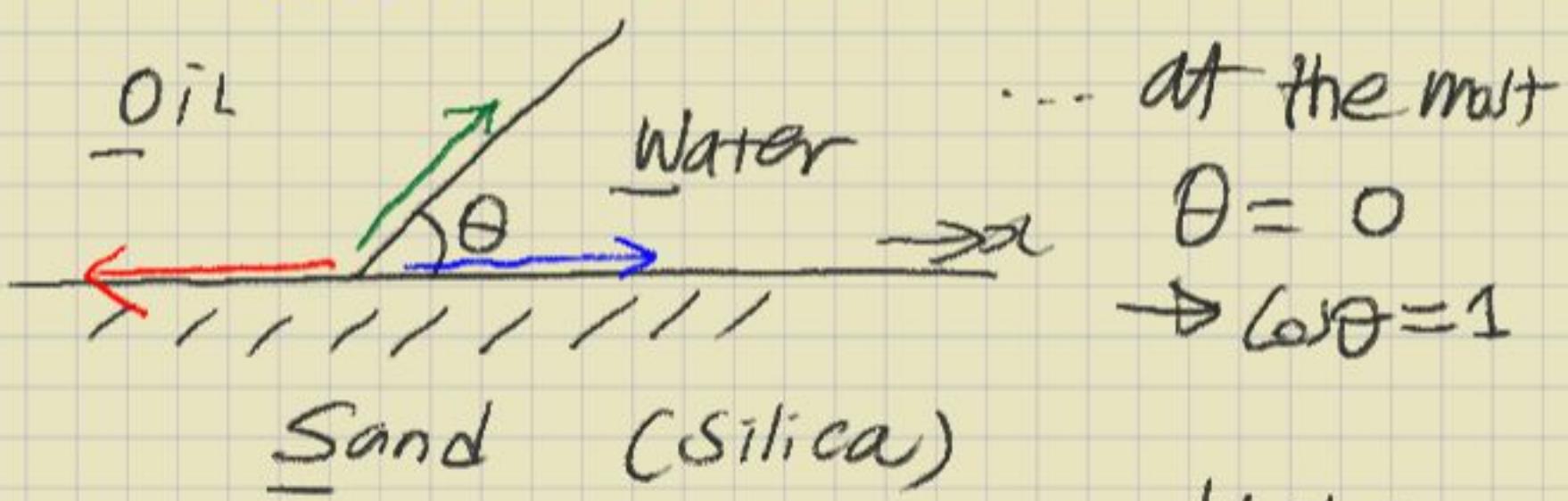


Lecture 4. Gas/liquid interface free surface

Contact angles, wettability.

- Young's equation \rightarrow force balance in st-dir

$$\sigma_{so} = \sigma_{sw} + \sigma_{ow} \cos\theta$$



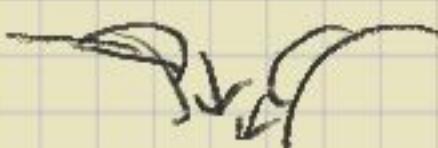
\rightarrow this is force balance

Water
spreads

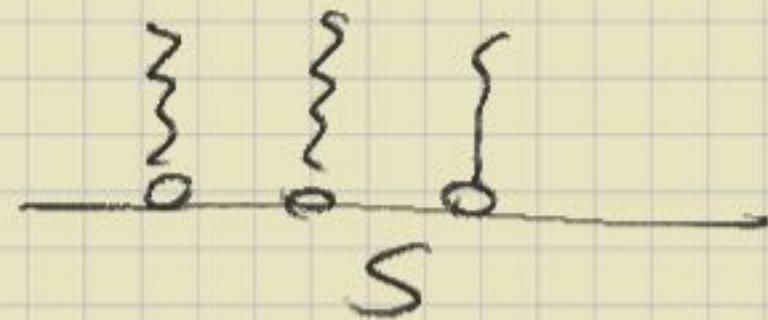
- If $\sigma_{so} > \sigma_{sw} + \sigma_{ow}$ spontaneously

e.g. on clean glass or sand

- Wetting fluid occupies small pores.



- Adsorption of surfactants on a solid can cause it to be oil wet

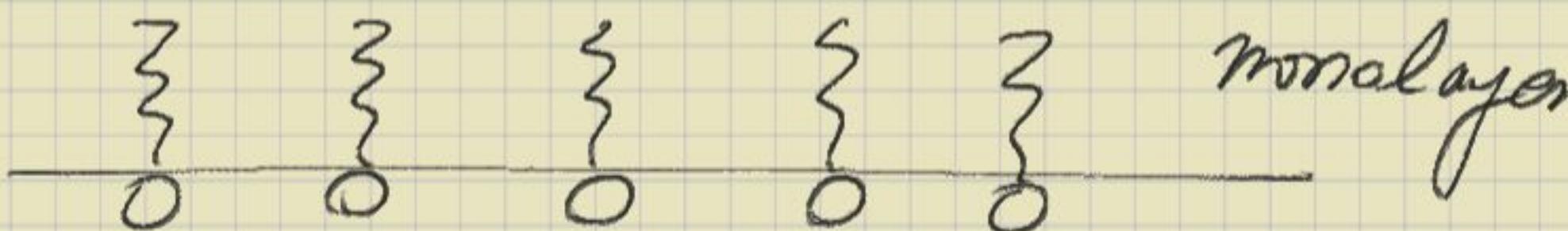


Formation can be

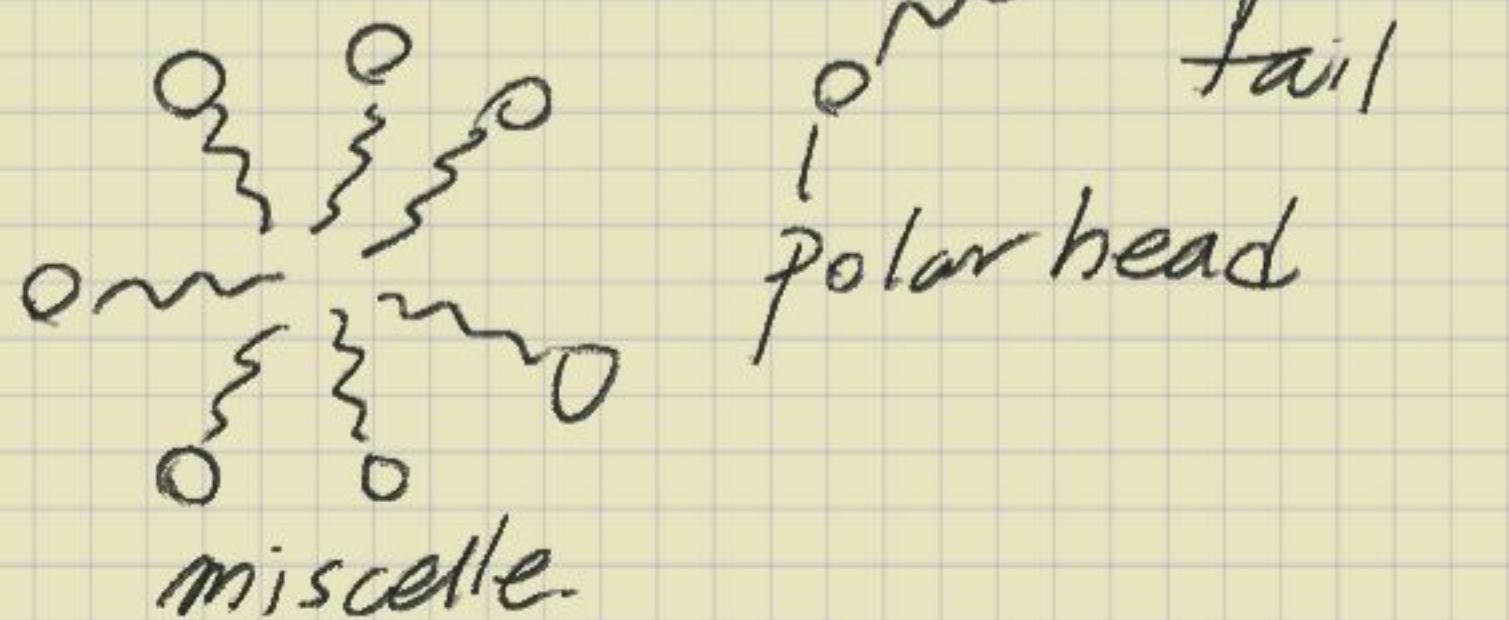
water-wet, oil-wet, or mixed.

Idealized diagram showing surfactant behavior

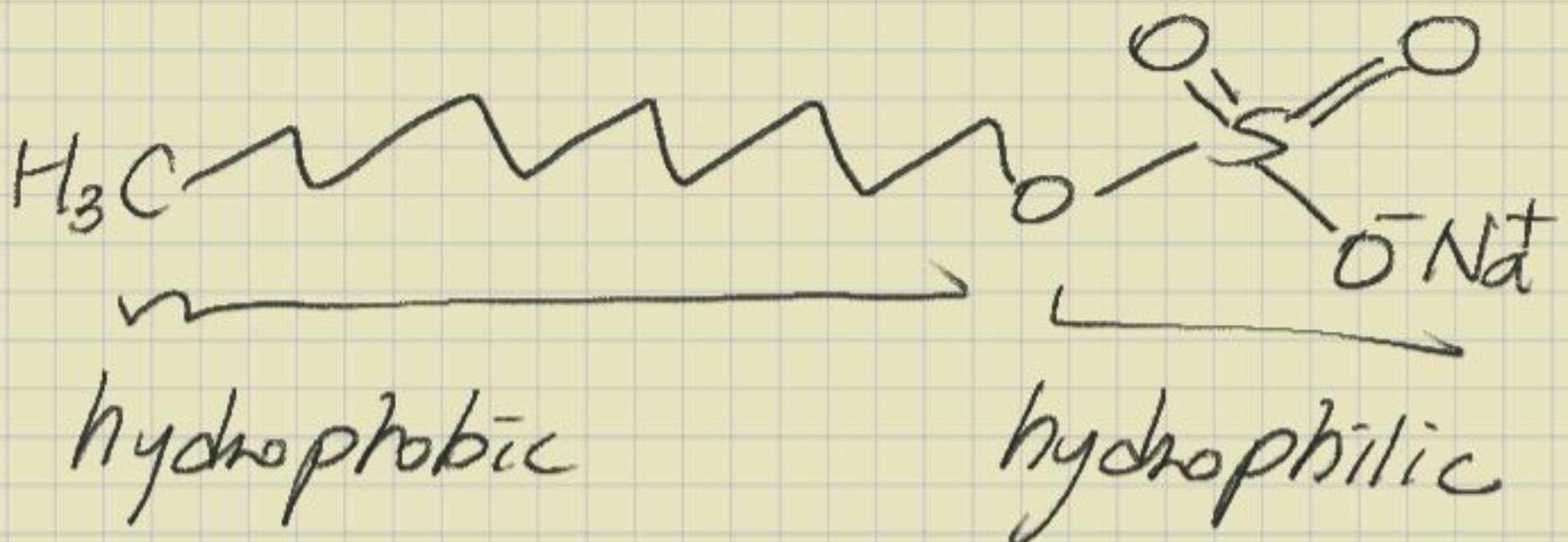
oil or air



water



Example) SDS (Sodium dodecyl sulfate)

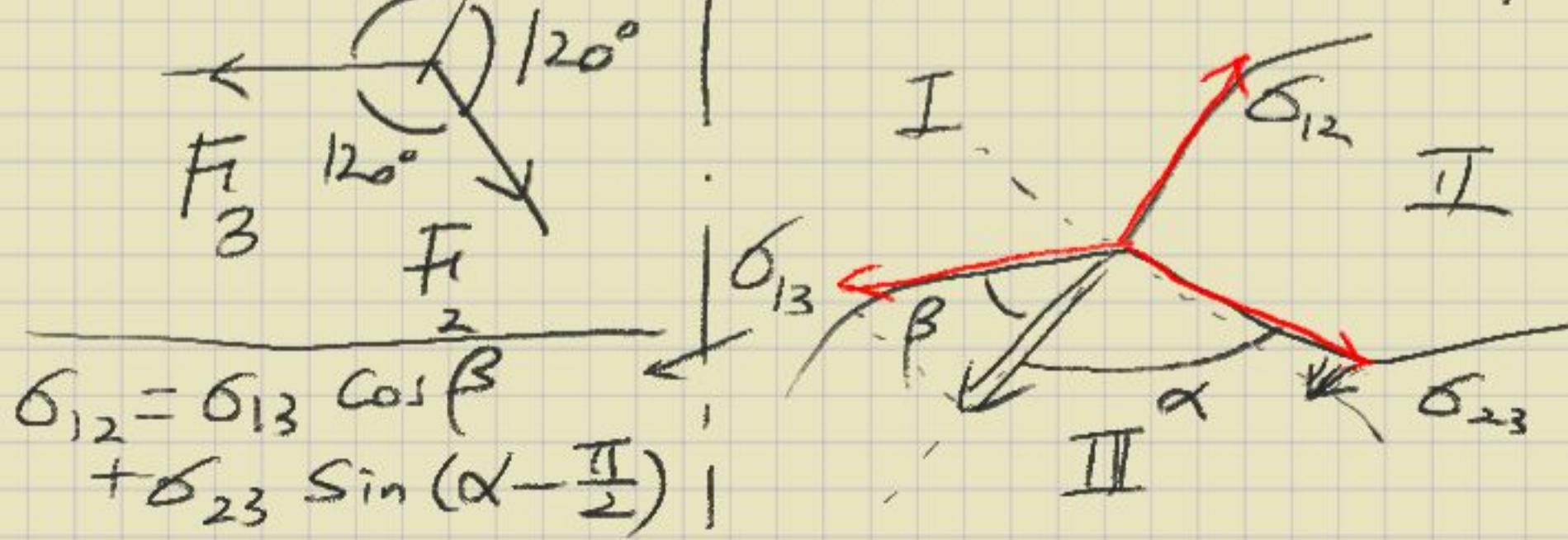


also protein (egg yolk)

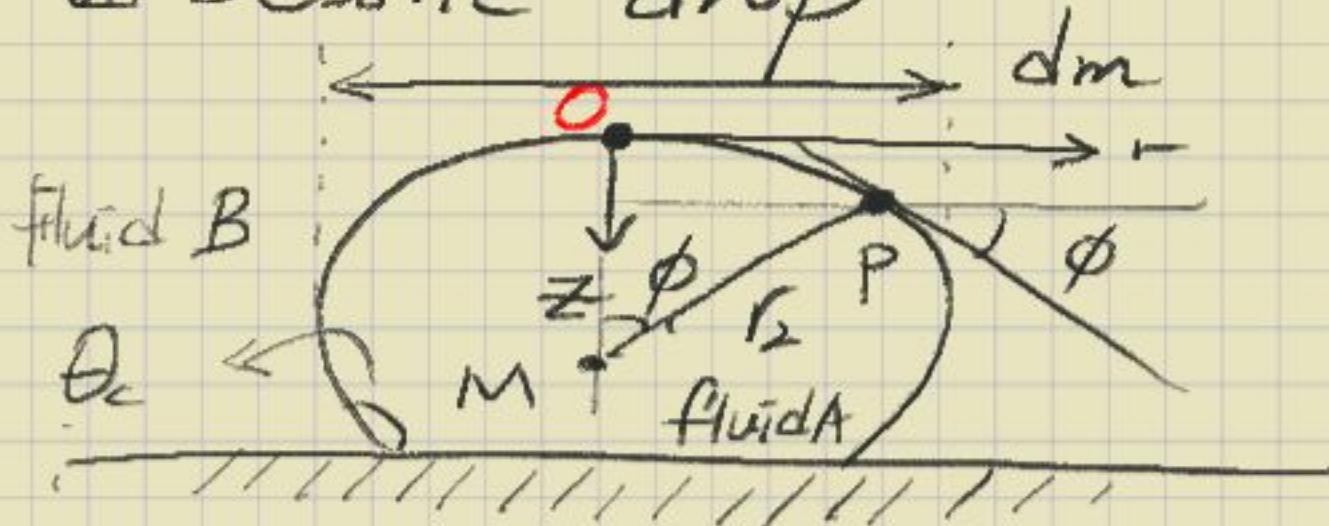
block copolymer

④ Soap bubble film movie

| What about if
| there were three distinct phase?



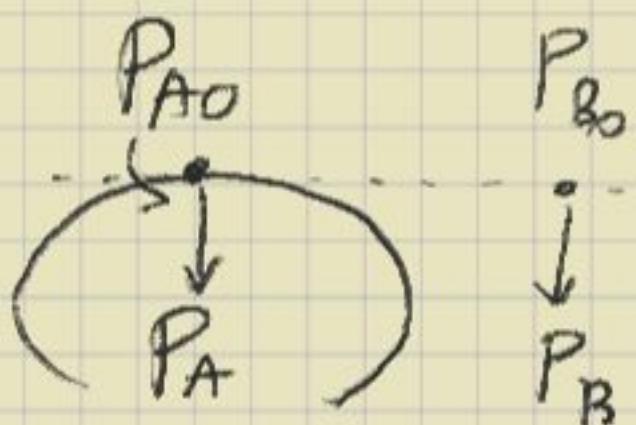
□ Sessile drop



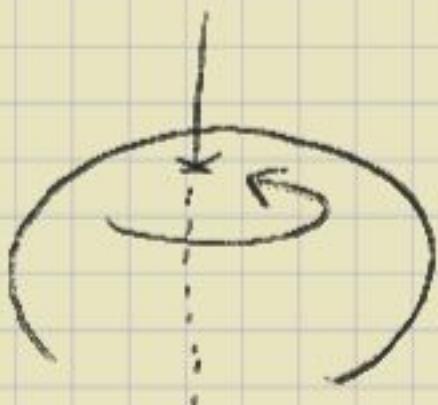
Contact angle

From the equation of hydro statics

$$\textcircled{+} \quad \begin{aligned} P_A &= P_{A0} + \rho_A g z \\ P_B &= P_{B0} + \rho_B g z \end{aligned}$$



If the drop is axisymmetric,



the radius of curvature r at O

is the same for all orientation.

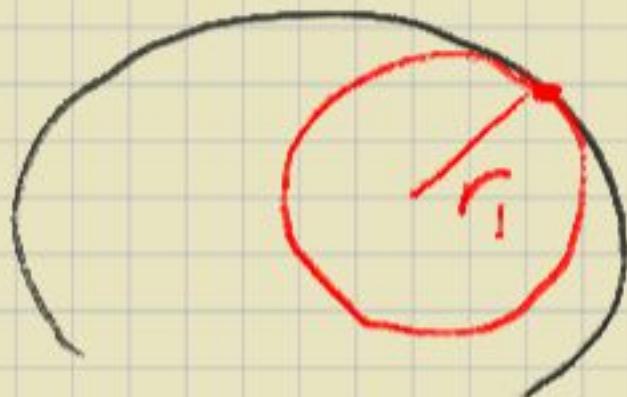
$$\textcircled{D} \quad -P_A - P_{B0} = \frac{2\sigma}{r} \quad (\text{Note that } \sigma = \gamma)$$

Combining ④ & ⑤:

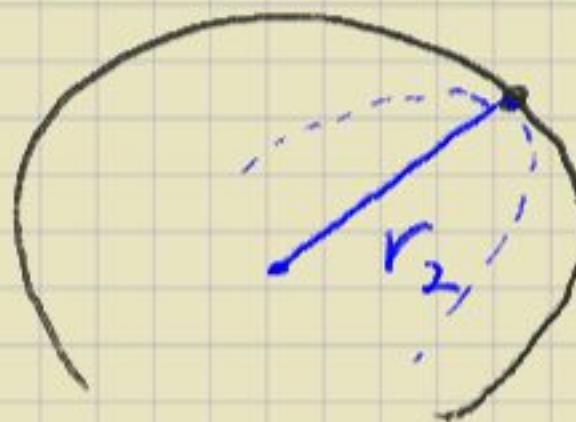
$$\sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \frac{2\sigma}{b} + (\rho_A - \rho_B)gz$$

... 

Note that)



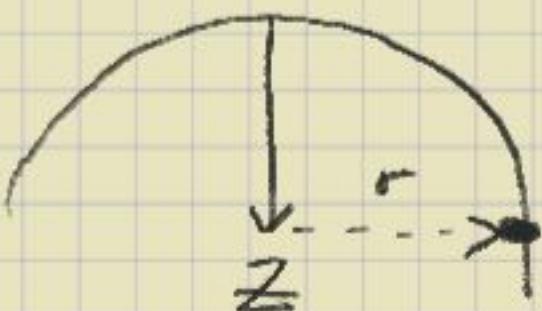
in plane



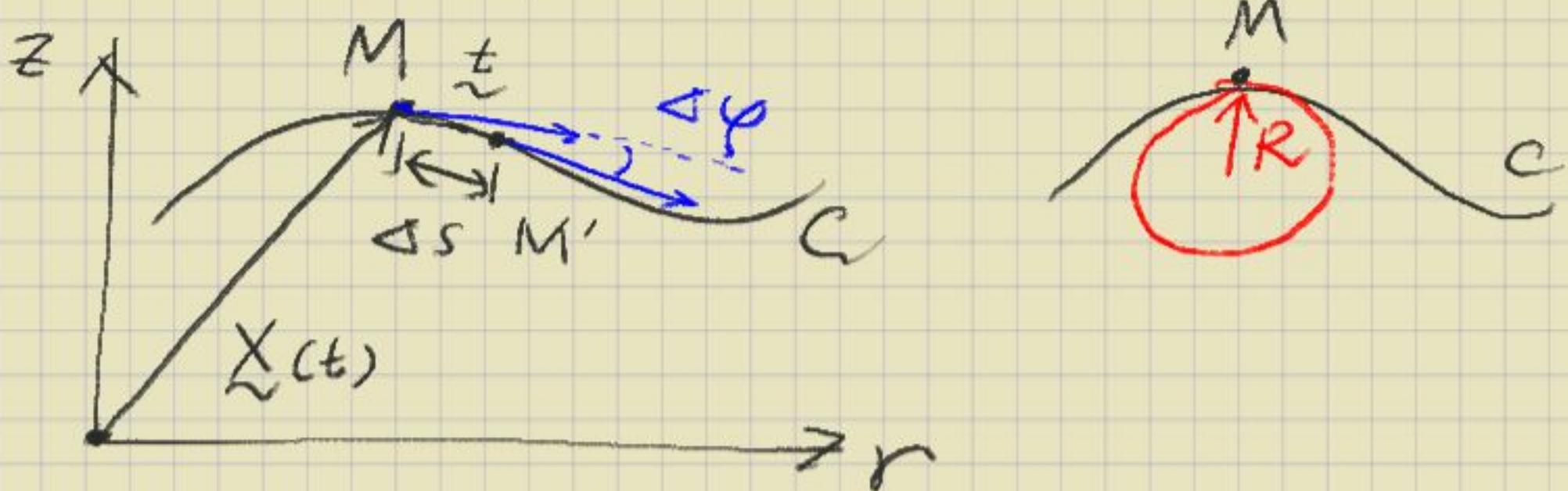
out-of-plane

Now we can derive

$$\frac{1}{r_i} = \frac{dz/dr^2}{[1 + (dz/dr)^2]^{3/2}}$$



Curvature of plane curves



Curvature of the curve

$$= \lim_{\Delta s \rightarrow 0} \frac{\Delta\varphi}{\Delta s} = \frac{d\varphi}{ds} = K$$

\$\therefore\$ avg. curvature of curve.

While $\tilde{x}(t) = i \tilde{x}(t) + j \tilde{y}(t) + k \tilde{z}(t)$
 \$\hookrightarrow\$ curve parameter \$t\$

arc length

$$s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

$\tilde{t} = \frac{dx}{ds}$ tangent vector

\$\rightarrow\$ check curvature.pdf

Using tangent vector:

$$\kappa = \pm \left| \frac{d\tilde{t}}{ds} \right| = \pm \left| \frac{d^2\tilde{x}}{ds^2} \right|$$

From Frenet formula

$$\kappa = \frac{\frac{dx}{dt} \times \frac{d^2x}{dt^2}}{\left| \frac{dx}{dt} \right|^3}$$

$$= \frac{\frac{dr}{dt} \frac{dz}{dt} - \frac{dz}{dt} \frac{dr}{dt^2}}{\left[\left(\frac{dr}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right]^{3/2}}$$

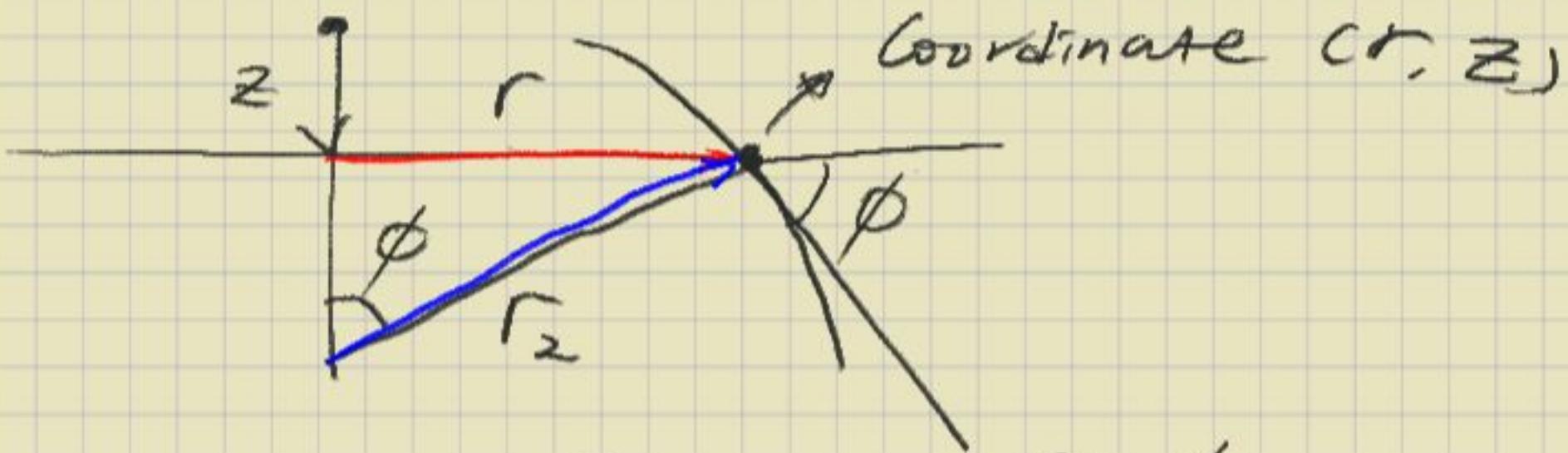
If $z = f(r)$ (explicit function)

$$\kappa = \frac{\frac{d^2z}{dr^2}}{\left[1 + \left(\frac{dz}{dr} \right)^2 \right]^{3/2}}$$

Back to sessile drop

$$\frac{1}{r} = K = \frac{\frac{d^2z}{dr^2}}{\left[1 + \left(\frac{dz}{dr}\right)^2\right]^{3/2}} \quad \square \quad [1]$$

But how to obtain r_2 ?



$$\sin \phi = \frac{r}{r_2} = \frac{\sin \phi}{(\sin^2 \phi + \cos^2 \phi)^{1/2}}$$

$$= \frac{\sin \phi / \cos \phi}{\left[1 + (\sin \phi / \cos \phi)^2\right]^{1/2}}$$

$$= \frac{\tan \phi}{(1 + \tan^2 \phi)^{1/2}}$$

therefore

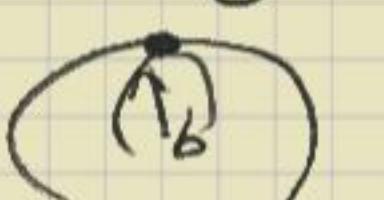
$$\frac{1}{r_2} = \frac{\frac{dz}{dr}}{r [1 + (\frac{dz}{dr})^2]^{1/2}} \quad \dots \boxed{2}$$

If we introduce dimensionless

Variable

$$Z = \frac{z}{b}$$

$$R = \frac{r}{b}$$

radius of curvature at O


Plugging $\boxed{1}$ & $\boxed{2}$ into Δ

$$\dots \left(= \frac{dZ}{dR} \right)$$

$$\frac{Z''}{[1 + (Z')^2]^{3/2}} + \frac{Z'}{R[1 + (Z')^2]^{1/2}} = 2 + \beta Z \quad \dots \boxed{3}$$

where

$$\beta = (\rho_A - \rho_B) \frac{gb^2}{\sigma}$$

two BCs:

$$Z = 0 \quad \& \quad Z' = 0 \quad \text{at } R = 0$$

Eq ① cannot be solved analytically except for certain limiting case.

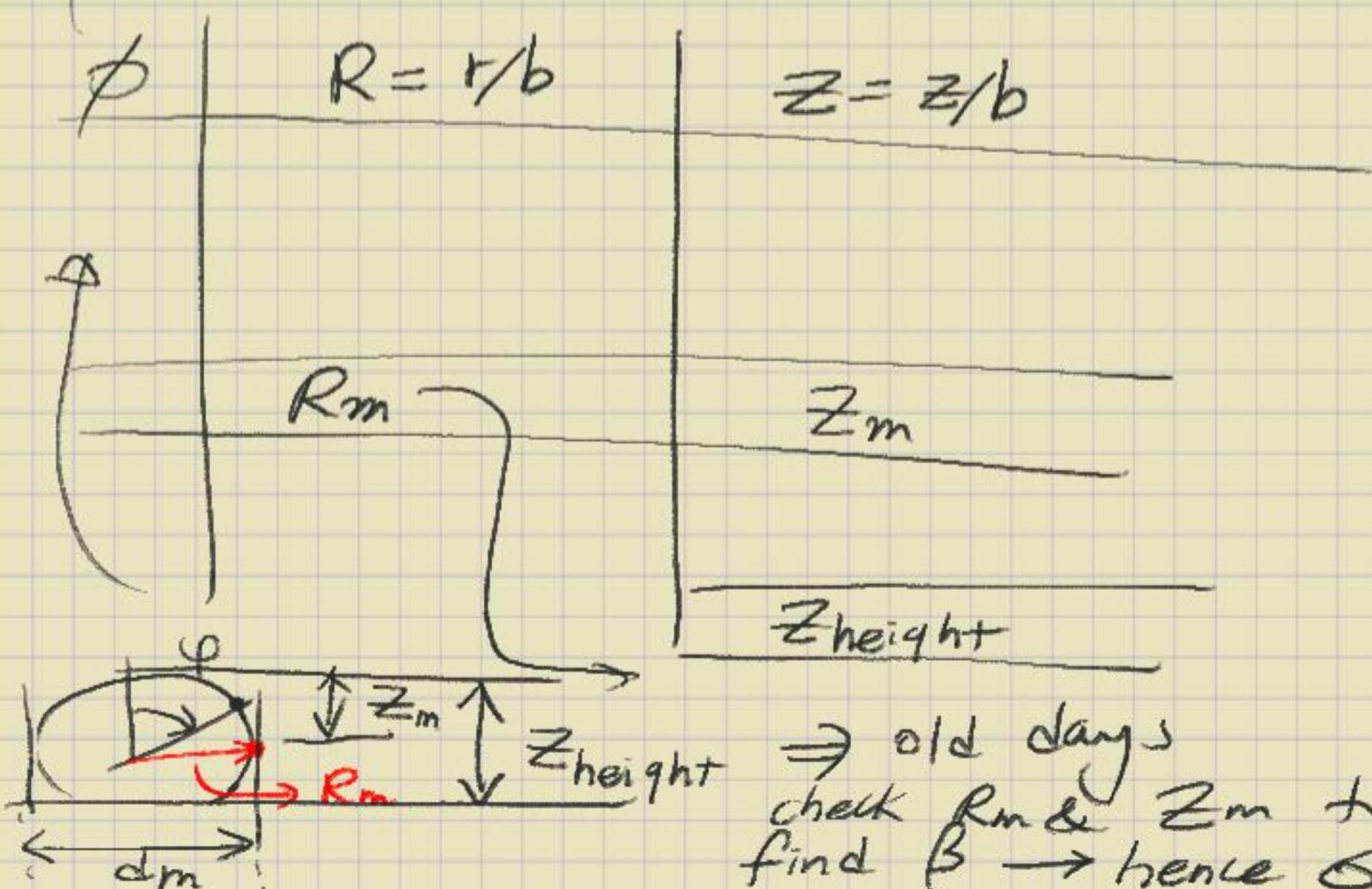
→ Numerical method is required.

Usually results are reported

$Z(R)$ for any specified β
(ρ, σ, g, b)

c.f) Bashforth & Adams (1893)

$$\beta = 20$$



In these days.

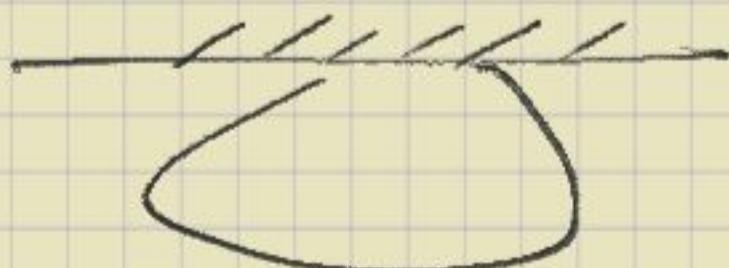
all workers use a video camera & image analysis techniques

→ Compare image vs numerical solution
to find δ value.

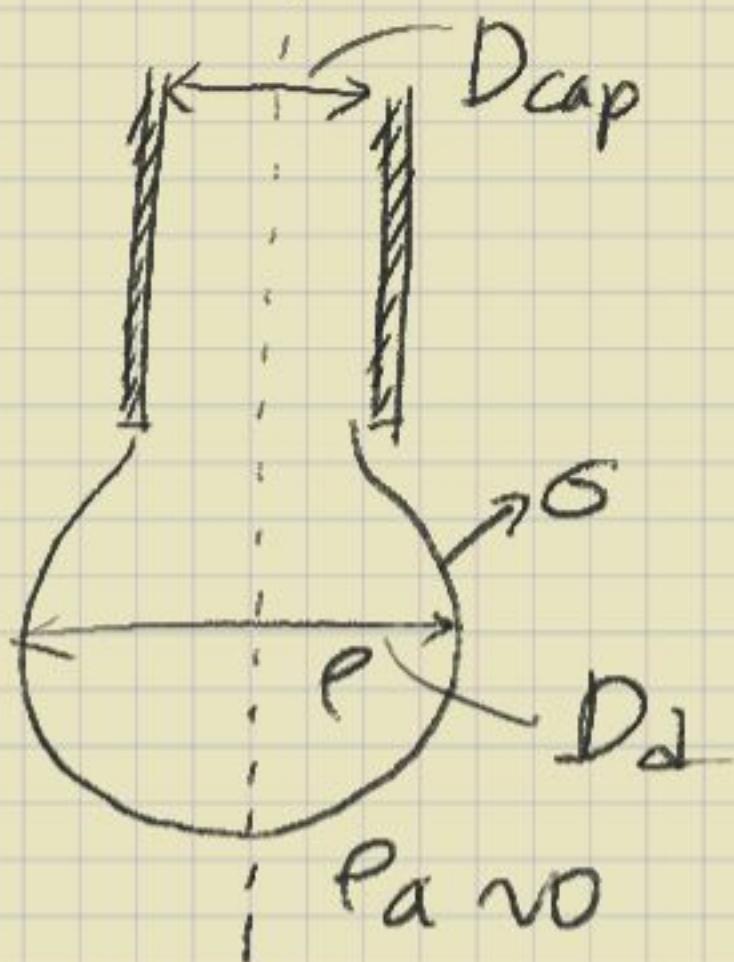
Is $\beta < 0$ possible? Yes

① $\Delta \rho < 0 \rightarrow$ floating bubble

② $g < 0 \rightarrow$ "hanging" drop



Dispensing drop from capillary tube



(View Trefethen's movies)

Alcohol comes out as smaller drops than water...

how much small?

⇒ How does the dispensed drop size D_d scale with respect to properties of the fluid?

$$\left(\text{drop volume} : \frac{\pi D_d^3}{6} \right)$$

I. Relevant properties of fluid

ρ, σ, μ no Neglect here
(quasi static case)

II Other parameters affects D_d

γ, D_{cap}

→ So we seek a relationship

between $D_d, \gamma, \sigma, g, D_{cap}$

$$\Rightarrow D_d = f_n(\rho, \sigma, g, D_{cap}) \dots \textcircled{A}$$

In general,

$$D = f_n(D_d, \rho, \sigma, g, D_{cap})$$

↳ can be

$$\sigma = f_n(D_d, \rho, g, D_{cap}) \Rightarrow \begin{matrix} \text{measuring} \\ \text{surface} \\ \text{tension.} \end{matrix}$$

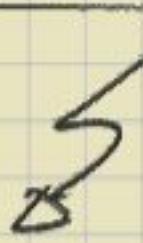
We will do experiments.

How to minimize # of exp?

• Assumptions)

∅ Such f_n exist

∅ it cannot depend on units



Introducing dimensionless number
 N_i

$N_1 = f_n$ of N_2, N_3, \dots

$$N_i = D_d^{m_1} D_{cap}^{m_2} \sigma^{m_3} g^{m_4} \rho^{m_5}$$

↓ where $m_i >, =, < 0$

Dimensionless

| D_d | ρ | D_{cap} | σ | g | |
|----------|-----------------|-----------|-----------------|-----------------|------------------------------|
| unit L | $\frac{M}{L^3}$ | L | $\frac{M}{T^2}$ | $\frac{L}{T^2}$ | Buckingham (π -theorem) |

5 variables] \Rightarrow $5 - 3 = 2$
3 units dimensionless group

• Since we want D_d ,
most convenient to isolate into
one group.

Also 'g' is an important in this
system, (cause drop to fall)

Pick combination of three variables
that include all three units e.g.

D_{cap} , ρ , g

• Get two dimensionless group.

D_p with D_{cap}, σ, ρ

g with D_{cap}, σ, ρ

$$\textcircled{1} N_1 = D_d D_{cap}^a \rho^b \sigma^c$$

\rightarrow all power of dimension
must add to 0.

$$T: 0 = -2c \quad \boxed{c=0}$$

$$M: 0 = b + c \quad \boxed{b=0}$$

$$L: 0 = 1 + a - 3b \quad \boxed{a=-1}$$

$$\therefore N_1 = \frac{D_d}{D_{cap}} \quad \dots \quad \begin{matrix} \text{dimensionless} \\ \text{diameter} \end{matrix}$$

$$\textcircled{2} N_2 = g D_{cap}^d \rho e \sigma f$$

$$T: 0 = -2 - 2f \quad \boxed{f=-1}$$

$$M: 0 = e + f \quad \boxed{e=1}$$

$$L: 0 = 1 + d - 3e \quad \boxed{d=2}$$

$$\therefore N_2 = \frac{\rho g D_{cap}^2}{\sigma} \quad \begin{matrix} \text{usually} \\ \text{called} \end{matrix}$$

$= B_o$

Bond number.

- Weight of drop - ①

$$\frac{\rho g \pi D_d^3}{6} \quad (\text{assume sphere})$$

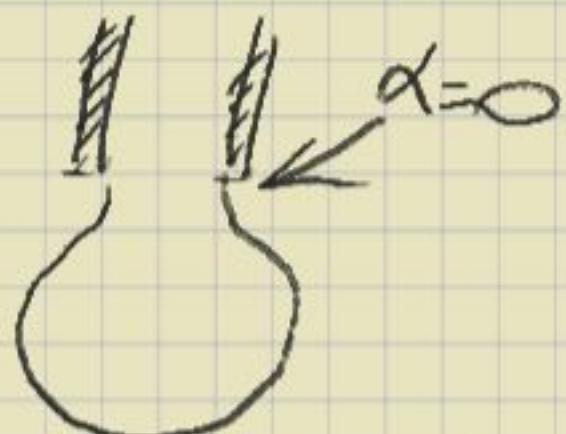


- Surface tension force - ②

$$\sigma \pi D_{cap} \cos \alpha$$

just before fall

$$\alpha \rightarrow 0 \Rightarrow \cos \alpha = 1$$



① & ② are balanced

just before fall.

$$\frac{\rho g \pi D_d^3}{6} = \sigma \pi D_{cap}$$

$$\Rightarrow \left(\frac{D_d}{D_{cap}} \right)^3 = \frac{6\sigma}{\rho g D_{cap}^2}$$

$$\Rightarrow \frac{D_d}{D_{cap}} = \left(\frac{6}{B_0} \right)^{1/3}$$

But in reality

drop is not sphere,



we need a correction factor

$$\psi = f(B_o).$$

$$\frac{D_d}{D_{Co}} = \left(\frac{6\varphi}{B_o} \right)^{1/3}$$

Usually numerical computations

are performed to produce

data for $\psi = f(B_o)$

→ usually produced as
a table.