3/29/21 Lagrangian / Hamiltonian description for single particle motion

$$L = (m\vec{v} + q\vec{A})\cdot\dot{\vec{x}} - H(\vec{v},\vec{x}) \qquad H = \frac{1}{2}mv^2 + q\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) - \frac{\partial L}{\partial \vec{x}} = 0 \iff m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

By small "m/q" expansion & gyro-average
(Read Littlejohn, 1979 - 1983)

→ Guiding-center particle Lagrangian up to the 1st order

$$\bar{L} = q\vec{A}^*\cdot\dot{\vec{X}} + \frac{m\mu}{q}\dot{\gamma} - H \qquad H = \frac{1}{2}mv_\parallel^2 + \mu B + q\phi$$

(fields evaluated at guiding center $\vec{X}$
γ : gyrophase

$$\vec{A}^* = \vec{A} + \frac{mv_\parallel}{q}\hat{b} \qquad \vec{B}^* = \nabla\times\vec{A}^*$$

→ can adopt any (non-canonical) coordinate system
→ phase space variables

$(\vec{X}, v_\parallel, \mu, \gamma)$ in the paper
or $(\vec{X}, U, \mu, \gamma)$ $\quad U \equiv \frac{1}{2}mv_\parallel^2 + \mu B + q\phi$
or $(r, \theta, \phi, U, \mu, \gamma)$ in a cylinder
or $(\psi, \theta, \varphi, U, \mu, \gamma)$ in a toroidal system

e.g.) $(\vec{X}, U, \mu, \gamma)$ $\quad L = q\vec{A}^*\cdot\dot{\vec{X}} + \frac{m\mu}{q}\dot{\gamma} - U$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}}\right) - \frac{\partial L}{\partial \gamma} = 0 \quad \rightarrow \quad \frac{d\mu}{dt} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{U}}\right) - \frac{\partial L}{\partial U} = 0 \quad \rightarrow \quad \frac{m}{\cancel{q}}\frac{\partial v_\parallel}{\partial U}\hat{b}\cdot\cancel{q}\dot{\vec{X}} - 1 = 0 \rightarrow \hat{b}\cdot\dot{\vec{X}} = v_\parallel$$

$$v_\parallel = \pm\sqrt{\left(\frac{2}{m}\right)(U - \mu B - q\phi)} \qquad \frac{\partial v_\parallel}{\partial U} = \frac{1}{mv_\parallel}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{X}}}\right) - \frac{\partial L}{\partial \vec{X}} = 0 \quad \rightarrow \quad \frac{d}{dt}(\cancel{q}\vec{A}^*) - \nabla(\cancel{q}\vec{A}^*\cdot\dot{\vec{X}}) = 0$$

$$\cancel{\frac{\partial}{\partial t}(q\vec{A}^*)} + \cancel{\dot{\vec{x}}\cdot\vec{\nabla}\vec{A}^*} - \dot{\vec{x}}\times(\vec{\nabla}\times\vec{A}^*) - \cancel{\dot{\vec{x}}\cdot\vec{\nabla}\vec{A}^*} = 0$$

For time-independent field. $\quad \dot{\vec{x}}\times\vec{B}^* = 0$

$$\hat{b}\times(\dot{\vec{x}}\times\vec{B}^*) = 0$$

$$\dot{\vec{x}}(\hat{b}\cdot\vec{B}^*) = \vec{B}^*(\hat{b}\cdot\dot{\vec{x}}) = v_\parallel \vec{B}^*$$

$$\dot{\vec{x}} = \frac{v_\parallel}{B^*_\parallel}\vec{B}^* \qquad \vec{B}^* = \vec{B} + \frac{m}{q}\vec{\nabla}\times(v_\parallel\hat{b})$$

$$= \vec{B} + \vec{\nabla}\times(\rho_\parallel\vec{B})$$

Morozov
Solovev
Boozer

$$\vec{v}_{gc} = \frac{v_\parallel(\vec{B}+\vec{\nabla}\times(\rho_\parallel\vec{B}))}{B(1+\underbrace{\{\hat{b}\cdot\vec{\nabla}\times(\rho_\parallel\vec{B})\}}_{\text{Banos correction } \propto J_\parallel}/B)} \qquad \rho_\parallel = \frac{mv_\parallel}{qB}$$

Read White's book ch. 3

2. Northrop's proof for J-invariance

second adiabatic invariant

$$J = \oint v_\parallel\, d\ell = \oint \sqrt{\left(\tfrac{2}{m}\right)(U-\mu B-q\phi)}\, d\ell$$

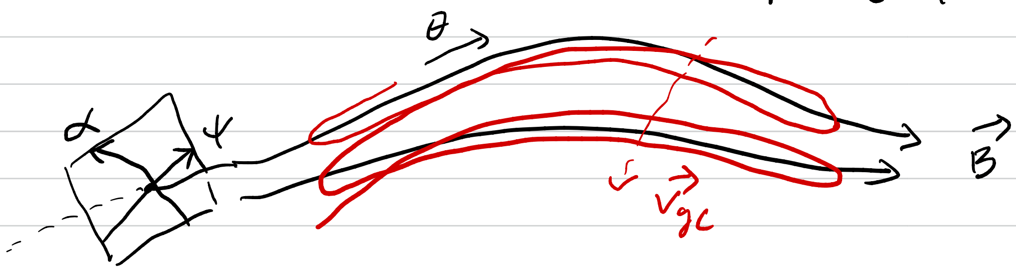
Bounce frequency ω_b

$$\frac{2\pi}{\omega_b} = \tau_b = \oint \frac{d\ell}{v_\parallel}$$

Bounce average

$$\langle A\rangle_b = \oint \frac{A\,d\ell}{v_\parallel}\Big/\oint\frac{d\ell}{v_\parallel} = \frac{\omega_b}{2\pi}\oint\frac{A\,d\ell}{v_\parallel}$$

J conserved? for time-independent field but drifting particles



$$\vec{B} = \vec{\nabla}\psi\times\vec{\nabla}\alpha \qquad J(\psi, \alpha)$$

needs one more coordinate along with $\vec{B}$

Let that be θ

(ψ, α, θ) (Clebsch coordinates (α, β, ℓ))

* Jacobian $J_c^{-1} = \vec{\nabla}\theta \cdot (\vec{\nabla}\psi \times \vec{\nabla}\alpha) = \vec{B} \cdot \vec{\nabla}\theta$

Field lines $\dfrac{d\ell}{B} = \dfrac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta} \left(= \dfrac{d\psi}{\vec{B}\cdot\vec{\nabla}\psi} \right) = J_c\, d\theta$

$$\langle A \rangle_b = \frac{\omega_b}{2\pi} \oint A \frac{J_c B}{v_\parallel} d\theta \qquad J = \oint J_c B v_\parallel \, d\theta$$

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\alpha = B_\psi \vec{\nabla}\psi + B_\alpha \vec{\nabla}\alpha + B_\theta \vec{\nabla}\theta$$

"contra" "covariant" rep.

$$B_\alpha = \vec{B} \cdot \frac{\partial \vec{x}}{\partial \alpha} = J_c \vec{B} \cdot (\vec{\nabla}\theta \times \vec{\nabla}\psi)$$

dual relation

$$B_\theta = \vec{B} \cdot \frac{\partial \vec{x}}{\partial \theta} = J_c \vec{B} \cdot (\vec{\nabla}\psi \times \vec{\nabla}\alpha) = J_c B^2$$

* $\alpha = q\theta - \varphi$
safety factor
($\vec{B} = \vec{\nabla}\varphi \times \vec{\nabla}\psi + q \vec{\nabla}\psi \times \vec{\nabla}\theta$)

$$\left\langle \frac{dJ}{dt} \right\rangle_b = \frac{\partial J}{\partial \psi} \left\langle \frac{d\psi}{dt} \right\rangle_b + \frac{\partial J}{\partial \alpha} \left\langle \frac{d\alpha}{dt} \right\rangle_b$$
$$= \frac{\partial J}{\partial \psi} \langle \vec{v}_{gc} \cdot \vec{\nabla}\psi \rangle_b + \frac{\partial J}{\partial \alpha} \langle \vec{v}_{gc} \cdot \vec{\nabla}\alpha \rangle_b$$

$$\vec{v}_{gc} = v_\parallel \hat{b} + \left(\frac{v_\parallel}{B}\right) \vec{\nabla} \times (\rho_\parallel \vec{B})$$

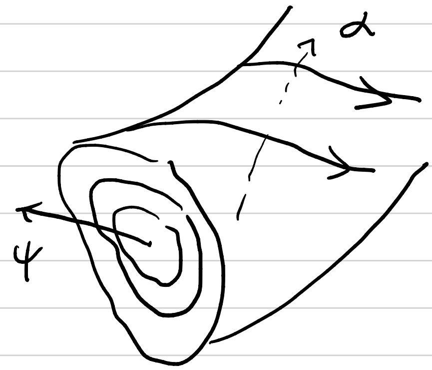
$$\vec{v}_{gc} \cdot \vec{\nabla}\psi = \left(\frac{v_\parallel}{B}\right) \vec{\nabla}\psi \cdot (\vec{\nabla} \times (\rho_\parallel \vec{B})) = \frac{v_\parallel}{B} \vec{\nabla} \cdot (\vec{\nabla}\psi \times \rho_\parallel \vec{B})$$
$$= \left(\frac{v_\parallel}{B}\right) \vec{\nabla} \cdot \left(\rho_\parallel B_\alpha (\vec{\nabla}\psi \times \vec{\nabla}\alpha) + \rho_\parallel B_\theta (\vec{\nabla}\psi \times \vec{\nabla}\theta) \right)$$
$$= \left(\frac{v_\parallel}{B}\right) \vec{\nabla} \cdot \left(\frac{\rho_\parallel B_\alpha}{J_c} \frac{\partial \vec{x}}{\partial \theta} - \frac{\rho_\parallel B_\theta}{J_c} \frac{\partial \vec{x}}{\partial \alpha} \right)$$
$$= \frac{v_\parallel}{J_c B} \left(\frac{\partial}{\partial \theta} (\rho_\parallel B_\alpha) - \frac{\partial}{\partial \alpha} (\rho_\parallel B_\theta) \right)$$

$$\langle \vec{v}_{gc} \cdot \vec{\nabla}\psi \rangle_b = \frac{\omega_b}{2\pi} \left[\cancel{\oint \frac{\partial}{\partial \theta} (\rho_\parallel B_\alpha) d\theta} - \frac{\partial}{\partial \alpha} \oint \rho_\parallel B_\theta \, d\theta \right]$$

$$= -\frac{\omega_b}{2\pi} \cdot \frac{\partial}{\partial \alpha} \oint \frac{m v_\parallel}{q \cancel{B}} J_c \cancel{B} \, d\theta$$

$$= -\frac{\omega_b}{2\pi} \frac{m}{q} \frac{\partial J}{\partial \alpha}$$

$$\langle \vec{v}_{gc} \cdot \vec{\nabla} \alpha \rangle_b = \frac{\omega_b}{2\pi} \frac{m}{q} \frac{\partial J}{\partial \psi}$$



$$\therefore \quad \left\langle \frac{dJ}{\partial \psi} \right\rangle_b = \left(\frac{\omega_b}{2\pi}\right)\left(\frac{m}{q}\right)\left[-\frac{\partial J}{\partial \psi}\frac{\partial J}{\partial \alpha} + \frac{\partial J}{\partial \alpha}\frac{\partial J}{\partial \psi} \right] = 0$$