

① Round-off error

노트 제목

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① Have a well-conditioned matrix, but the solution procedure is bad.

→ We can suggest improvement in algorithm.

② Have an ill-conditioned matrix which is close to being singular.

→ no hope if it is due to round-off error.

* Computers store floating point numbers as product of a fractional part f with d digits and exponential part 10^s .
 $0.1 \leq |f| < 1$

$$\text{ex)} \quad 27.648 \rightarrow \underline{0.87648} \times 10^2$$

$$100 \rightarrow \underline{0.1} \times 10^3$$

rounding is performed on f → round-off error

$$\text{ex) } \begin{cases} 0.01x_1 + x_2 = 1 \\ x_1 - x_2 = 0 \end{cases} \rightarrow \text{exact sol. } x_1 = x_2 = \frac{1}{1.01} = 0.990099\dots$$

Do this example prob. w/ a computer that carries

② significant figures → $x_1 = x_2 = 0.99$ (best sol.)

GE

$$\rightarrow \begin{cases} 0.01x_1 + x_2 = 1 \\ -101x_2 = -100 \end{cases}$$

$\underbrace{-0.101 \times 10^3 x_2}_{\rightarrow -0.10 \times 10^3 x_2} \rightarrow x_2 = 1$

$x_1 = 0$

wrong!

But do the same prob.

$$\left\{ \begin{array}{l} 1x_1 - x_2 = 0 \\ 0.01x_1 + x_2 = 1 \end{array} \right. \xrightarrow{\text{GE}} \quad \begin{array}{l} x_1 - x_2 = 0 \\ 1.01x_2 = 1 \rightarrow x_2 = 1 \end{array} \quad \begin{array}{l} x_1 = 1 \\ \text{accurate!} \end{array}$$

pivot

\Rightarrow choice of pivots is very important

By row changes, make the pivot the largest element
in the column.

ex) $\left\{ \begin{array}{l} 1x_1 + 100x_2 = 100 \\ 1x_1 - x_2 = 0 \end{array} \right. \xrightarrow{\text{GE}} \xrightarrow{\text{BS}} \quad \begin{array}{l} x_1 = 0 \\ x_2 = 1 \end{array} \quad \} \text{wrong}$

\Rightarrow we need scaling.

Normalize each eq. in advance s.t. the largest element in each row is 1.

$$\xrightarrow{\text{Scaling}} \begin{array}{l} \boxed{0.01} x_1 + x_2 = 1 \\ x_1 - x_2 = 0 \end{array} \xrightarrow{\text{Pivot}} \begin{array}{l} x_1 - x_2 = 0 \\ 0.01x_1 + x_2 = 1 \end{array} \xrightarrow{\text{GE}} \xrightarrow{\text{BS}} x_1 = x_2 = 1$$

② example of ill-conditioned matrix

$$\left\{ \begin{array}{l} x_1 + x_2 = 2 \\ x_1 + 1.0001x_2 = 2 \end{array} \right. \xrightarrow{\text{exact sol.}} \left\{ \begin{array}{l} x_1 = 2 \\ x_2 = 0 \end{array} \right.$$

perturb the RHS

$$\left\{ \begin{array}{l} x_1 + x_2 = 2 \\ x_1 + 1.0001x_2 = 2.0001 \end{array} \right. \xrightarrow{\text{exact sol.}} \left\{ \begin{array}{l} x_1 = 1 \\ x_2 = 1 \end{array} \right.$$

slight change in $b \rightarrow$ large change in x .

→ No numerical method can avoid this sensitivity
to small perturbations.

Q : Given $Ax = b$, what is the change in x
c.r.t. a change in the parameter of the system b ?

$$Ax = b$$

error in b , $\delta b \rightarrow$ find δx .

$$A(x + \delta x) = b + \delta b \quad & Ax = b$$

$$\rightarrow A\delta x = \delta b \rightarrow \delta x = A^{-1}\delta b$$

$$|\delta x| = |A^T \delta b| \leq \|A^{-1}\| |\delta b| \quad (\text{Schwartz inequality})$$

$$Ax = b \rightarrow |\underline{b}| = |Ax| \leq \underline{\|A\| |x|}$$

$$\rightarrow \frac{|\delta x|}{\|A\| |x|} \leq \frac{\|A^T\| |\delta b|}{|\underline{b}|}$$

$$\|A\|^2 = \lambda_{\max}(AA^T)$$

$\|A\|$: norm

$$\rightarrow \boxed{\frac{|\delta x|}{|x|} \leq \underbrace{\|A\| \|A^{-1}\|}_{\text{condition number of } A} \frac{|\delta b|}{|\underline{b}|}}$$

condition number of $A = \frac{\lambda_{\max}}{\lambda_{\min}}$
amplification factor

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 + 1.0001x_2 = 2 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.0001 \end{pmatrix} \rightarrow \|A\| = 2.00005 \approx 2$$

$$A^{-1} = \begin{pmatrix} 10001 & -10000 \\ -10000 & 10000 \end{pmatrix} \rightarrow \|A^{-1}\| = 20,000$$

$$\rightarrow \frac{|\delta x|}{|x|} \leq 2 \times 20000 \frac{|\delta b|}{|b|} = \underbrace{40,000} \frac{|\delta b|}{|b|}$$

$\gg 1$

matrix A is stiff or ill-conditioned.

If the condition number times the order of round-off accuracy of machine is order of 1. \rightarrow concerned

⑥

Cayley - Hamilton theorem

Every matrix satisfies its own characteristic eqn.

$$\begin{matrix} A \\ \textcircled{n \times n} \end{matrix} : P(\lambda) = \det(A - \lambda I) = 0 \quad \downarrow$$

$$\rightarrow \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_{n-1} \lambda + c_n = 0$$

$$\Rightarrow \boxed{A^n + c_1 A^{n-1} + c_2 A^{n-2} + \dots + c_{n-1} A + c_n I = 0} \quad n: \text{inter}$$

$$P(A) = 0$$

$$\underbrace{T}_{2 \times 2} = f(D) =$$

$$= c_0 I + c_1 D$$

$$\boxed{\alpha I + \beta D + \gamma D^2 + \delta D^3 + \dots + c_2 D^2 + c_3 D^3 + \dots + c_n D^n}$$

- Any power of an $n \times n$ matrix can be represented as a polynomial of degree up to $n-1$.

Ex) $A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$ $\sqrt{A} = ?$

$\lambda_1 = 1, \lambda_2 = 4$

$$\sqrt{A} = \alpha_0 I + \alpha_1 A \rightarrow \begin{aligned} \sqrt{\lambda_1} &= \alpha_0 + \alpha_1 \lambda_1 \\ \sqrt{\lambda_2} &= \alpha_0 + \alpha_1 \lambda_2 \end{aligned} \rightarrow \begin{aligned} \alpha_0 &= 2/3 \\ \alpha_1 &= 1/3 \end{aligned}$$

$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$

$P(A) = (A - \lambda_1 I)(A - \lambda_2 I) = 0$

$\rightarrow \sqrt{A} = \frac{2}{3}I + \frac{1}{3}A = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 2 \end{pmatrix}$

- Any function of a matrix can be represented as a polynomial of degree up to $n-1$.

(x) $A = \begin{pmatrix} \pi & 3\pi \\ 2\pi & 2\pi \end{pmatrix}$, $\cos A = ?$ $\lambda_1 = k\pi, \lambda_2 = -\pi$

$$\cos A = \alpha_0 I + \alpha_1 A \rightarrow \cos \lambda_1 = \alpha_0 + \alpha_1 \lambda_1 \quad \downarrow \quad \alpha_0 = -\frac{3}{5}$$

$$\cos \lambda_2 = \alpha_0 + \alpha_1 \lambda_2 \quad \rightarrow \quad \alpha_1 = \frac{2}{5\pi}$$

$$\rightarrow \cos A = \begin{pmatrix} -\frac{1}{5} & \frac{6}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$\text{ex)} \quad A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

\sqrt{A} ?

$$\sqrt{A} = \alpha_0 A + \beta_0 I$$

$$\sqrt{\lambda} = \alpha_0 \lambda_1 + \beta_0$$

$$\lambda_1 = \lambda_2 = 2$$

double root

$$P(\lambda) = (\lambda - \lambda_1)^2 (\lambda - \lambda_2) \cdots (\lambda - \lambda_{n-1})$$

$$P' \Big|_{\lambda=\lambda_1} = 2(\lambda - \lambda_1) \underbrace{\cdots + \cdots + \cdots}_{\lambda=\lambda_1} = 0$$

$$\left\{ \begin{array}{l} \sqrt{\lambda} = \alpha_0 \lambda + \beta_0 \\ \frac{1}{2} \lambda^{-1/2} = \alpha_0 \end{array} \right. \rightarrow \begin{array}{l} \alpha_0 \\ \beta_0 \end{array} \rightarrow \sqrt{A} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

$$\begin{array}{c} Ax = b \\ \cancel{x = A^{-1}b} \end{array}$$

Ch 1. Interpolation