

• Operation counts for obtaining solution of $Ax = b$

$Ax = b$

A : full matrix

$n \times n$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Diagram annotations: A blue box highlights the first column $a_{11}, a_{21}, \dots, a_{n1}$. A red box highlights the second column $a_{22}, a_{32}, \dots, a_{n2}$. A blue line connects a_{21} to a_{22} , and another blue line connects a_{21} to a_{nn} . The fraction $\frac{a_{21}}{a_{11}}$ is written to the left of the matrix. The label 'GE' is written vertically on the left side.

To eliminate a_{21} : $1D, nM, nA$

To " the first col.: $(n-1)D$, $n(n-1)M$, $n(n-1)A$

for second col.: $(n-2)D$, $(n-1)(n-2)M$, $(n-1)(n-2)A$

for $n-1^{\text{th}}$ col. : $\underbrace{1D}$, $\underbrace{2M}$, $2A$

Total divisions : $\sum_{k=1}^{n-1} k = \frac{1}{2}n(n-1)$

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

" multiplications : $\sum_{k=1}^{n-1} k(k+1) = \frac{1}{3}(n^3 - n)$

$\Rightarrow \frac{1}{2}n(n-1)D, \frac{1}{3}(n^3 - n)M, \frac{1}{3}(n^3 - n)A$

\Rightarrow Gauss elimination requires $\Theta(n^3/3)$ $\sim \Theta(n^3)$

Backward sweep

$$\begin{cases} x_n = b_n / a_{nn} \\ x_j = (b_j - \sum_{k=j+1}^n a_{jk} x_k) / a_{jj} \\ j = n-1, n-2, \dots, 1 \end{cases}$$

for each j , $1D, (n-j)M, (n-j)A$

total $\sum_{j=1}^{n-1} (n-j) = \frac{1}{2} n(n-1) M, \frac{1}{2} n(n-1) A$

total divisions: $n \div$

$\hookrightarrow O(n^2/2)$

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negligible as compared to GE requiring $O(n^3/3)$ for $n \gg 1$.

• Gauss - Jordan elimination \rightarrow exactly same operation counts.

* Tri-diagonal matrix

$$\begin{pmatrix} b_1 & c_1 & 0 & \dots & 0 \\ a_2 & b_2 & c_2 & \dots & 0 \\ & a_3 & b_3 & c_3 & 0 \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

To eliminate $a_2, 1 \div, 2M, 2A$
 To make U, $(n-1) \div, 2(n-1)M,$

$GE \rightarrow O(n)$

$2(n-1)A.$



$$\begin{pmatrix} b_1 & c_1 & & 0 \\ 0 & b_2' & c_2' & \\ & & \ddots & c_{n-1}' \\ 0 & & -b_{n-1}' & b_n' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2' \\ \vdots \\ f_n' \end{pmatrix} \quad \text{backward sweep}$$

$$x_n = f_n' / b_n'$$

$$x_j = (f_j - c_j' x_{j+1}) / b_j', \quad j = n-1, \dots, 1$$

$$\rightarrow n \text{ } \oplus, (n-1) \text{ } M, (n-1) \text{ } A \rightarrow \Theta(n)$$

$$\text{Total operations: } (2n-1) \oplus, 3(n-1) M, 3(n-1) A$$

$\Rightarrow \Theta(n)$ operations for tri-diagonal matrix system

① Computation of inverse matrix

$$AB = I \quad B = [\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n]$$

$$AB = [A\underline{b}_1, A\underline{b}_2, \dots, A\underline{b}_n] = I$$

$$\rightarrow Ab_1 = e_1, Ab_2 = e_2, \dots, Ab_j = e_j, \dots$$

$$\text{Augmented matrix } [A \ e_1 \ e_2 \ \dots \ e_n] = [A \ I]$$

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_j$$

perform Gauss-Jordan on this augmented matrix

$$\text{ex) } A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, A^{-1} = ?$$

tri-diagonal matrix

$$\left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{array} \right)$$

I
↑

$\rightarrow \dots \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{3}{5} \end{pmatrix} = A^{-1}$: full matrix
 inverse of a banded matrix is a full matrix.

Operation counts for A^{-1} : n^3 of each M & A
 (full matrix A) \sim similar to that of AB

If A^{-1} is not needed, it should not be computed.

• LU decomposition

$$Ax = b \xrightarrow{GE} Ux = c$$

$$Ax = b \rightarrow x = A^{-1}b$$

$$\downarrow$$

$$LUx = b \rightarrow Lz = b \quad \mathcal{O}(n^2)$$

$$\underbrace{\quad}_z \quad Ux = z \quad \mathcal{O}(n^2)$$

Operation of multiplication of one row by a constant and subtract from another row can be performed by a matrix multiplication.

e.g. multiply 1st row by e_{21} ($= -a_{21}/a_{11}$) and add to 2nd row,

$$E_{21} = \begin{pmatrix} 1 & & & \\ e_{21} & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \rightarrow E_{21}a_1 = \begin{pmatrix} a_{11} \\ 0 \\ a_{31} \\ \vdots \\ a_{n1} \end{pmatrix}$$

$$E_{31} = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ e_{31} & & & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \quad e_{31} = -a_{31}/a_{11}$$

$$E_{31} E_{21} = \begin{pmatrix} 1 & & & \\ e_{21} & 1 & & \\ e_{31} & & & \\ 0 & & \ddots & \\ 0 & & & 1 \end{pmatrix} \quad E_{31} E_{21} a_1 = \begin{pmatrix} a_{11} \\ 0 \\ 0 \\ a_{41} \\ \vdots \\ a_{n1} \end{pmatrix}$$

To eliminate the 1st col.,

$$E_1 = \begin{pmatrix} 1 & & & \\ e_{21} & 1 & & \\ e_{31} & & & \\ \vdots & & \ddots & \\ e_{n1} & & & 1 \end{pmatrix} \quad e_{j1} = -a_{j1}/a_{11} \quad E_1 A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a'_{22} & \dots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & \dots & a'_{nn} \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 1 & & & \\ e_{21} & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \rightarrow E_{21}^{-1} = \begin{pmatrix} 1 & & & \\ -e_{21} & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \quad E_{21}^{-1} E_{21} A = A$$

multiply 1st row by $-e_{21}$ and add to 2nd row.

$$\rightarrow E_1^{-1} = \begin{pmatrix} 1 & & & \\ -e_{21} & 1 & & \\ -e_{31} & & \ddots & \\ \vdots & & & \\ -e_{n1} & & & 1 \end{pmatrix}, \quad E_2^{-1} = \dots, \quad E_3^{-1} = \dots$$

lower triangular matrix
w/ 1's on the main
diagonal.

$$EA = U \rightarrow E_{n-1} E_{n-2} \dots E_1 A = U$$

$$\rightarrow A = \underbrace{E_1^{-1} E_2^{-1} \dots E_{n-2}^{-1} E_{n-1}^{-1}}_{L} U = LU \rightarrow O(n^3)$$

$Ux = z$ backward sweep $\mathcal{O}(n^2)$ to get x .

LU decomposition $\rightarrow \mathcal{O}(n^3)$ expensive!

$Ax = b \rightarrow x = A^{-1}b$
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 $LUx = b$) direct solution.

what if $a_{11} = 0$?

$$\begin{pmatrix} 0 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow$$

$$\Rightarrow Ax = b$$

$$\Rightarrow PAx = b' \rightarrow LUx = b'$$

row exchange

\downarrow
permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow PA = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 2 & -2 & 1 \end{pmatrix}$$