Engineering Mathematics 2

Lecture 4

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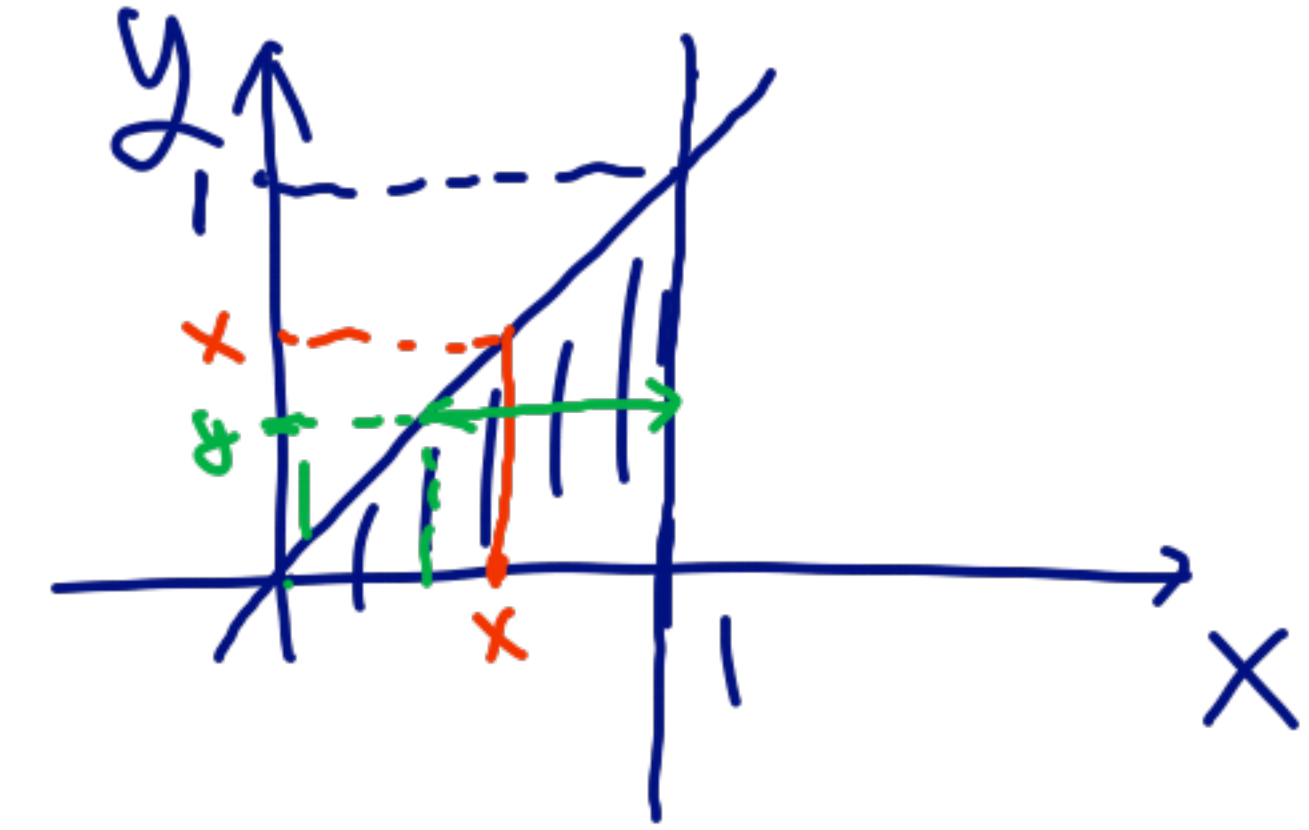
1st Exam on next Wednesday
(23 September)

- Online using eTL
- 30 questions for 75 minutes
- Academic integrity

Previously, we discussed, $\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$

- gradient of scalar function
- divergence
- curl
- line integral
- path independence, conservative vector field, potential

10.3 Double integrals



$$\iint_R f(x,y)\,dx\,dy$$

Example: Integrate $\underline{f(x,y) = 8xy}$ over a region surrounded by $\underline{y=0,\ y=x,\ x=1}$

$$\int_0^1 \underbrace{\int_0^x 8xy\,dy}\,dx = \int_0^1 4x^3\,dx = 1$$

$$\int_0^1 \int_y^1 8xy\,dx\,dy = \int_0^1 (4y-4y^3)\,dy = 2-1 = 1$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(u,v),\, y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

in which $J = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

e.g. polar coordinate: $x = r\cos\theta$, $y = r\sin\theta$

$dA = dx\,dy$

$$J = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$dA = r\,dr\,d\theta$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Example

Calculate the area of a quarter circle with radius 1

$$\iint r\,dr\,d\theta$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} \int_0^1 r\,dr}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\,d\theta$$

$$= \frac{\pi}{4}$$

10.4 Green's theorem

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_c F_1\,dx + F_2\,dy$$

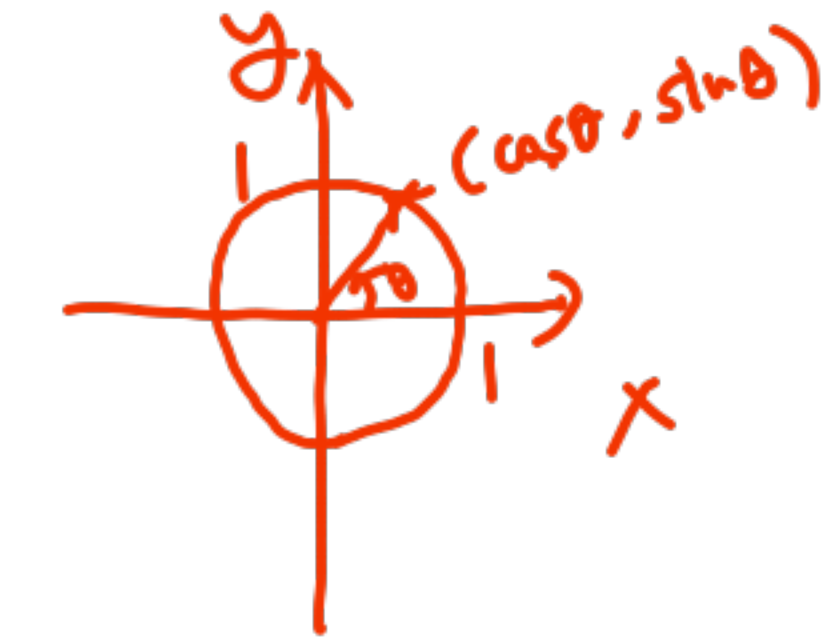
or $$\iint_R (\nabla \times \vec{F}) \cdot \vec{k}\,dx\,dy = \oint_c \vec{F} \cdot d\vec{r}$$

Proof by carrying out double integral

Example 1:

$x = \cos\theta \rightarrow dx = -\sin\theta \, d\theta$

$y = \sin\theta \rightarrow dy = \cos\theta \, d\theta$



Given $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$

Verify Green's thm over the unit circle with the centre at the origin.

$$LHS = \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \iint \left[(\cancel{2y} + 2) - (\cancel{2y} - 7) \right] dx\,dy$$

$$= 9\pi$$

$$RHS = \oint_c F_1\,dx + F_2\,dy$$

$$= \int_0^{2\pi} (\sin^2\theta - 7\sin\theta)(-\sin\theta\,d\theta) + (2\cos\theta\sin\theta + 2\cos\theta)\cos\theta\,d\theta$$

$$= 9\pi$$

Example 2:

$\vec{F} = [ay, bx]$, $b - a = 1$, Green's thm gives formulas to get area:

$$A = \iint_R \left(\underbrace{\frac{\partial F_2}{\partial x}}_{=b} - \underbrace{\frac{\partial F_1}{\partial y}}_{=a} \right) dx\, dy$$

$$= \oint_C \underbrace{F_1}_{ay} dx + \underbrace{F_2}_{bx} dy$$

i) $a = 0, b = 1$: $A = \oint_c x\, dy$

ii) $a = 1, b = 0$: $A = -\oint_c y\, dx$

iii) $-a = b = \frac{1}{2}$: $A = \frac{1}{2}\left(\oint x\, dy - y\, dx \right)$

Example 4: $\vec{F} = \left[-\frac{\partial w}{\partial y}, \frac{\partial w}{\partial x}\right]$. show that

$$\iint_R \nabla^2 w \, dx\, dy = \oint_c \frac{\partial w}{\partial n} \, ds$$

$$\frac{d\vec{r}}{ds} = \left[\frac{dx}{ds}, \frac{dy}{ds}\right]$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) dx\,dy = \iint_R \nabla^2 w \, dx\,dy$$

Green's thm

$$= \oint \left(F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds}\right) ds$$

$$= \oint \left(-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds}\right) ds$$

$$\oint \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right] \cdot \left[\frac{dy}{ds}, -\frac{dx}{ds}\right]$$

$$= \oint \nabla w \cdot \vec{n} \, ds$$

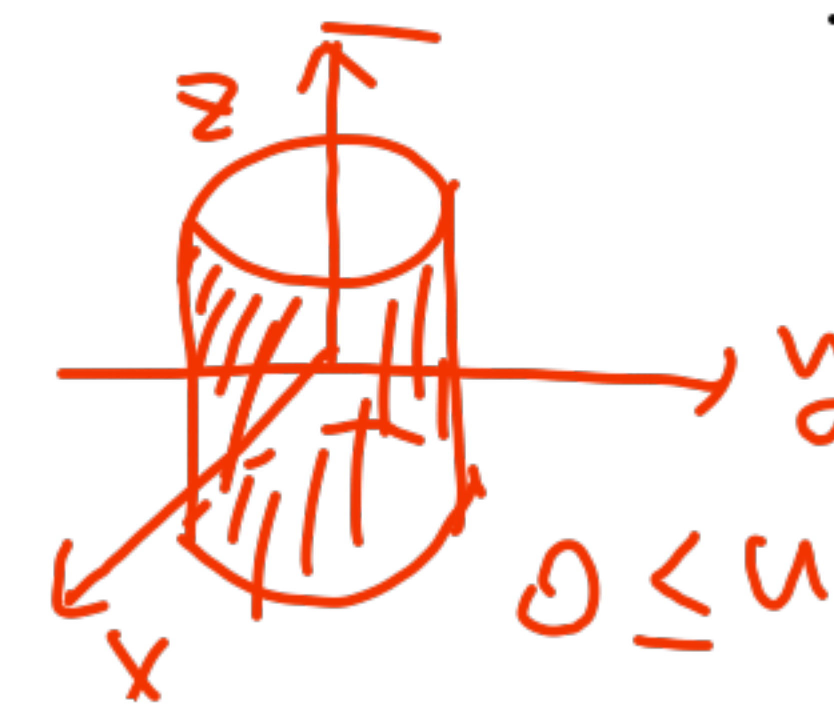
$$= \oint \frac{\partial w}{\partial n} \, ds$$

10.5 Surface for surface integral

* Parametric representation of a surface S in space

$$\vec{r}(u,v) = [\,x(u,v),\ y(u,v),\ z(u,v)\,]$$

– Cylinder $(x^2+y^2=a^2,\ |z|\le 1)$



$$\vec{r}(u,v) = [\,a\cos u,\ a\sin u,\ v\,]$$

$0 \le u \le 2\pi,\ |v| \le 1$

– Sphere $(x^2+y^2+z^2=a^2)$

$$\vec{r}(u,v) = [\,a\cos v\cos u,\ a\cos v\sin u,\ a\sin v\,],\quad 0\le u\le 2\pi,\ -\pi \le v \le \pi$$

$$\vec{r}(u(t), v(t))$$

* $\dfrac{d\vec{r}}{dt} = \dfrac{\partial r}{\partial u}\dfrac{du}{dt} + \dfrac{\partial r}{\partial v}\dfrac{dv}{dt} = \vec{r}_u \dfrac{du}{dt} + \vec{r}_v \dfrac{dv}{dt}$

* $\vec{r}_u$ and $\vec{r}_v$ span the tangent plane of the surface

* $\vec{N} = \vec{r}_u \times \vec{r}_v$ is surface normal vector.

$$\vec{n} = \frac{N}{|N|}$$

10-6 Surface integrals

* Flux integral

S: $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}(\vec{r}(u,v))\cdot\vec{N}(u,v)\,du\,dv$$

* Direction cosine : $\vec{n} = [\cos\alpha, \cos\beta, \cos\gamma]$

$\alpha = \vec{n}\cdot\vec{i}$, $\beta = \vec{n}\cdot\vec{j}$, $\gamma = \vec{n}\cdot\vec{k}$

Example 2:

$$\iint_S \vec{F}\cdot\vec{n}\,dA \quad \text{for } \vec{F}=[x^2,0,3y^2] \text{ and}$$

$\vec{r}=[u,v,1-u-v]$

$S:\ x+y+z=1$ in the first octant.

$x=u,\ y=v,\ z=1-u-v$

$$=\iint \vec{F}(\vec{r}(u,v))\cdot\vec{N}\,du\,dv$$

$$=\int_0^1\int_0^u [u^2,0,3v^2]\cdot[1,1,1]\,dv\,du$$

$$=\int_0^1\underbrace{\int_0^u (u^2+3v^2)\,dv}\,du$$

$$=\int_0^1 2u^3\,du=\frac{1}{2}$$

$$\vec{N}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -1\\ 0 & 1 & -1\end{vmatrix}=[1,1,1]$$

Surface integral of a scalar function

$$\iint_S G(\vec{r})\,dA = \iint_R G(\vec{r}(u,v))\,|\vec{N}|\,du\,dv$$

For $G=1$, $A = \iint_R |\vec{N}|\,du\,dv$

$$= \iint_R |\vec{r}_u \times \vec{r}_v|\,du\,dv$$

For $\underline{z = f(x,y)}$, $A = ?$

$\vec{r} = [x, y, f(x,y)]$

$\underline{x = u}$, $\underline{y = v}$

$$A = \iint_R |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$= \iint_R |\vec{r}_x \times \vec{r}_y| \, dx\, dy$$

$$= \iint_R \sqrt{1 + f_x^2 + f_y^2} \, dx\, dy$$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0, & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left[-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right]$$

$$|\vec{N}| = \sqrt{1 + f_x^2 + f_y^2}$$