\rho^2 \vec{\nabla} B$$

$$\frac{m}{q}\frac{d\vec{v}_0}{dt} = \vec{v}_1 \times \vec{B} + E_\parallel \hat{b} - \frac{1}{2}\rho^2 \omega \vec{\nabla} B \qquad \frac{1}{2}\rho^2\omega \cdot q = \mu$$

Leading order

parallel : $\boxed{m\frac{dv_{\parallel 0}}{dt} = qE_\parallel - \mu(\hat{b}\cdot\vec{\nabla})B}$ $\leftarrow \left(\frac{d\vec{V}}{dt}\cdot\hat{b} \simeq \frac{dv_{\parallel 0}}{dt} + \mathcal{O}(\epsilon) \right)$

perp : $\vec{v}_1 = \underbrace{\frac{m}{qB^2}\left(\vec{B} \times \frac{d\vec{v}_0}{dt}\right)}_{\text{inertial drift}} + \underbrace{\frac{\mu}{qB^2}(\vec{B}\times\vec{\nabla}B)}_{\vec{\nabla}B \text{ drift}}$

$$\frac{d\vec{v}_0}{dt} = \frac{d}{dt}\left(V_{\parallel 0}\hat{b} + \frac{\vec{E}\times\vec{B}}{B^2}\right) = \cancel{\hat{b}\frac{dV_{\parallel 0}}{dt}} + V_{\parallel 0}\frac{d\hat{b}}{dt} + \frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right)$$

gone by $\vec{B}\times$

$$= V_{\parallel 0}\left(\frac{\partial\hat{b}}{\partial t} + (V_{\parallel 0}\hat{b} + \vec{V}_E)\cdot\vec{\nabla}\hat{b}\right) + \frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right)$$

$V_{\parallel 0} \gg V_\perp$

$$\approx V_{\parallel 0}\frac{\partial\hat{b}}{\partial t} + V_{\parallel 0}^2\,\hat{b}\cdot\vec{\nabla}\hat{b} + \frac{\frac{d}{dt}\vec{E}_\perp\times\vec{B}}{B^2} + \vec{E}_\perp\times\frac{d}{dt}\left(\frac{\vec{B}}{B^2}\right) + \text{non-linear terms}$$

$$\vec{B}\times\frac{d\vec{v}_0}{dt} \approx V_{\parallel 0}^2(\vec{B}\times\vec{\kappa}) + \frac{d\vec{E}_\perp}{dt} + \sim$$

$$\boxed{\vec{v}_0 + \vec{v}_1 = \frac{\vec{E}\times\vec{B}}{B^2} + \frac{mV_{\parallel 0}^2}{qB^2}(\vec{B}\times\vec{\kappa}) + \frac{\mu}{qB^2}(\vec{B}\times\vec{\nabla}B) + \frac{m}{qB^2}\frac{d}{dt}\vec{E}_\perp + \sim}$$

(B)

$$\epsilon' : \Big\{ 2\omega\frac{\partial^2\vec{\rho}}{\partial t\partial\gamma} + \frac{d\omega}{dt}\frac{\partial\vec{\rho}}{\partial\gamma} + \omega(\vec{v}_0\cdot\vec{\nabla})\frac{\partial\vec{\rho}}{\partial\gamma} + \frac{d\vec{v}_0}{dt}$$

$$= \frac{q}{m}\Big[(\vec{\rho}\cdot\vec{\nabla})\vec{E}_\perp + \vec{E}_\parallel + \vec{v}_1\times\vec{B} + \frac{\partial\vec{\rho}}{\partial t}\times\vec{B}$$

$$+ \vec{v}_0\times(\vec{\rho}\cdot\vec{\nabla})\vec{B} + \omega\frac{\partial\vec{\rho}}{\partial\gamma}\times(\vec{\rho}\cdot\vec{\nabla})\vec{B}\Big]\Big\}$$

Take the gyro-average with $\frac{\partial\vec{\rho}}{\partial\gamma}$ $\quad\left\langle\frac{\partial\vec{\rho}}{\partial\gamma}\cdot\{\}\right\rangle$

$$2\omega\left\langle\frac{\partial\vec{\rho}}{\partial\gamma}\cdot\frac{\partial^2\vec{\rho}}{\partial t\partial\gamma}\right\rangle + \frac{d\omega}{dt}\left\langle\frac{\partial\vec{\rho}}{\partial\gamma}\cdot\frac{\partial\vec{\rho}}{\partial\gamma}\right\rangle + \omega(\vec{v}_0\cdot\vec{\nabla})\left\langle\frac{\partial\vec{\rho}}{\partial\gamma}\cdot\frac{\partial\vec{\rho}}{\partial\gamma}\right\rangle$$

$$= \frac{q}{m}\left\langle\frac{\partial\vec{\rho}}{\partial\gamma}\cdot(\vec{\rho}\cdot\vec{\nabla})\vec{E}_\perp\right\rangle + \frac{q}{m}\left\langle\frac{\partial\vec{\rho}}{\partial\gamma}\cdot\frac{\partial\vec{\rho}}{\partial t}\times\vec{B}\right\rangle$$

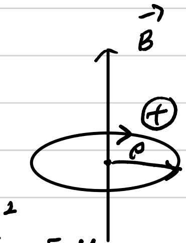
$$+ \frac{q}{m}\left\langle\frac{\partial\vec{\rho}}{\partial\gamma}\cdot\vec{v}_0\times(\vec{\rho}\cdot\vec{\nabla})\vec{B}\right\rangle$$

$$\longrightarrow \qquad \frac{d\mu}{dt} = 0 \qquad \mu\text{-invariance}$$

- Heuristic : Goldston ch 3.3, 3.5
- Kruskal : Extended WKB approxi. (1958)
- Little John : Lagrangian + Lie algebra (1983)

3/24/21 Adiabatic Invariants / conservation in particle motion

1. Magnetic moment $\mu = \dfrac{\frac{1}{2} m v_\perp^2}{B}$



Classical def. = (current) × (Area)

$$\mathcal{M} = IA = \frac{q\omega_c}{2\pi}\pi\rho^2 = \frac{1}{2} q\omega_c\rho^2 = \frac{1}{2} q\frac{qB}{m}\frac{m^2v_\perp^2}{q^2B^2} = \mu$$

Adiabatic μ-invariance $\dfrac{d\mu}{dt} = 0$

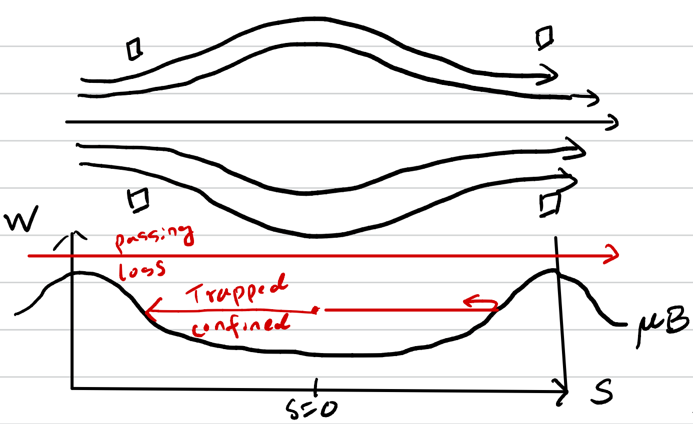
→ Flux conservation

$$\psi_{gyro} = \pi\rho^2 B = \pi\frac{2\mu}{q\omega_c}B = \frac{2\pi m}{q^2}\mu$$

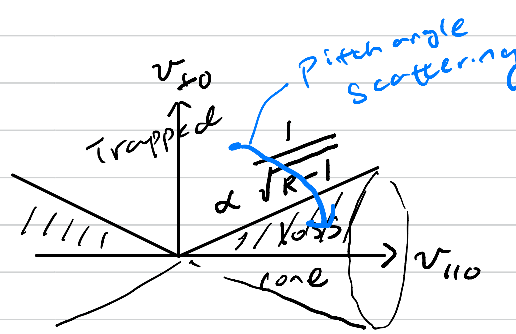
→ Compression (or expansion) by slow change in B

Read Goldston ch 4.2

2. Application to magnetic mirror



$\underline{\phi = 0}$



with energy conservation

$$W = \frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 + q\phi$$
$$= \frac{1}{2}mv_\parallel^2 + \mu B + q\phi$$

$$\frac{1}{2}mv_\parallel^2 = W - \mu B - q\phi$$

$$\rightarrow m\frac{dv_\parallel}{dt} = -q\nabla_\parallel\phi - \mu\nabla_\parallel B$$

$$\left(\text{if, } \frac{dW}{dt} = 0,\ \frac{d\mu}{dt} = 0\right)$$

$\frac{1}{2}mv_\parallel^2 = W - \mu B$

At midplane ($s=0$)

Let, $W_{\parallel 0} = \frac{1}{2}mv_{\parallel 0}^2$

$W_{\perp 0} = \frac{1}{2}mv_{\perp 0}^2 = \mu B_{min}$

passing / Trap: $W_0 = W_{\parallel 0} + W_{\perp 0} \gtrless \mu B_{max}$

$$\frac{W_{\parallel 0}}{W_{\perp 0}} \gtrless \frac{B_{max}}{B_{min}} - 1$$

$\frac{B_{max}}{B_{min}} \equiv R$

* $\dfrac{dW}{dt} \neq 0$, $\dfrac{d\mu}{dt} = 0$ due to drift ($O(\epsilon)$)

— Trapped particles can enter loss cone due to $\left\{\begin{array}{l}\text{pitch-angle} \\ \text{energy}\end{array}\right\}$ scattering $\left\{\begin{array}{l}W = \text{const} \\ W \neq \text{const}\end{array}\right.$

- Electrons will be lost faster
  , but the ambipolar electron potential will be developed

3. Adiabatic Invariants in particle motion

Hamiltonian system $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

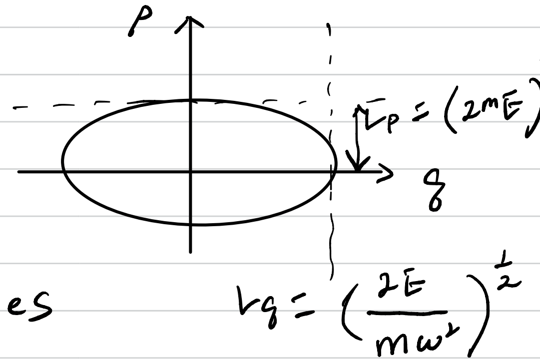
$H(q_1 \cdots q_k, p_1 \cdots p_k)$

For nearly periodic motion w.r.t (p_i, q_i)

action $I = \oint p\,dq$ is adiabatically invariant

* Consider a simple harmonic oscillation

$H(p,q) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = E \rightarrow \ddot{q} + \omega^2 q = 0$



$I = \oint p\,dq$

$= \pi L_p L_q = \frac{2\pi E}{\omega}$

conserved even if $\omega(t)$ changes slowly $\frac{1}{\omega}\frac{d\omega}{dt} \ll \omega$

in a sense for phase space conservation

eigen states "

$\frac{E}{\omega} = \hbar(N + \frac{1}{2})$

For gyromotion $E = \frac{1}{2} m v_\perp^2$ $\omega_c = \frac{qB}{m}$

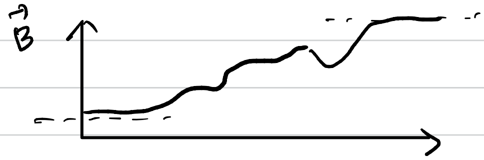
$I = \oint p\,dq = 2\pi \cdot \frac{\frac{1}{2} m v_\perp^2}{qB/m} = 2\pi \frac{m}{q} \mu$

Kulsrud (1958) for all order w.r.t $\epsilon \sim \frac{1}{\omega^2}\frac{d\omega}{dt}$

$$\mu \sim \sum_{n=0}^{\infty} \epsilon^n \mu_n = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots \quad (\epsilon \to 0) \quad \mu_0 = \frac{\frac{1}{2}mv_\perp^2}{B}$$

Asymptotic → $\mu - \sum_{n=0}^{\infty} \epsilon^n \mu_n \sim \mathcal{O}(e^{-p/\epsilon})$ $(p>0)$

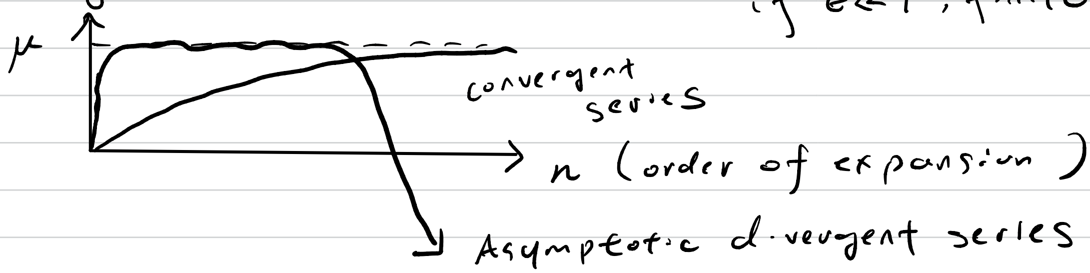
if, $\vec{B}, \vec{E}$ begins with const → ends with other consts.



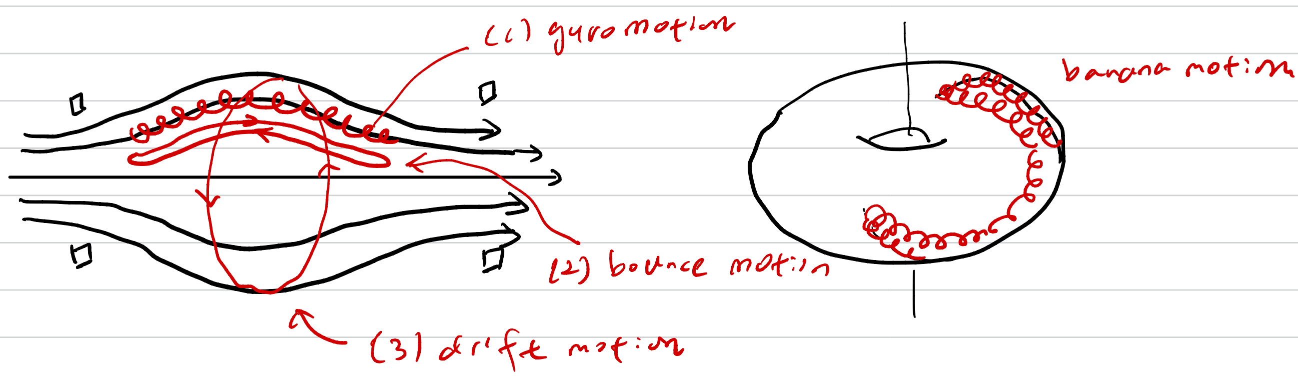
$$\mu \sim \left(\sum_{n=0}^{\infty} \epsilon^n \mu_n = \mu_0 \right)$$

Even if not, likely → $\left(\mu - \sum_{n=0}^{\infty} \epsilon^n \mu_n\right) \sim (\mu - \mu_0) \sim e^{-p/\epsilon}$

if $\epsilon \ll 1$, finite.



★ Three adiabatic invariants in mirror / tokamak / earth



	Ion, 1keV, 1T	Invariant	Validity (time scale of interest)
(1) gyromotion	$\omega_c \sim 10^8/s$	$\mu = \frac{1}{2}mv_\perp^2/B$	$\omega_c \gg \omega$
(2) bounce motion	$\omega_b \sim 10^6/s$	$J = \oint v_\parallel ds$	$\omega_c \gg \omega_b \gg \omega$
// $E\times B$ motion			
(3) drift motion	$\omega_B \sim 10^4/s$	$\Phi = \oint \psi\, d\theta$	$\omega_b \gg \omega_B \gg \omega$

4. comments on energy conservation (including drift motion)

$$\frac{dW}{dt} = q\vec{V}_{gc}\cdot\vec{E} + \mu\frac{\partial \vec{B}}{\partial t} + O(\epsilon^2)$$

$$\frac{d}{dt}\left(\frac{1}{2}mv_{\parallel}^2\right) = \vec{V}_{gc}\cdot(q\vec{E} - \mu\vec{\nabla}B)$$

$$= -\vec{v}_{gc}\cdot\left(q\vec{\nabla}\phi + \mu\vec{\nabla}B + q\frac{\partial\vec{A}}{\partial t}\right)$$

- important for drift-kinetic equation.
- $\vec{v}_{gc} = V_{\parallel 0}\hat{b} + \frac{\vec{E}\times\vec{B}}{B^2} + \vec{v}_1$ (all first-order drift)
- can be proved from 3/22 Lecture