Dynamics (동역학)

Lecture 4: System of Particles

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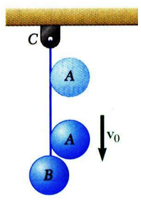
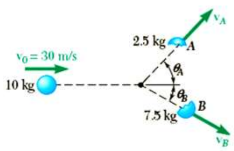
Introduction

- In the current chapter, we will study the motion of *systems of particles under systems of forces*.
- The <u>*effective force*</u> of a particle is the product of its mass and acceleration. It will be shown that the *system of external forces* acting on a system of particles is *equipollent* with the *system of effective forces* of the system (internal forces gone).

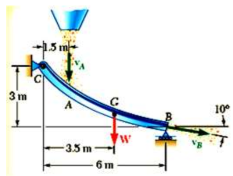
$$\boxed{\sum \vec{F}_j^i - \sum m_i \vec{a}_i = 0, \quad -m_i \vec{a}_i \equiv inertial\ foce\ vector}$$



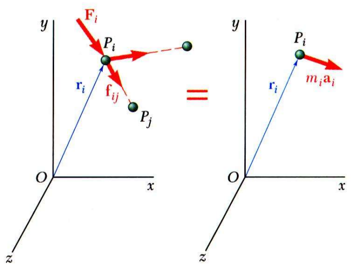
- The *mass center* of a system of particles will be defined and its motion described.



- Application of the *work-energy principle* and the *impulse-momentum principle* to a system of particles will be described. Result obtained are also applicable to a system of rigidly connected particles, i.e., *rigid body*.
- Analysis methods will be presented for *variable systems of particles*, i.e., systems in which the particles included in the system change.



System Dynamics and Equivalence



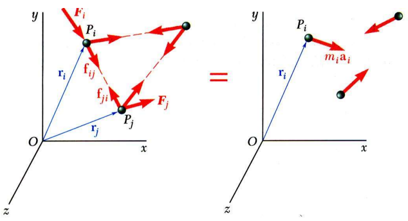
- Newton's second law for each particle P_i in a system of n-particles,

$$\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i \vec{a}_i$$

$$\vec{r}_i \times \vec{F}_i + \sum_{j=1}^{n} \left(\vec{r}_i \times \vec{f}_{ij}\right) = \vec{r}_i \times m_i \vec{a}_i$$

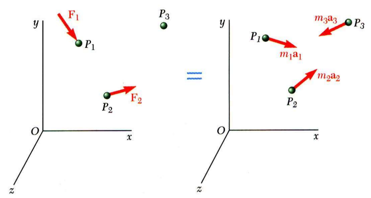
$\vec{F}_i$ = external force $\quad \vec{f}_{ij}$ = internal forces

$m_i \vec{a}_i$ = effective force



- The system of external and internal forces on a particle is *equivalent* to the effective force of the particle.
- The system of external and internal forces acting on the entire system of particles is *equivalent* to the system of effective forces.
- Equivalent system of external/internal forces $\{f_{ij}, F_i\}$ completely specifies both overall motion & internal deformation of the system of effective forces $\{m_i \vec{a}_i\}$

System Dynamics and Equipollence



- Summing over all the particles,

$$\sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{f}_{ij} = \sum_{i=1}^{n} m_i \vec{a}_i$$

$$\sum_{i=1}^{n} \left(\vec{r}_i \times \vec{F}_i\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\vec{r}_i \times \vec{f}_{ij}\right) = \sum_{i=1}^{n} \left(\vec{r}_i \times m_i \vec{a}_i\right)$$

- Since the internal forces occur in equal and opposite collinear pairs, their resultant force and moment are all zero, i.e., from $\vec{f}_{pq} = -\vec{f}_{qp}$ and $\vec{r}_p - \vec{r}_q \,//\, \vec{f}_{pq}$

$$\sum_{i=1,j=1} \vec{f}_{ij} = \dots + \cancel{\vec{f}_{pq}} + \dots + \cancel{\vec{f}_{qp}} + \cdots$$

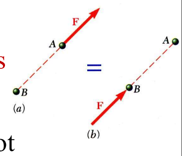
$$\sum_{i=1} \vec{r}_i \times \vec{f}_{ij} = \cdots + \vec{r}_p \times \vec{f}_{pq} + \cdots \vec{r}_q \times \vec{f}_{qp} + \cdots$$

$$= \cdots + \cancel{(\vec{r}_p - \vec{r}_q)} \times \vec{f}_{pq} + \dots$$

$$\sum \vec{F}_i = \sum m_i \vec{a}_i$$

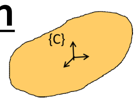
$$\sum \left(\vec{r}_i \times \vec{F}_i\right) = \sum \left(\vec{r}_i \times m_i \vec{a}_i\right)$$

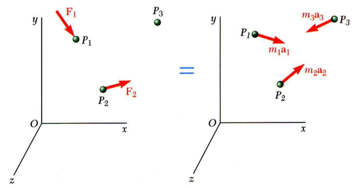
- The system of external forces $\{F_i\}$ and the system of effective forces $\{m_i \vec{a}_i\}$ are *equipollent (same for the resultant)* but not *equivalent*.



- For rigid-body, equivalence = equipollence; for deformable body, not (two equipollent force systems can produce different deformation).

System Linear & Angular Momentum





$$\sum \vec{F}_i = \sum m_i \vec{a}_i$$
$$\sum \left(\vec{r}_i \times \vec{F}_i\right) = \sum \left(\vec{r}_i \times m_i \vec{a}_i\right)$$

- Total linear momentum of the system of particles,

$$\vec{L} = \sum_{i=1}^{n} m_i \vec{v}_i$$
$$\dot{\vec{L}} = \sum_{i=1}^{n} m_i \dot{\vec{v}}_i = \sum_{i=1}^{n} m_i \vec{a}_i$$

- Resultant of the external forces is equal to rate of change of linear momentum of the system of particles,

$$\boxed{\sum \vec{F} = \dot{\vec{L}}}$$

- Total angular momentum about a fixed point O of system of particles,

$$\vec{H}_O = \sum_{i=1}^{n} \left(\vec{r}_i \times m_i \vec{v}_i\right)$$
$$\dot{\vec{H}}_O = \sum_{i=1}^{n} \left(\dot{\vec{r}}_i \times m_i \vec{v}_i\right) + \sum_{i=1}^{n} \left(\vec{r}_i \times m_i \dot{\vec{v}}_i\right)$$
$$= \sum_{i=1}^{n} \left(\vec{r}_i \times m_i \vec{a}_i\right)$$

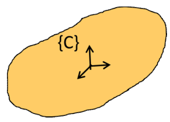
- Moment resultant about a fixed point O of the external forces is equal to the rate of change of angular momentum of the system of particles,

$$\boxed{\sum \vec{M}_O = \dot{\vec{H}}_O}$$

Linear Momentum of Mass Center

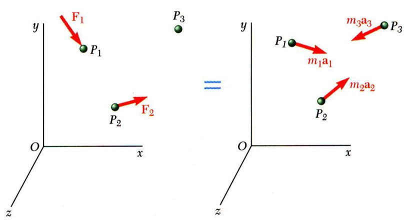
- Mass center G of the system of particles is defined by position vector $\vec{r}_G$ which satisfies

$$m\vec{r}_g = \sum_{i=1}^{n} m_i \vec{r}_i, \quad m = m_1 + m_2 + \ldots m_n$$



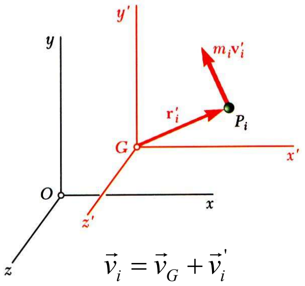
- Differentiating twice in a Newtonian frame,

$$m\dot{\vec{r}}_G = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i$$
$$m\vec{v}_G = \sum_{i=1}^{n} m_i \vec{v}_i = \vec{L}$$
$$m\vec{a}_G = \dot{\vec{L}} = \sum \vec{F}_i$$



- The mass center moves as if the entire mass and all of the external forces were concentrated at that point.
- The mass center "abstracts" the collective motion or "represents" overall beabviour of all the particles.

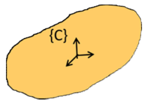
Angular Momentum about Mass Center



$$\vec{v}_i = \vec{v}_G + \vec{v}_i'$$
$$\vec{a}_i = \vec{a}_G + \vec{a}_i'$$

- Consider the centroidal frame *Gx'y'z'*, which is attached at the mass center.
- This centroidal frame *Gx'y'z'* is in general not a Newtonian frame; can accelerate or rotate w.r.t. a Newtonian frame *Oxyz.*

- The angular momentum of the system of particles about the mass center,



$$\vec{H}'_G = \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{v}_i')$$

$$\dot{\vec{H}}'_G = \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{a}_i') = \sum_{i=1}^{n} (\vec{r}_i' \times m_i (\vec{a}_i - \vec{a}_G))$$

$$= \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{a}_i) - \left(\cancel{\sum_{i=1}^{n} m_i \vec{r}_i'} \right) \times \vec{a}_G$$

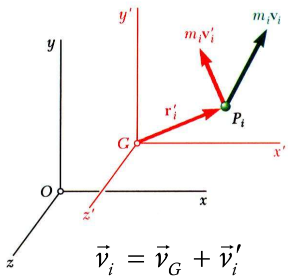
$$= \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{a}_i) = \sum_{i=1}^{n} (\vec{r}_i' \times \vec{F}_i) + \sum_i \sum_j \cancel{r_i' \times f_{ij}}$$

$$= \sum \vec{M}_G$$

- The moment resultant about *G* of the external forces is equal to the rate of change of angular momentum about *G* of the system of particles (similar to the rigid-body rotation dynamics).

$$\boxed{\dot{\vec{H}}'_G = \sum \vec{M}_G}$$

Angular Momentum about Mass Center



$$\vec{v}_i = \vec{v}_G + \vec{v}_i'$$

- The previous $\vec{H}_G'$ is the angular momentum about *G* of the particles in their relative motion to the centroid frame *Gx'y'z':*

$$\boxed{\vec{H}'_G = \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{v}_i')}$$

- Angular momentum about *G* of particles in their absolute motion in a Newtonian *Oxyz* frame of reference.

$$\vec{H}_G = \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{v}_i)$$
$$= \sum_{i=1}^{n} (\vec{r}_i' \times m_i (\vec{v}_G + \vec{v}_i'))$$
$$= \left(\cancel{\sum_{i=1}^{n} m_i \vec{r}_i'} \right) \times \vec{v}_G + \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{v}_i')$$
$$\vec{H}_G = \vec{H}'_G = \sum \vec{M}_G$$

- Angular momentum about *G* of the particle momenta can be calculated either using the absolute motion in a Newtonian frame $\vec{H}_G$ or the relative motion in the centroid frame $\vec{H}_G'$, since $\vec{H}_G = \vec{H}_G'$.

Conservation of Momentum

$$\sum \vec{F}_i = \sum m_i \vec{a}_i$$
$$\sum \left(\vec{r}_i \times \vec{F}_i\right) = \sum \left(\vec{r}_i \times m_i \vec{a}_i\right)$$

$$\dot{\vec{L}} = \sum \vec{F}, \quad \dot{\vec{H}}_o = \sum \vec{M}_o$$
$$\dot{\vec{L}} = m\dot{\vec{v}}_G = \sum \vec{F}, \quad \dot{\vec{H}}'_G = \sum \vec{M}'_G$$

- If no external forces act on the particles of a system, then the total linear momentum and angular momentum about a fixed point O are conserved.

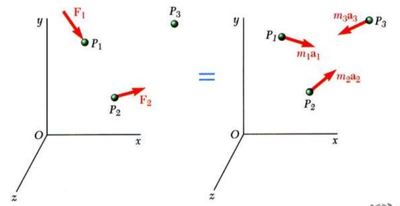
$$\dot{\vec{L}} = \sum \vec{F} = 0 \qquad \dot{\vec{H}}_O = \sum \vec{M}_O = 0$$
$$\vec{L} = \text{constant} \qquad \vec{H}_O = \text{constant}$$

- In some applications, such as problems involving central forces,

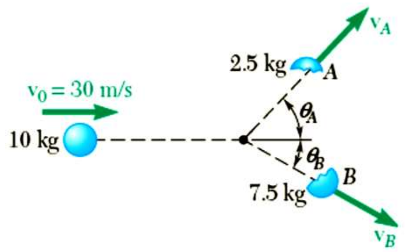
$$\dot{\vec{L}} = \sum \vec{F} \neq 0 \qquad \dot{\vec{H}}_O = \sum \vec{M}_O = 0$$
$$\vec{L} \neq \text{constant} \qquad \vec{H}_O = \text{constant}$$

- Concept of conservation of momentum also applies to the analysis of the mass center motion: if no external forces act on the particles of the system,

$$\dot{\vec{L}} = \sum \vec{F} = 0 \qquad \dot{\vec{H}}_G = \sum \vec{M}_G = 0$$
$$\vec{L} = m\vec{v}_G = \text{constant}$$
$$\vec{v}_G = \text{constant} \qquad \vec{H}'_G = \vec{H}_G = \text{constant}$$



Sample Problem 14.2



A 10-kg projectile is moving with a velocity of 30 m/s when it explodes into 2.5 and 7.5-kg fragments. Immediately after the explosion, the fragments travel in the directions $\theta_A = 45^o$ and $\theta_B = 30^o$. Determine the velocity of each fragment. (unknowns: 2)

SOLUTION:

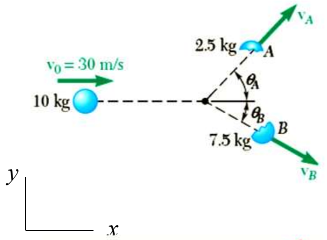
- Since there are no external forces, the linear momentum of the system is conserved.
- Write separate component equations for the conservation of linear momentum.
- Solve the equations simultaneously for the fragment velocities.

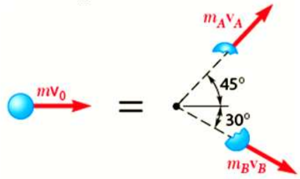
$$\sum \vec{F}_i = \sum m_i \vec{a}_i$$
$$\sum \left(\vec{r}_i \times \vec{F}_i\right) = \sum \left(\vec{r}_i \times m_i \vec{a}_i\right)$$

Sample Problem 14.2

SOLUTION:

- Since there are no external forces, the linear momentum of the system is conserved.





- Write separate component equations for the conservation of linear momentum.

$$m_A \vec{v}_A + m_B \vec{v}_B = m\vec{v}_0$$

$$2.5\vec{v}_A + 7.5\vec{v}_B = 10\vec{v}_0$$

x components:

$$25v_A \cos 45° + 7.5v_B \cos 30° = 10(30)$$

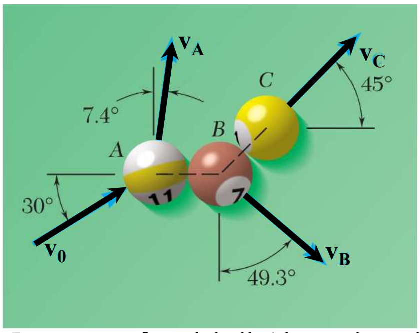
y components:

$$2.5v_A \sin 45° - 7.5v_B \sin 30° = 0$$

- Solve the equations simultaneously for the fragment velocities.

$$\boxed{v_A = 62.2\,\text{m/s} \qquad v_B = 29.3\,\text{m/s}}$$

Group Problem Solving



In a game of pool, ball *A* is moving with a velocity $\mathbf{v}_0$ when it strikes balls *B* and *C*, which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and that $v_0 = 4$ m/s and $v_C = 2$ m/s, determine the magnitude of the velocity of (*a*) ball *A*, (*b*) ball *B*. (unknowns: 2)

Strategy:

- Since there are no external forces, the linear momentum of the system is conserved.
- Write separate component equations for the conservation of linear momentum.
- Solve the equations simultaneously for the pool ball velocities.

Group Problem Solving [2]

Modeling And Analysis:

Write separate component equations for the conservation of linear momentum

x: $m(4)\cos 30° = mv_A \sin 7.4° + mv_B \sin 49.3° \; + m(2)\cos 45°$

$0.12880v_A + 0.75813v_B = 2.0499 \quad (1)$

y: $m(4)\sin 30° = mv_A \cos 7.4° - mv_B \cos 49.3° \; + m(2)\sin 45°$

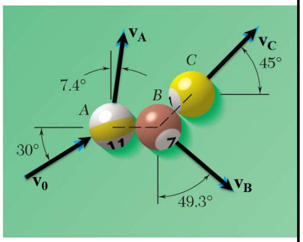
$0.99167v_A - 0.65210v_B = 0.5858 \quad (2)$

Two equations, two unknowns - solve

$0.65210\,(0.12880v_A + 0.75813v_B = 2.0499\,)$

$+\;0.75813\,(0.99167v_A - 0.65210v_B = 0.5858\,)$

$0.83581\,v_A = 1.78085$



Sub into (1) or (2) to get v_B

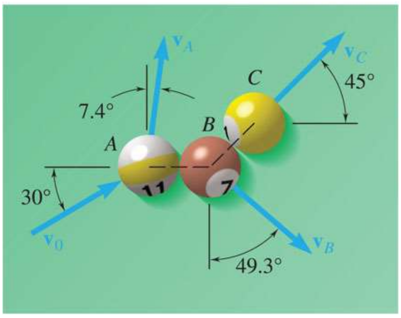
$\boxed{v_A = 2.13 \text{ m/s} \quad v_B = 2.34 \text{ m/s}}$

Concept Question [3]

Reflect and Think:

In a game of pool, ball A is moving with a velocity $\mathbf{v}_0$ when it strikes balls B and C, which are at rest and aligned as shown.

After the impact, what is true about the overall center of mass of the system of three balls?



a) The overall system CG will move in the same direction as v_0

b) The overall system CG will stay at a single, constant point

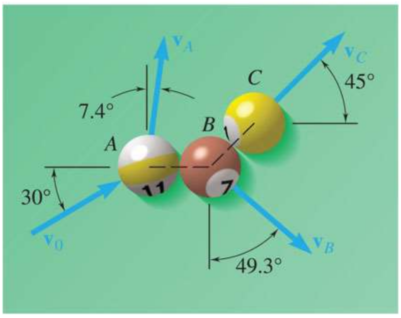
c) There is not enough information to determine the CG location

Concept Question [4]

Reflect and Think:

In a game of pool, ball *A* is moving with a velocity $\mathbf{v}_0$ when it strikes balls *B* and *C*, which are at rest and aligned as shown.

After the impact, what is true about the overall center of mass of the system of three balls?



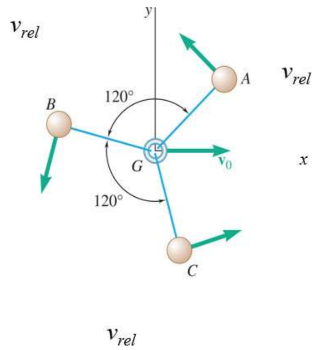
a) The overall system CG will move in the same direction as v_0

b) The overall system CG will stay at a single, constant point

c) There is not enough information to determine the CG location

Concept Question [1]

Three small identical spheres *A*, *B*, and *C*, which can slide on a horizontal, frictionless surface, are attached to three 200-mm-long strings, which are tied to a ring *G*. Initially, each of the spheres rotate clockwise about the ring with a relative velocity of v_{rel} and the ring moves along the *x*-axis with a velocity $\mathbf{v_0}$= (0.4 m/s)**i.**



Which of the following is true?

a) The linear momentum of the system is in the positive x direction.

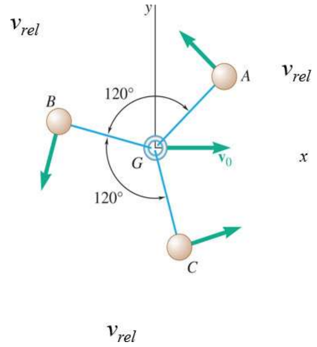
b) The angular momentum of the system is in the positive y direction.

c) The angular momentum of the system about G is zero.

d) The linear momentum of the system is zero.

Concept Question 2

Three small identical spheres A, B, and C, which can slide on a horizontal, frictionless surface, are attached to three 200-mm-long strings, which are tied to a ring G. Initially, each of the spheres rotate clockwise about the ring with a relative velocity of v_{rel} and the ring moves along the x-axis with a velocity $\mathbf{v}_0$= (0.4 m/s)**i**.



Which of the following is true?

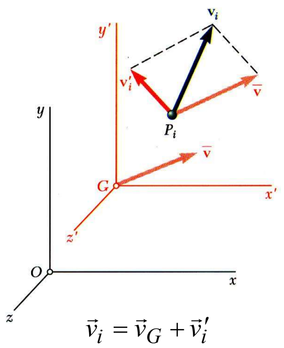
a) The linear momentum of the system is in the positive x direction.

b) The angular momentum of the system is in the positive y direction.

c) The angular momentum of the system about G is zero.

d) The linear momentum of the system is zero.

Kinetic Energy



$$\vec{v}_i = \vec{v}_G + \vec{v}_i'$$

$$mr_G = \sum m_i r_i, \qquad m = \sum m_i$$

$$\rightarrow mv_G = \sum m_i v_i = \sum m_i (v_G + v_i')$$

$$\rightarrow mv_G = mv_G + \sum m_i\, v_i'$$

$$\rightarrow \sum m_i v_i' = 0$$

- Kinetic energy of a system of particles,

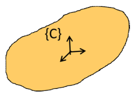
$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i (\vec{v}_i \bullet \vec{v}_i) = \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i^2$$

- Expressing the velocity using mass center velocity and relative velocity from that:

$$T = \tfrac{1}{2}\sum_{i=1}^{n}\left[m_i(\vec{v}_G + \vec{v}_i')\bullet(\vec{v}_G + \vec{v}_i')\right]$$

$$= \tfrac{1}{2}\left(\sum_{i=1}^{n} m_i\right)v_G^2 + \vec{v}_G \bullet \sum_{i=1}^{n} m_i \vec{v}_i' + \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i'^2$$

$$= \tfrac{1}{2} m\bar{v}_G^2 + \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i'^2$$



- Kinetic energy is equal to the kinetic energy of mass center plus kinetic energy relative to the mass center (related to the rotational motion for rigid-body).

Work-Energy Principle of System of Particles

- Principle of work and energy can be applied to each particle P_i ,

$$\boxed{T_{i,1} + U_{i,1\to 2} = T_{i,2}}$$

where $U_{i,1\to 2}$ represents the work done by the internal forces $\vec{f}_{ij}$ and the resultant external force $\vec{F}_i$ acting on P_i .

- Principle of work/energy of the entire system by adding the energetics of all the particles with the work done by all external and internal forces.

$$\boxed{(\tfrac{1}{2} m\vec{v}_G^2 + \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i'^2)_1 + \sum U_{1\to 2} = (\tfrac{1}{2} m\vec{v}_G^2 + \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i'^2)_2}$$

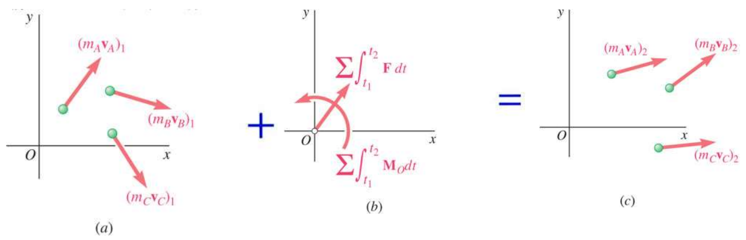
- Although internal forces $\vec{f}_{ij}$ and $\vec{f}_{ji}$ are opposite/equal, thus, equipollent and not affecting linear and angular momentums, their work is not canceled out (e.g., deformation energy; zero work if no relative motion (e.g., constraints)).

- If the forces acting on the particles are conservative, the work is equal to the change in potential energy and

$$\boxed{T_1 + V_1 = T_2 + V_2}$$

which is the conservation of energy for the system of particles.

Impulse/Momentum Principle of Sys. of Particles



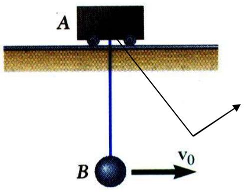
$$\sum \vec{F} = \dot{\vec{L}} \qquad \sum \vec{M}_O = \dot{\vec{H}}_O$$

$$\sum \int_{t_1}^{t_2} \vec{F}dt = \vec{L}_2 - \vec{L}_1 \qquad \sum \int_{t_1}^{t_2} \vec{M}_O dt = \vec{H}_2 - \vec{H}_1$$

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F}dt = \vec{L}_2 \qquad \vec{H}_1 + \sum \int_{t_1}^{t_2} \vec{M}_O dt = \vec{H}_2$$

- The momenta of the particles at time t_1 and the impulse of the forces from t_1 to t_2 form a system of vectors *equipollent* to the system of momenta of the particles at time t_2 (internal forces not affecting the momenta).

Sample Problem 14.4



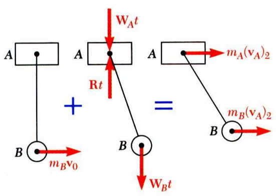
Ball *B*, of mass m_B, is suspended from a cord, of length *l*, attached to cart *A*, of mass m_A, which can roll freely on a frictionless horizontal tract. While the cart is at rest, the ball is given an initial velocity $v_0 = \sqrt{2gl}$.

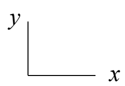
Determine *(a)* the velocity of *B* as it reaches it maximum elevation, and *(b)* the maximum vertical distance *h* through which *B* will rise. (3 unknowns: v_A, v_B, θ; 3 equations: momentum, energy and constraint)

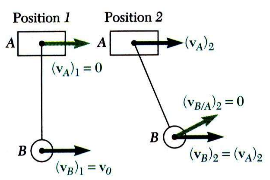
SOLUTION:

- With no external horizontal forces, it follows from the impulse-momentum principle that horizontal momentum is conserved. This relation can be solved for the velocity of *B* at its maximum elevation.
- The conservation of energy principle can be applied to relate the initial kinetic energy to the maximum potential energy. The maximum vertical distance is determined from this relation.

Sample Problem 14.4







SOLUTION:

- With no external horizontal forces, it follows from the impulse-momentum principle that the horizontal component of momentum is conserved. This relation can be solved for the velocity of *B* at its maximum elevation.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2$$

x component equation:

$$m_A v_{A,1} + m_B v_{B,1} = m_A v_{A,2} + m_B v_{B,2}$$

Velocities at positions 1 and 2 are

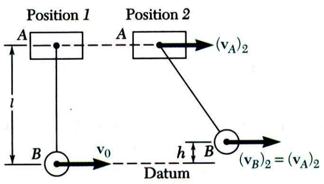
$$v_{A,1} = 0 \quad v_{B,1} = v_0$$

$$\boxed{v_{B,2} = v_{A,2} + v_{B/A,2} = v_{A,2}}$$ (velocity of *B* relative to *A* is zero at position 2)

$$m_B v_0 = (m_A + m_B) v_{A,2}$$

$$\boxed{v_{A,2} = v_{B,2} = \frac{m_B}{m_A + m_B} v_0}$$

Sample Problem 14.4



- The conservation of energy principle can be applied to relate the initial kinetic energy to the maximum potential energy.

$$T_1 + V_1 = T_2 + V_2$$

Position 1 - Potential Energy: $V_1 = m_A g l$

Kinetic Energy: $T_1 = \frac{1}{2} m_B v_0^2$

Position 2 - Potential Energy: $V_2 = m_A g l + m_B g h$

Kinetic Energy: $T_2 = \frac{1}{2}(m_A + m_B) v_{A,2}^2$

$$\tfrac{1}{2} m_B v_0^2 + m_A g l = \tfrac{1}{2}(m_A + m_B) v_{A,2}^2 + m_A g l + m_B g h$$

$$h = \frac{v_0^2}{2g} - \frac{m_A + m_B}{m_B}\frac{v_{A,2}^2}{2g} = \frac{v_0^2}{2g} - \frac{m_A + m_B}{2g\, m_B}\left(\frac{m_B}{m_A + m_B} v_0\right)^2$$

$$h = \frac{v_0^2}{2g} - \frac{m_B}{m_A + m_B}\frac{v_0^2}{2g}$$

$$\boxed{h = \frac{m_A}{m_A + m_B}\frac{v_0^2}{2g}}$$

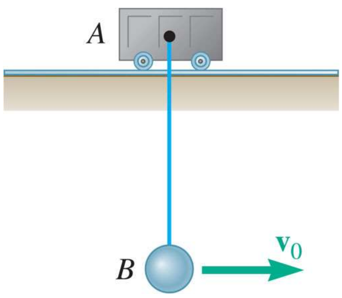
Or, alternatively (more rigorously):

$$m_A v_A(0) + m_B v_B(0) = m_A v_A(t) + m_B v_B(t) \sin\theta$$

$$\tfrac{1}{2} m_A v_A^2(0) + \tfrac{1}{2} m_B v_B^2(0) + mgh(0) = \tfrac{1}{2} m_A v_A^2(t) + \tfrac{1}{2} m_B v_B^2(t) + mgh(t)$$

$$h(t) - h(0) = L(1 - \cos\theta)$$

Sample Problem 14.5



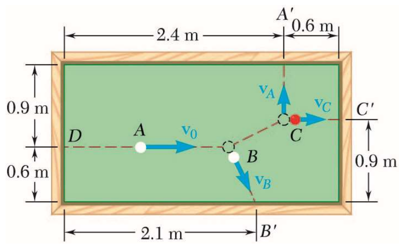
Strategy:

- Recalling that $v_0^2 < 2gl$, it follows from the last equation that $h < 1$; this verifies that B stays below A, as assumed in the solution.
- For $m_A \gg m_B$, the answers reduce to

  $(v_B)_2 = (v_A)_2 = 0$ *and* $h = v_0^2 / 2g$;

  B oscillates as a simple pendulum with A fixed.
- For $m_A \ll m_B$, they reduce to $(v_B)_2 = (v_A)_2 = v_0$ and $h = 0g$; A and B move with the same constant velocity v_0.

Sample Problem 14.5



Ball A has initial velocity $v_0 = 3$ m/s parallel to the axis of the table. It hits ball B and then ball C which are both at rest. Balls A and C hit the sides of the table squarely at A' and C' and ball B hits obliquely at B'.

Assuming perfectly elastic collisions, determine velocities v_A, v_B, and v_C with which the balls hit the sides of the table.

(unknowns: 4)

SOLUTION:

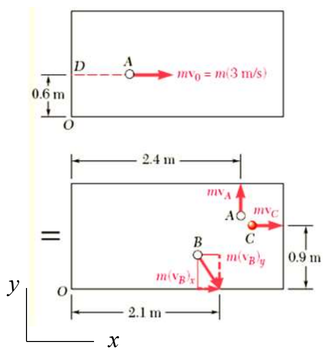
- There are four unknowns: v_A, $v_{B,x}$, $v_{B,y}$, and v_C.
- Solution requires four equations: linear momentum conservation (two component equations), angular momentum (about which point?), and energy.
- Write the conservation equations in terms of the unknown velocities and solve simultaneously.

Sample Problem 14.5

SOLUTION:

- There are four unknowns: v_A, $v_{B,x}$, $v_{B,y}$, and v_C.

$$\vec{v}_A = v_A \vec{j}$$
$$\vec{v}_B = v_{B,x}\vec{i} + v_{B,y}\vec{j}$$
$$\vec{v}_C = v_C \vec{i}$$



- The conservation of momentum and energy equations,

$$\vec{L}_1 + \sum \int \vec{F} dt = \vec{L}_2$$

$$mv_0 = mv_{B,x} + mv_C \qquad 0 = mv_A - mv_{B,y}$$

$$\vec{H}_{O,1} + \sum \int \vec{M}_O dt = \vec{H}_{O,2}$$

$$-(0.6\,\mathrm{m})mv_0 = (2.4\,\mathrm{m})mv_A - (2.1\,\mathrm{m})mv_{B,y} - (0.9\,\mathrm{m})mv_C$$

angular/linear momentum of B invariant after impact

$$T_1 + V_1 = T_2 + V_2$$

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv_A^2 + \tfrac{1}{2}m\left(v_{B,x}^2 + v_{B,y}^2\right) + \tfrac{1}{2}mv_C^2$$

Solving the first three equations in terms of v_C,

$$v_A = v_{B,y} = 3v_C - 20 \qquad v_{B,x} = 10 - v_C$$

Substituting into the energy equation,

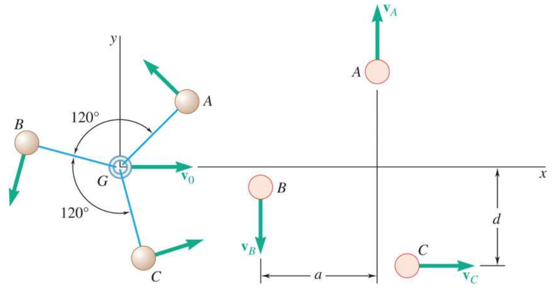
$$2(3v_C - 6)^2 + (3 - v_C)^2 + v_C^2 = 9$$

$$20v_C^2 - 78v_C + 72 = 0$$

$$\boxed{\begin{array}{l} v_A = 1.2\,\mathrm{m/s} \quad v_C = 2.4\,\mathrm{m/s} \\ \vec{v}_B = (2\vec{i} - 4\vec{j})\,\mathrm{m/s} \quad v_B = 1.342\,\mathrm{m/s} \end{array}}$$

- same answer if the hitting point of the ball B is chosen for H.

Group Problem Solving 3



Three small identical spheres *A*, *B*, and *C*, which can slide on a horizontal, frictionless surface, are attached to three 200-mm-long strings, which are tied to a ring *G*. Initially, the spheres rotate clockwise about the ring with a relative velocity of 0.8 m/s and the ring moves along the *x*-axis with a velocity $\mathbf{v_0}$= (0.4 m/s)**i.** Suddenly, the ring breaks and the three spheres move freely in the *xy* plane with *A* and *B* following paths parallel to the *y*-axis at a distance a= 346 mm from each other and *C* following a path parallel to the *x* axis. Determine (*a*) the velocity of each sphere, (*b*) the distance *d*. (unknowns: 4)

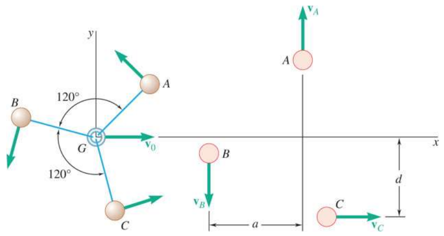
Group Problem Solving 4

Given: $v_{Arel}= v_{Brel} = v_{Crel} = 0.8$ m/s, $\mathbf{v_0}$= (0.4 m/s)**i** , L= 200 mm, a= 346 mm

Find: v_A, v_B, v_C (after ring breaks), *d*

Strategy:

- There are four unknowns: v_A, v_B, v_B, *d*.
- Solution requires four equations: conservation principles for linear momentum (two component equations), angular momentum, and energy.
- Write the conservation equations in terms of the unknown velocities and solve simultaneously.



Modeling and Analysis:

Apply the conservation of linear momentum equation – find $\mathbf{L_0}$ before ring breaks

$$\mathbf{L}_0 = (3\,\mathrm{m})\overline{\mathbf{v}} = 3m(0.4\mathbf{i}) = \mathrm{m}(1.2\ \mathrm{m/s})\mathbf{i}$$

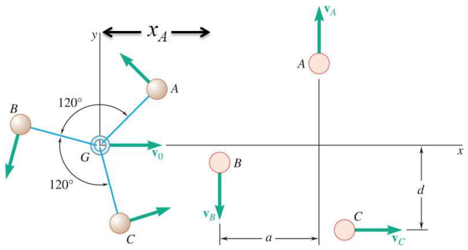
What is $\mathbf{L_f}$ (after ring breaks)?

$$\mathbf{L}_f = mv_A\mathbf{j} - mv_B\mathbf{j} + mv_C\mathbf{i}$$

Group Problem Solving [5]

Set $L_0 = L_f$

$$m(1.2\text{ m/s})\mathbf{i} = mv_C\mathbf{i} + m(v_A - v_B)\mathbf{j}$$



From the y components,

$$v_A = v_B$$

From the x components,

$$v_C = 1.200\text{ m/s} \quad \mathbf{v}_C = 1.200\text{ m/s}$$

Apply the conservation of angular momentum equation

H_0: $(H_G)_0 = 3mlv_{rel} = 3m(0.2\,\text{m})(0.8\text{ m/s}) = 0.480m$

H_f: $(H_G)_f = -mv_A x_A + mv_B(x_A + a) + mv_C d =$

Since $v_A = v_B$, and $v_C = 1.2$ m/s, then:

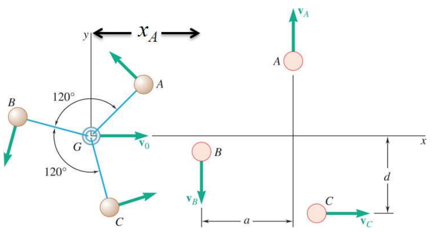
$$0.480m = 0.346mv_A + mv_C d$$

$$0.480 = 0.346v_A + 1.200d$$

$$d = 0.400 - 0.28833v_A$$

Group Problem Solving [6]

Need another equation- try work-energy, where $T_0 = T_f$



T_0:

$$T_0 = \frac{1}{2}(3\text{m})\bar{v}^2 + 3\left(\frac{1}{2}mv_{rel}{}^2\right)$$

$$= \frac{3}{2}\text{m}\left(v_0^2 + v_{rel}{}^2\right) = \frac{3}{2}[(0.4)^2 + (0.8)^2]m = 1.200m$$

T_f:

$$T_f = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2$$

Substitute in known values:

$$\frac{1}{2}\left[v_A^2 + v_A^2 + (1.200)^2\right] = 1.200$$

$$v_A^2 = 0.480$$

$$v_A = v_B = 0.69282\text{ m/s}$$

Solve for d:

$$d = 0.400 - 0.28833(0.69282) = 0.20024\text{ m}$$

$\mathbf{v}_A = 0.693\text{ m/s}\uparrow$	$\mathbf{v}_C = 1.200\text{ m/s}\rightarrow$
$\mathbf{v}_B = 0.693\text{ m/s}\downarrow$	$d = 0.200\text{ m}$

Variable Systems of Particles

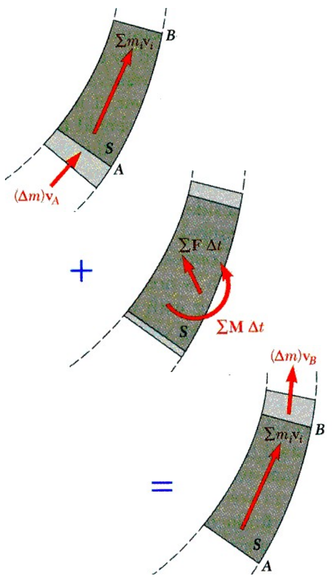
- Dynamics principles established so far were derived for constant systems of particles, i.e., systems which neither gain nor lose particles.

$$\sum \vec{F}_i = \sum m_i \vec{a}_i$$
$$\sum(\vec{r}_i \times \vec{F}_i) = \sum(\vec{r}_i \times m_i \vec{a}_i)$$

$$(\tfrac{1}{2} m\bar{v}_G^2 + \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i'^2)_1 + \sum U_{1\to 2} = (\tfrac{1}{2} m\bar{v}_G^2 + \tfrac{1}{2}\sum_{i=1}^{n} m_i v_i'^2)_2$$

- A large number of engineering applications require the consideration of variable systems of particles, e.g., hydraulic turbine, rocket engine, etc.
- For analyses, consider auxiliary systems which consist of the particles instantaneously within the system plus the particles that enter or leave the system during a short time interval. The auxiliary systems, thus defined, are constant systems of particles.

Steady Stream of Particles



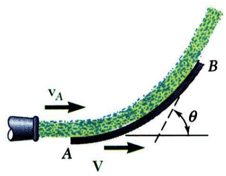
- System consists of a steady stream of particles against a vane or through a duct.
- Define auxiliary system including the particles which flow in and out over Δt *with moving boundary*.
- The auxiliary system is a constant system of particles over Δt.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2$$

$$[\sum m_i \vec{v}_i + (\Delta m)\vec{v}_A] + \sum \vec{F}\,\Delta t = [\sum m_i \vec{v}_i + (\Delta m)\vec{v}_B]$$

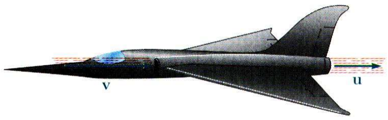
$$\sum \vec{F} = \frac{dm}{dt}(\vec{v}_B - \vec{v}_A)$$

Steady Stream of Particles: Applications

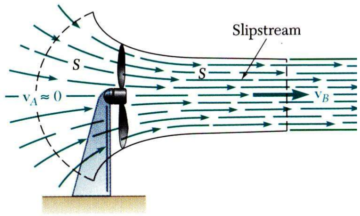


- Fluid Stream Diverted by Vane or Duct

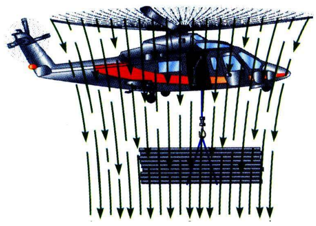
- Fluid Flowing Through a Pipe



- Jet Engine

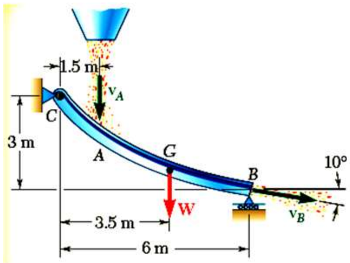


- Fan



- Helicopter

Sample Problem 14.6



Grain falls onto a chute at the rate of 120 kg/s. It hits the chute with a velocity of 10 m/s and leaves with a velocity of 7.5 m/s. The combined weight of the chute and the grain it carries is 3000 N with the center of gravity at *G*.

Determine the reactions at *C* and *B*. (unknowns: 3)

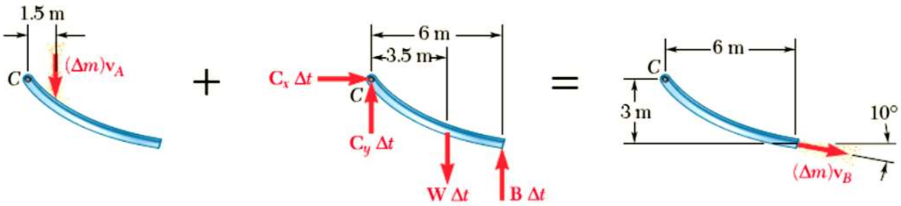
SOLUTION:

- Define a system consisting of the mass of grain on the chute plus the mass that is added and removed during the time interval Δt.
- Apply the principles of conservation of linear and angular momentum for three equations for the three unknown reactions.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2$$

$$\left[\sum m_i \vec{v}_i + (\Delta m)\vec{v}_A\right] + \sum \vec{F}\,\Delta t = \left[\sum m_i \vec{v}_i + (\Delta m)\vec{v}_B\right]$$

Sample Problem 14.6



SOLUTION:

- Define a system consisting of the mass of grain on the chute plus the mass that is added and removed during the time interval Δt.
- Apply the principles of conservation of linear and angular momentum for three equations for the three unknown reactions.

$$\vec{L}_1 + \sum\int \vec{F}dt = \vec{L}_2$$

$$C_x\Delta t = (\Delta m)v_B \cos 10°$$

$$-(\Delta m)v_A + (C_y - W + B)\Delta t = -(\Delta m)v_B \sin 10°$$

$$\vec{H}_{C,1} + \sum\int \vec{M}_C dt = \vec{H}_{C,2}$$

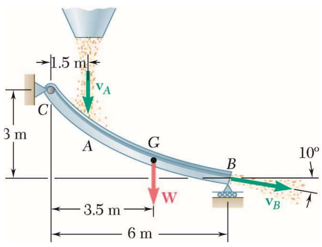
$$-1.5(\Delta m)v_A + (-3.5W + 6B)\Delta t$$
$$= 3(\Delta m)v_B \cos 10° - 6(\Delta m)v_B \sin 10°$$

Solve for C_x, C_y, and B with

$$\frac{\Delta m}{\Delta t} = \frac{120\,\mathrm{kg/s}}{16.1\,\mathrm{m/s^2}} = 7.45\,\mathrm{slug/s}$$

$$\boxed{B = 2340\,\mathrm{N} \quad \vec{C} = (886\vec{i} + 1704\vec{j})\,\mathrm{N}}$$

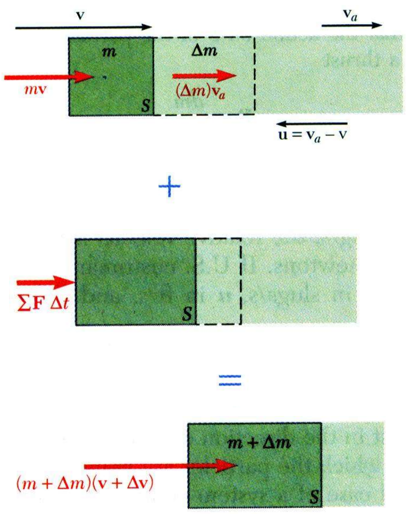
Sample Problem 14.7 3



Reflect and Think:

- This kind of situation is common in factory and storage settings. Being able to determine the reactions is essential for designing a proper chute that will support the stream safely. We can compare this situation to the case when there is no mass flow, which results in reactions of B_y = 1750 N, C_y = 1250 N, and C_x = 0 N.

Streams Gaining or Losing Mass



- Define auxiliary system to include particles of mass m within system at time t plus the particles of mass Δm which enter the system over time interval Δt.
- The auxiliary system is a constant system of particles.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2$$

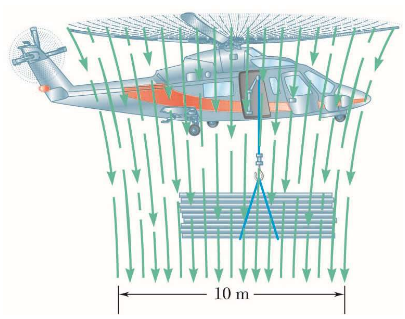
$$[m\vec{v} + (\Delta m)\vec{v}_a] + \sum \vec{F}\,\Delta t = (m + \Delta m)(\vec{v} + \Delta \vec{v})$$

$$\sum \vec{F}\,\Delta t = m\Delta\vec{v} + \Delta m(\vec{v} - \vec{v}_a) + (\Delta m)\Delta\vec{v}$$

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}\vec{u}$$

$$m\vec{a} = \sum \vec{F} - \frac{dm}{dt}\vec{u}$$

Group Problem Solving 7



Strategy:

- Calculate the time rate of change of the mass of the air.
- Determine the thrust generated by the airstream.
- Use this thrust to determine the maximum load that the helicopter can carry.

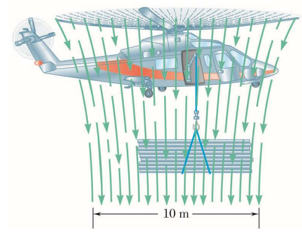
The helicopter shown can produce a maximum downward air speed of 25 m/s in a 10-m-diameter slipstream. Knowing that the weight of the helicopter and its crew is 18 kN and assuming ρ = 1.21 kg/m^3 for air, determine the maximum load that the helicopter can lift while hovering in midair.

Group Problem Solving 8

Modeling and Analysis:

Given: $v_B = 25$ m/s, W= 18,000 N, $\rho = 1.21$ kg/m^3

Find: Max load during hover

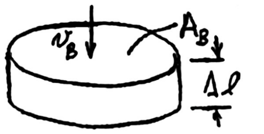


Choose the relationship you will use to determine the thrust

$$F = \frac{dm}{dt}(v_B - v_A)$$

Calculate the time rate of change (dm/dt) of the mass of the air.

mass = density × volume = density × area × length



$$\Delta m = \rho A_B(\Delta l) \quad = \rho A_B v_B(\Delta t)$$

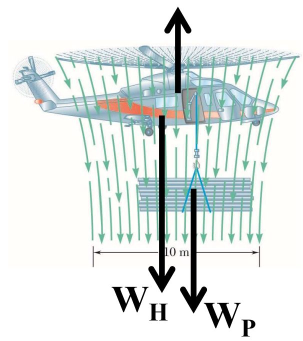
$$\frac{\Delta m}{\Delta t} = \rho A_B v_B = \frac{dm}{dt}$$

A_B is the area of the slipstream
v_B is the velocity in the slipstream.
Well above the blade, $v_A \approx 0$

Group Problem Solving 9

Use the relationship for dm/dt to determine the thrust

$$F = \frac{dm}{dt}(v_B - v_A) \qquad \frac{dm}{dt} = \rho A_B v_B$$



$$F = \rho A_B v_B^2$$
$$= \left(1.21\,\text{kg/m}^3\right)\left(\frac{\pi}{4}\right)(10\text{ m})^2(25\text{ m/s})^2$$
$$= 59{,}396\text{ N}$$

Use statics to determine the maximum payload during hover

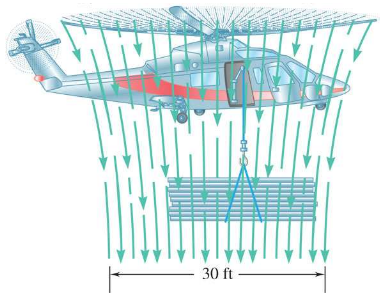
$$+\uparrow \; \Sigma F_y = F - W_H - W_P = 0$$

$$W_P = F - W_H = 59{,}396 - 18{,}000 = 41{,}395\text{ N}$$

$$\boxed{W = 41{,}400\text{ N}}$$

Concept Question 5

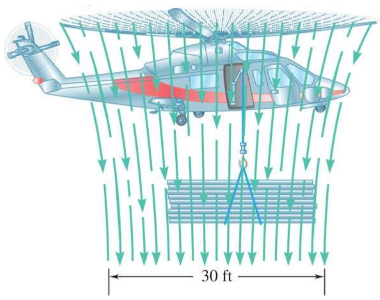
Reflect and Think:
In the previous problem with the maximum payload attached, what happens if the helicopter tilts (or pitches) forward?



a) The area of displaced air becomes smaller.
b) The volume of displaced air becomes smaller.
c) The helicopter will accelerate upward.
d) The helicopter will accelerate forward.

Concept Question 6

Reflect and Think:
In the previous problem with the maximum payload attached, what happens if the helicopter tilts (or pitches) forward?



a) The area of displaced air becomes smaller.
b) The volume of displaced air becomes smaller.
c) The helicopter will accelerate upward.
d) The helicopter will accelerate forward.

*The helicopter will also accelerate downward