

Lecture 5 (chapter 2)

Conservation equation

system

Draw a control surface :

① it defines what is inside and outside.

② Can be fixed or moving.

③ Can be fully inside the fluid, partially inside, etc...

④ If mass is allowed to come in and out \Rightarrow open system

otherwise \Rightarrow closed system

(the same as thermo)

Extensive property

A property that scales with
the system size.

If the system is homogeneous,

making the system size V_2
cuts the value of an extensive
property by V_2 .

Doubling the size, doubles the prop.
etc.

Ex) Volume, mass, momentum,
energy (mechanical & internal),
entropy, # of moles, etc.

Intensive properties

Independent of system size

Temperature, pressure, etc...

Specific properties

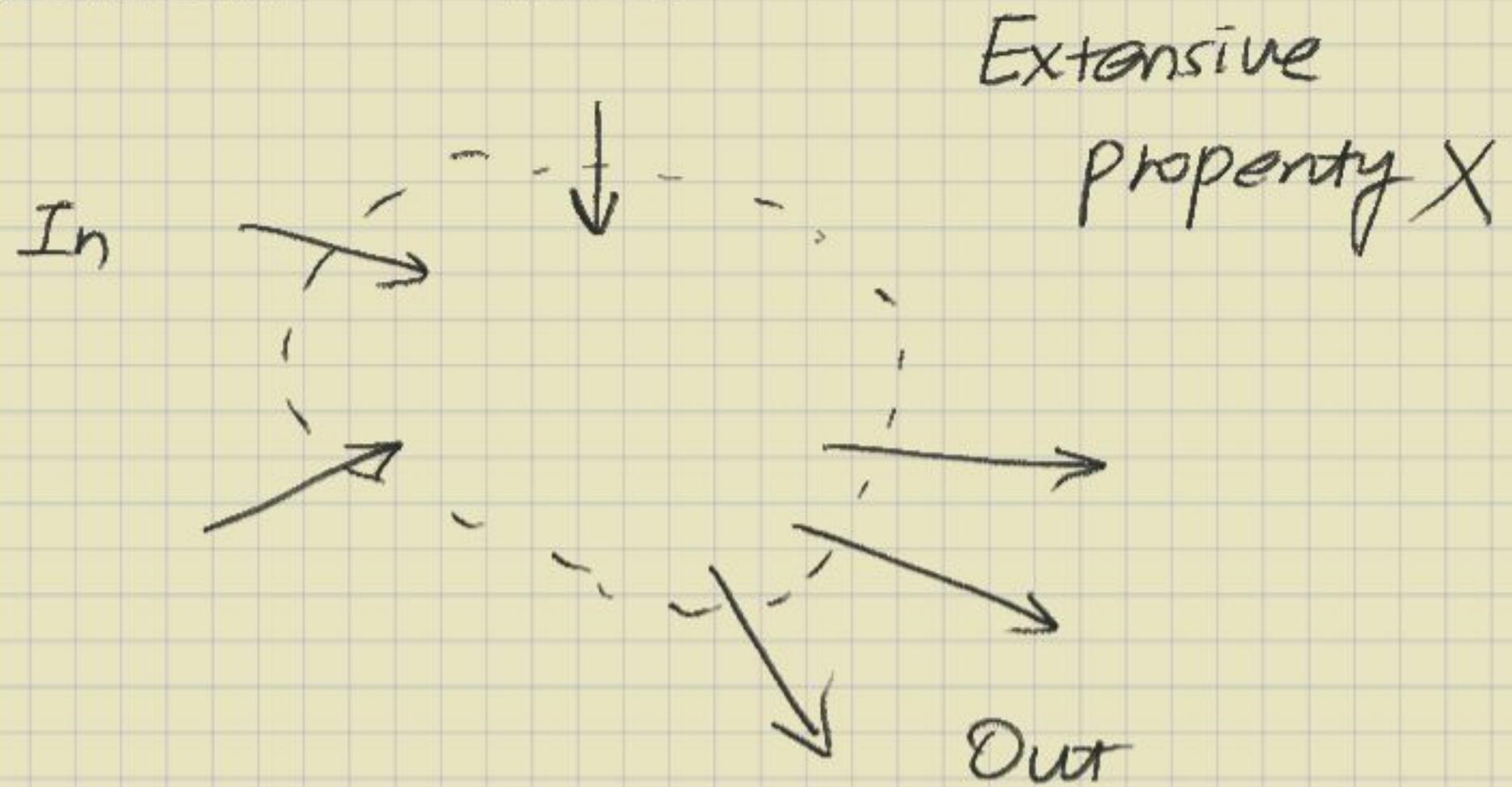
Ratio of two extensive properties

density, concentration, specific \bar{e}

Note: Some are "funny"

velocity = momentum per unit mass.

Balance laws



$$\frac{dX}{dt} = \underbrace{X_{in} - X_{out}}_{\text{fluxes} \times \text{Area}} + \underbrace{X_{cre} - X_{des}}_{\text{rates}}$$

→ Sometimes

$$= X_{gen}$$

Note) Some properties are conserved:
e.g. mass & energy

→ It does not mean $\frac{dX}{dt} = 0$

it means $X_{cre} - X_{des} = 0 = X_{gen}$

Steady-state problem

The properties of the system do not change with time

$$\Rightarrow \frac{dx}{dt} = 0 \quad (\text{true also for the intensive variable})$$

• Lumped parameter model

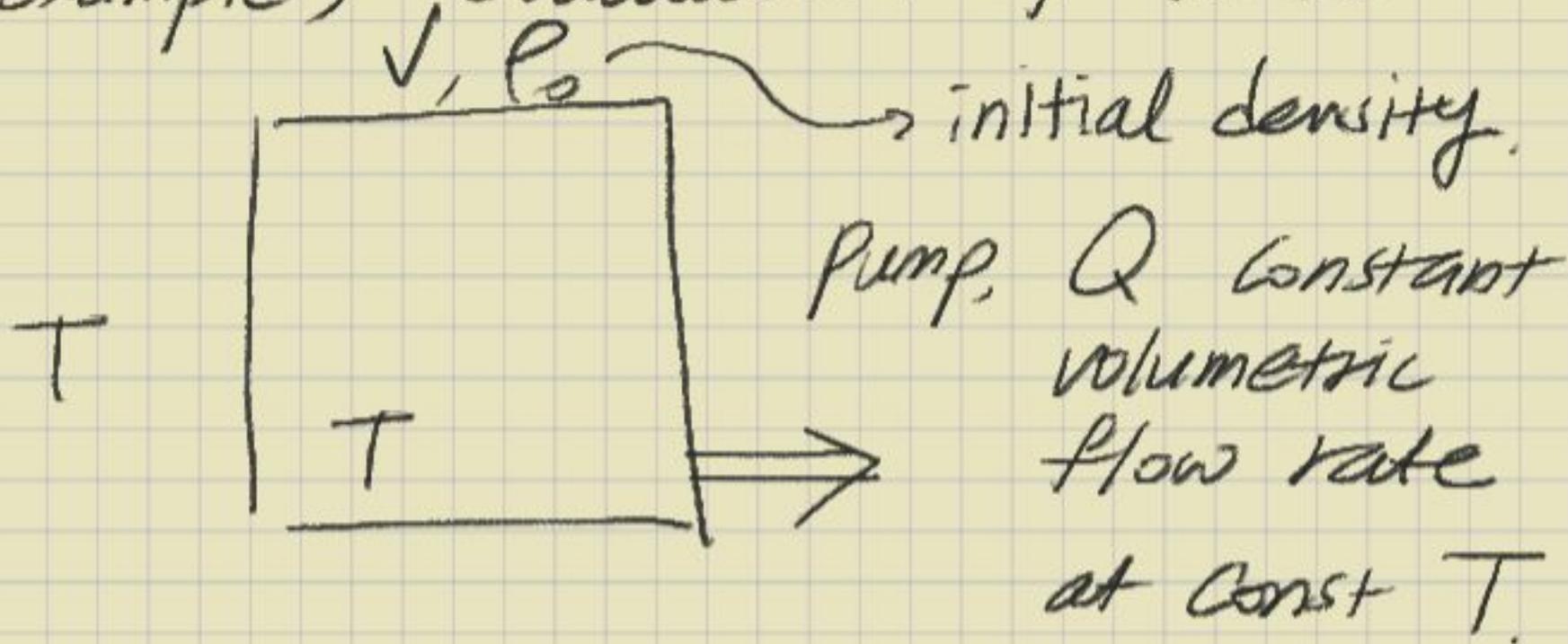
We deal only with the total quantity X rather than how a specific quantity (e.g. $\frac{X}{V}$ or $\frac{X}{M}$) is distributed in a system.

They give: ODEs for unsteady problems
algebraic equations for st. st. problems

Mass balance

$$\frac{dM}{dt} = m_{in} - m_{out}$$

Example) evacuation of tank



Question: how does pressure change with time?

$$\frac{dm}{dt} = -m_{out} = -\rho Q$$

($\rho \neq \text{const}$)

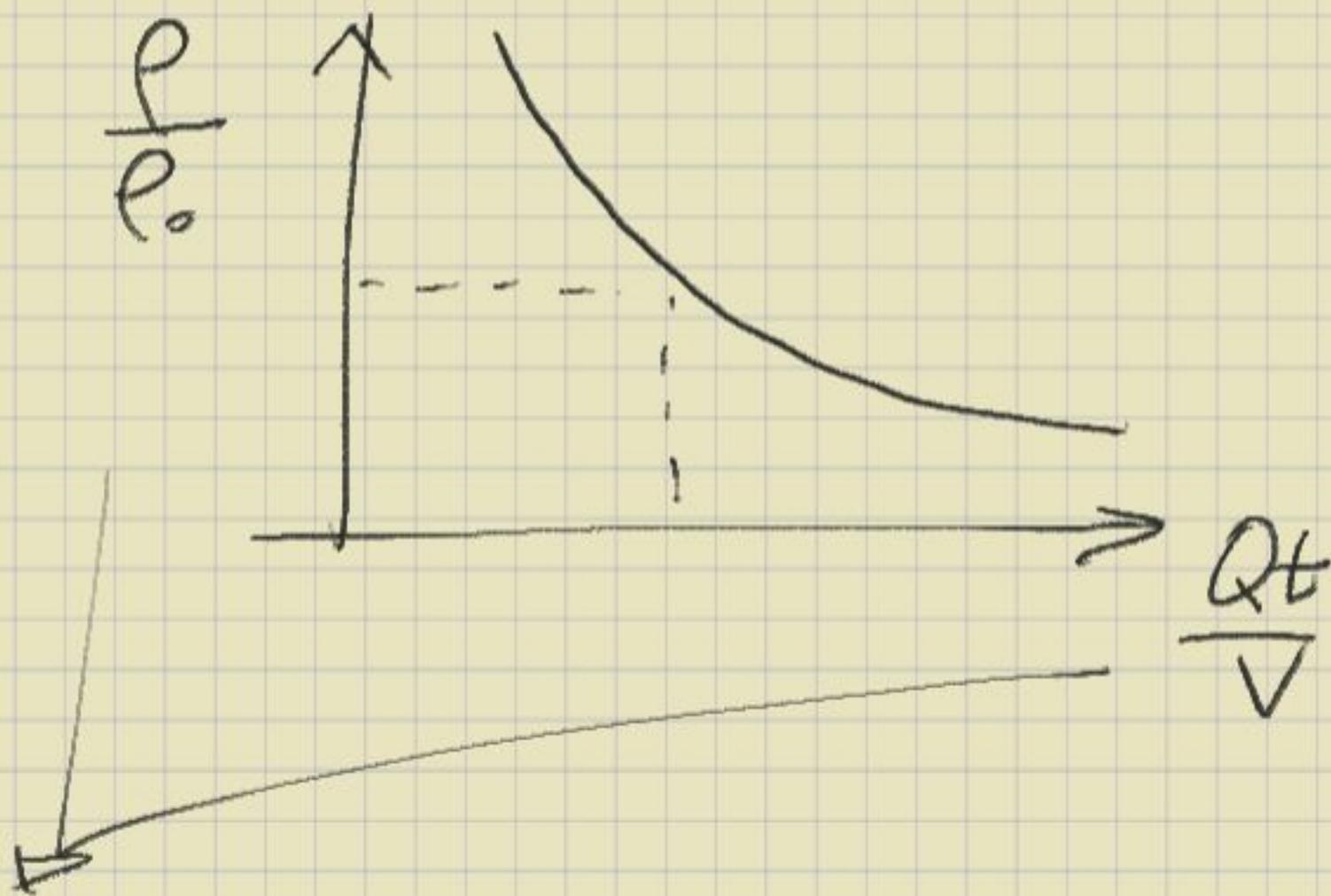
$$\frac{d\rho V}{dt} = -\rho Q \rightarrow \frac{d\rho}{dt} = -\rho \frac{Q}{V}$$

$$\Rightarrow \rho = \rho_0 e^{-\frac{Qt}{V}}$$

But for an ideal gas

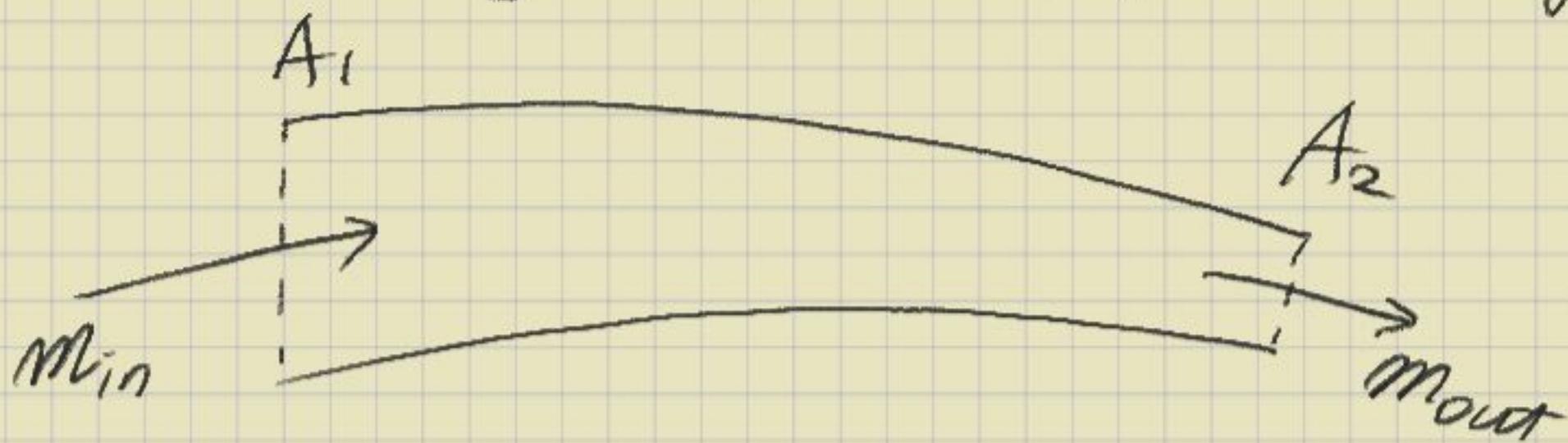
$$PV = nRT \Rightarrow \frac{P}{\rho} = \frac{M_w RT}{\rho} = \text{Constant}$$

$$\therefore \frac{P}{P_0} = \frac{\rho}{\rho_0} = e^{-\frac{Q_t}{V}}$$



Note dimensionless variables.

Flow through tapered pipe, steady



$$\frac{dM}{dt} = 0 = m_{in} - m_{out}$$

$$0 = \rho_1 A_1 V_1 - \rho_2 A_2 V_2$$

If flow is incompressible, $\rho_1 = \rho_2$

$$\bullet A_1 V_1 = A_2 V_2$$

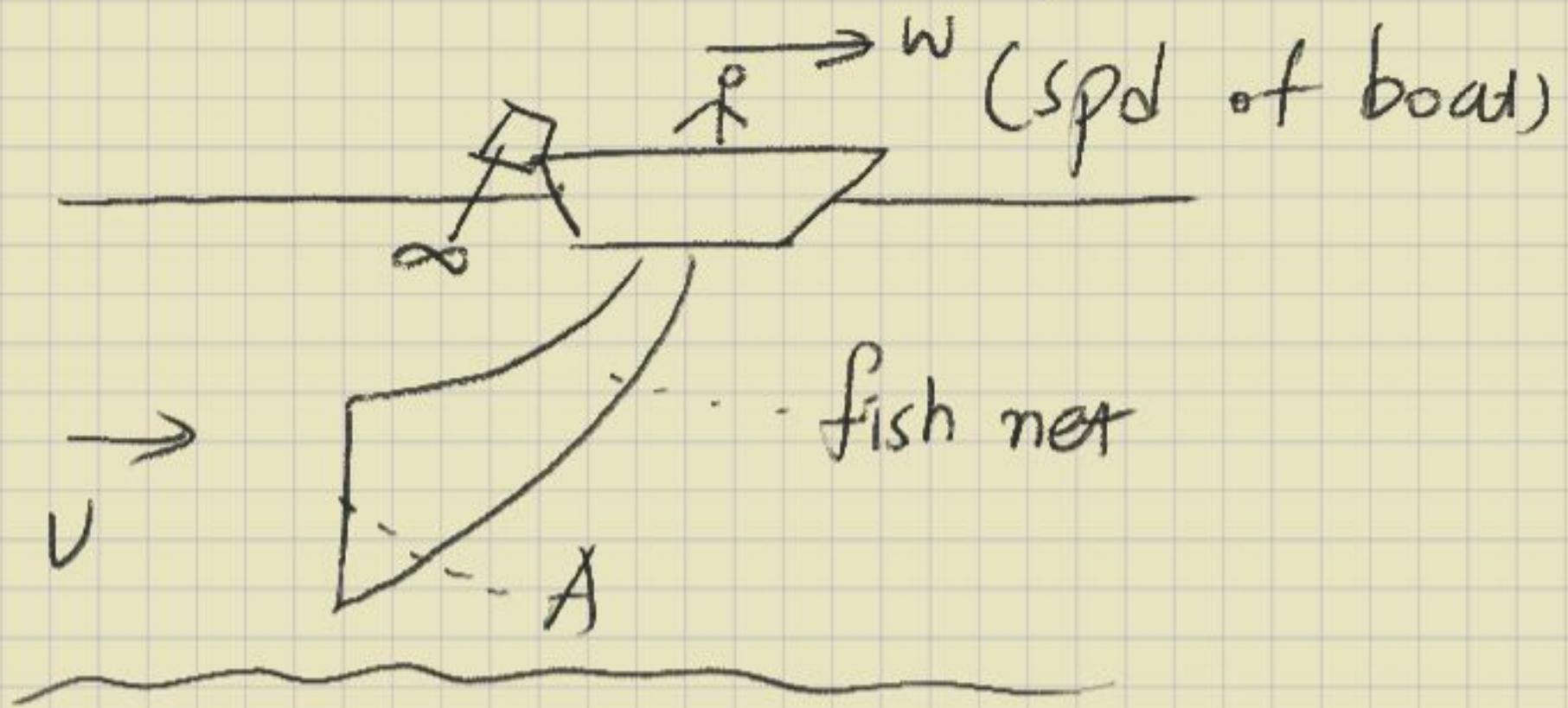
$$\bullet \frac{V_1}{V_2} = \frac{A_2}{A_1}$$

higher velocity
in smaller cross section

And $Q = Av = \text{const}$

Volumetric flow rate.

Flow rate through moving surface.



flow rate through surface A

$$Q = A(v - w)$$

relative speed.

(Negative, i.e. outwards
if $w > v$)

Momentum balance

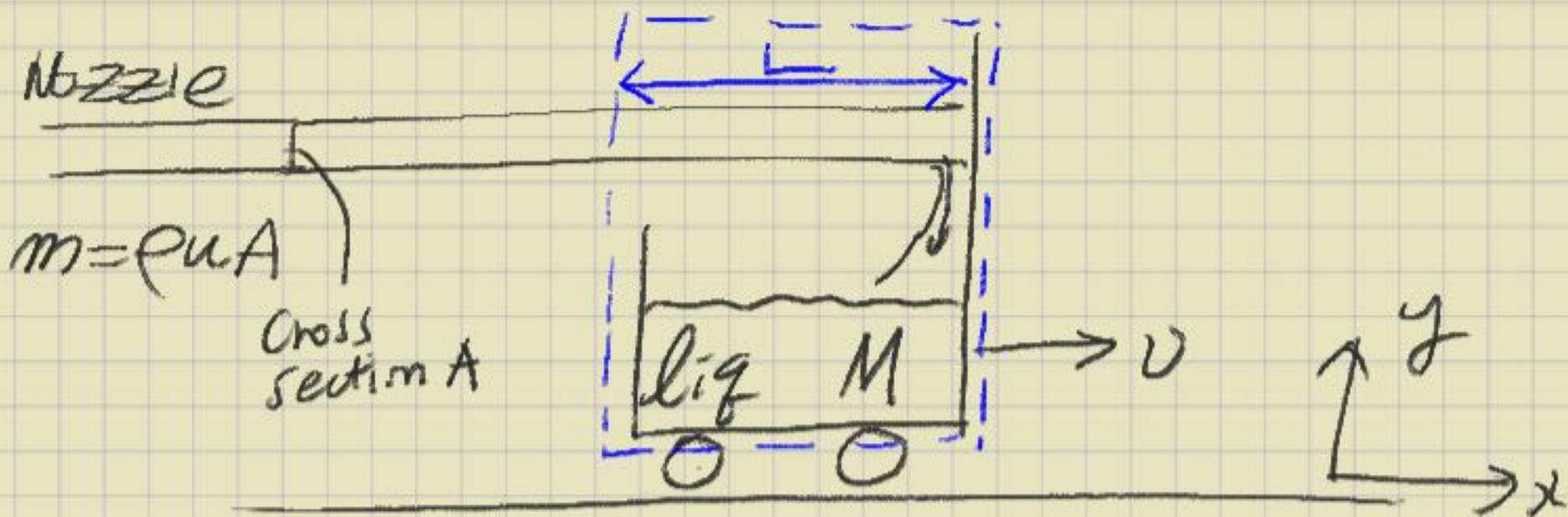
$$m \underline{\underline{a}} = \sum \underline{\underline{f}}^{\text{ext}} \quad (\text{closed system})$$

$$\frac{d}{dt} \underline{\underline{M}} = \underline{\underline{M}}_{\text{in}} - \underline{\underline{M}}_{\text{out}} + \sum \underline{\underline{f}}^{\text{ext}}$$

Note) $\underline{\underline{M}}$ is a vector:

hence this is a vector balance

• Moving trolley problem



trolley ($L \equiv \text{const}$)

or stationary ($\frac{dL}{dt} = v$)

① Moving C.V. ($L \equiv \text{const}$) case

Only non trivial component : x

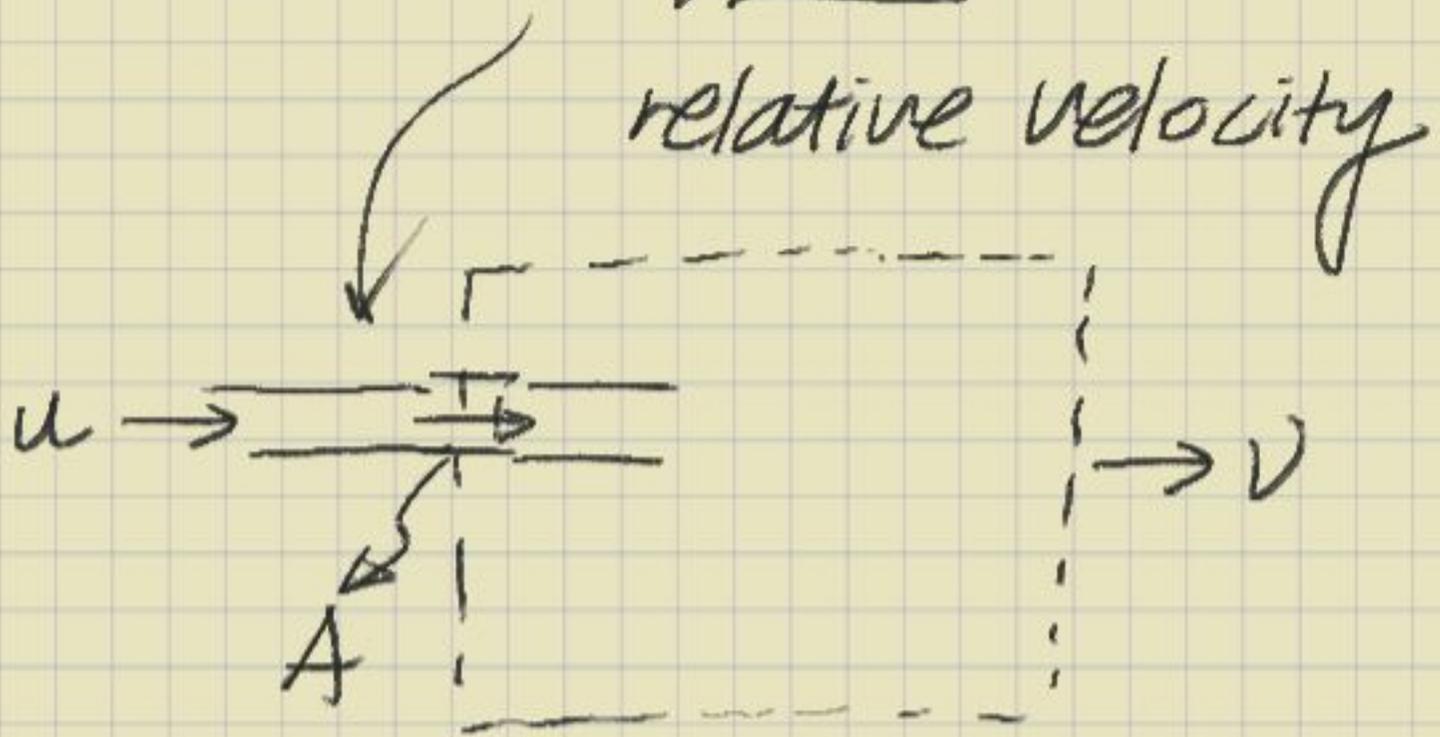
Find acceleration $a \equiv \frac{dv}{dt}$

Mass of trolley depends on time

$$M = M(t)$$

total mass : $M + \rho A L$

$$\frac{d}{dt} (M + \rho A L) = \rho A \underbrace{(u - v)}$$



$$\therefore \frac{dM}{dt} = \rho A (u - v)$$

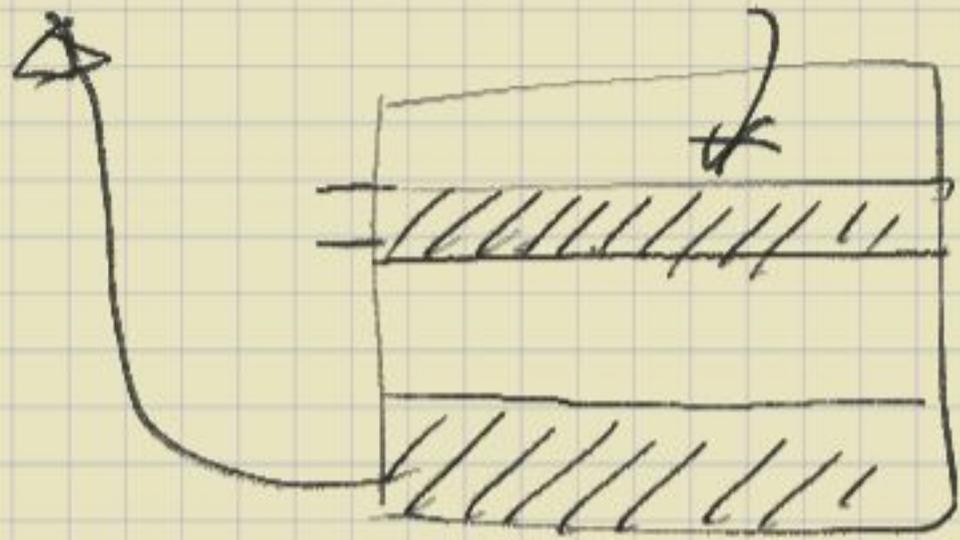
If control volume is moving
with velocity v .

$$\left(\frac{dL}{dt} = 0 \right)$$

Momentum balance:

total momentum in the box is

$$Mv + \rho A L u$$



mass flux \times
momentum per
unit mass

$$= \rho A (u-v) \times u$$

$$\frac{d}{dt} (Mv + \rho A L u) = \rho u A (u-v)$$

\uparrow Constant.

$$\left(\frac{dM}{dt} \right) v + M \frac{dv}{dt} = \rho u A (u-v)$$

from mass balance

$$\rho v A (u-v) + M \left(\frac{dv}{dt} \right) = \rho u A (u-v)$$

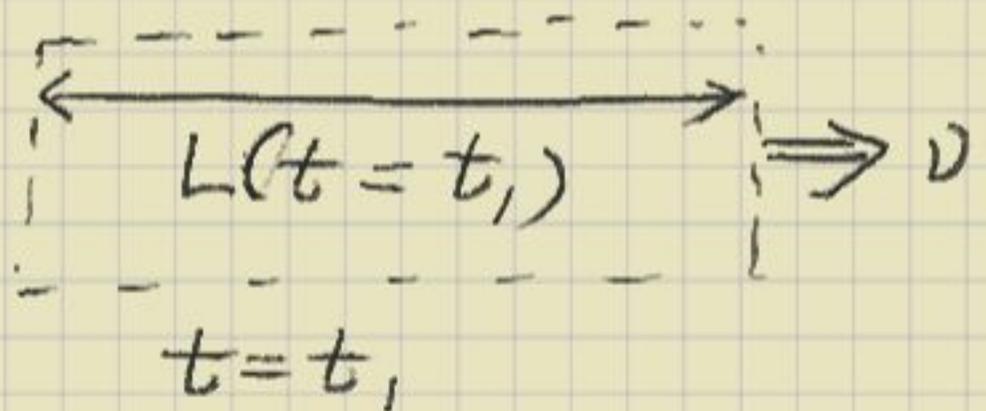
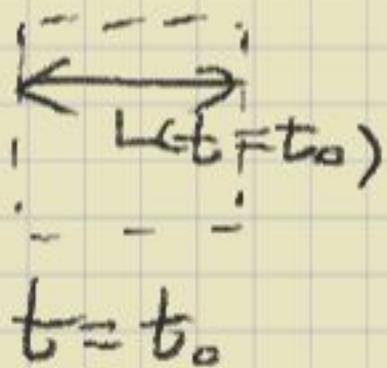
$$Ma = \rho u A (u-v) - \rho v A (u-v) = \rho A (u-v)^2$$

∴ the acceleration of the trolley is

$$a = \frac{\rho A}{M} (u - v)^2$$

The same problem w/ diff. C.V.

② Stationary C.V. $\frac{dL}{dt} = v$



Total mass: $M + \rho AL$

$$\frac{d}{dt} (M + \rho AL) = \underbrace{\rho u A}_{\text{In}} \quad \Rightarrow \quad \rho u$$

$$\Rightarrow \frac{dM}{dt} + \rho A \left(\frac{dL}{dt} \right) = \rho u A$$

"v"

$$\frac{dM}{dt} = \rho A (u - v)$$

same as before

$$\text{Total momentum: } \underbrace{Mv}_{\text{from Trolley}} + \underbrace{\rho A L u}_{\text{from water jet}}$$

from
Trolley

from
water jet

$$\frac{d}{dt} (Mv + \rho A L u) = \rho A u^2 \quad \left(= \underbrace{\rho A u}_{\text{mass flux}} \times \underbrace{\vec{u}}_{\text{mom. p.u.m}} \right)$$

$$v \frac{dM}{dt} + M \frac{dv}{dt} + \rho A u \left(\frac{dL}{dt} \right) = \rho A u^2$$

= v

$$Ma = \rho A u^2 - v \rho A (u - v) - \rho A u v$$

$$Ma = \rho A (u - v)^2$$

$$a = \frac{\rho A}{M} (u - v)^2$$

⇒ the same as before.

Energy balance

Recall from thermodynamics

$$E = E + U + K$$

total internal potential kinetic

In a closed system \rightarrow sign convention opposite to the book

$$\Delta E = \Delta Q + \Delta W$$

change in E heat flow into system work done on the system

\Rightarrow the first law of Thermodynamics

In a flowing system (open)

$$\frac{dE}{dt} = E_{in} - E_{out}$$

Convenient to work with
specific quantities

$$e \equiv \frac{E}{M} = \frac{E}{M} + \frac{U}{M} + \frac{K}{M}$$

$$= e + gz + \frac{u^2}{2}$$

$$\begin{cases} U = Mgz \\ K = \frac{1}{2}Mu^2 \end{cases}$$

E_{in} & E_{out} have ^① a convected

component and ^② a component

associated with doing work

or ^③ putting heat into the system.

Convected component $\mathcal{E}_{in}^c, \mathcal{E}_{out}^c$

$$\mathcal{E}_{in}^c = (\text{mass flow rate})_{in} (\text{specific } E)_{in}$$

$$\mathcal{E}_{out}^c = (\text{mass flow rate})_{out} (\text{specific } E)_{out}$$

mass flow rate ρQ , therefore

$$\mathcal{E}_{in}^c = (\rho Q)_{in} \left(e + gz + \frac{u^2}{2} \right)_{in}$$

$$\mathcal{E}_{out}^c = (\rho Q)_{out} \left(e + gz + \frac{u^2}{2} \right)_{out}$$

rate of working onto system

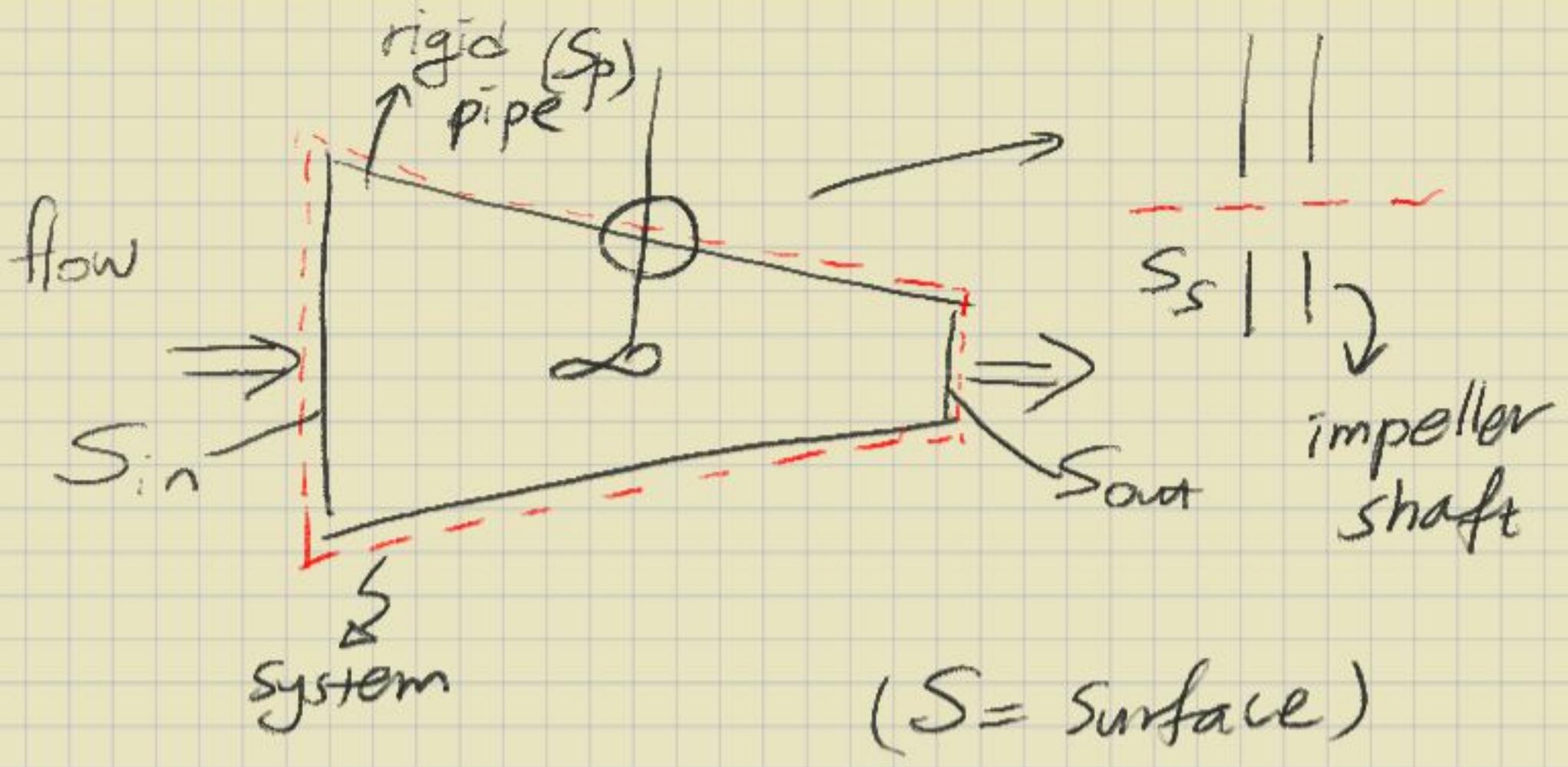
$$\frac{d\mathcal{E}}{dt} = \mathcal{E}_{in}^c - \mathcal{E}_{out}^c + \dot{Q} + \dot{W}$$

rate of heat flow

into system

\dot{W} is the trouble spot:

pay attention!



work = force \times displacement

rate of working = force \times velocity

(power)

$$\vec{f} \cdot \vec{v}$$

dot product

- No power is coming into the system through S_p , because they are rigid.
- Work is done through S_s and through the open flow surfaces S_{out} & S_{in}

- Work through S_s is usually called "shaft" work.

- In lump-parameter system, normally it is given or it must be calculated based on specs.

Example)

① I know everything at S_{in}
the power to a pump at S_s ,
I want to compute flow at S_{out} .

② I know my inflow condition at S_{in}
and desired condition at S_{out} ,
What power do I need to
drive a pump?

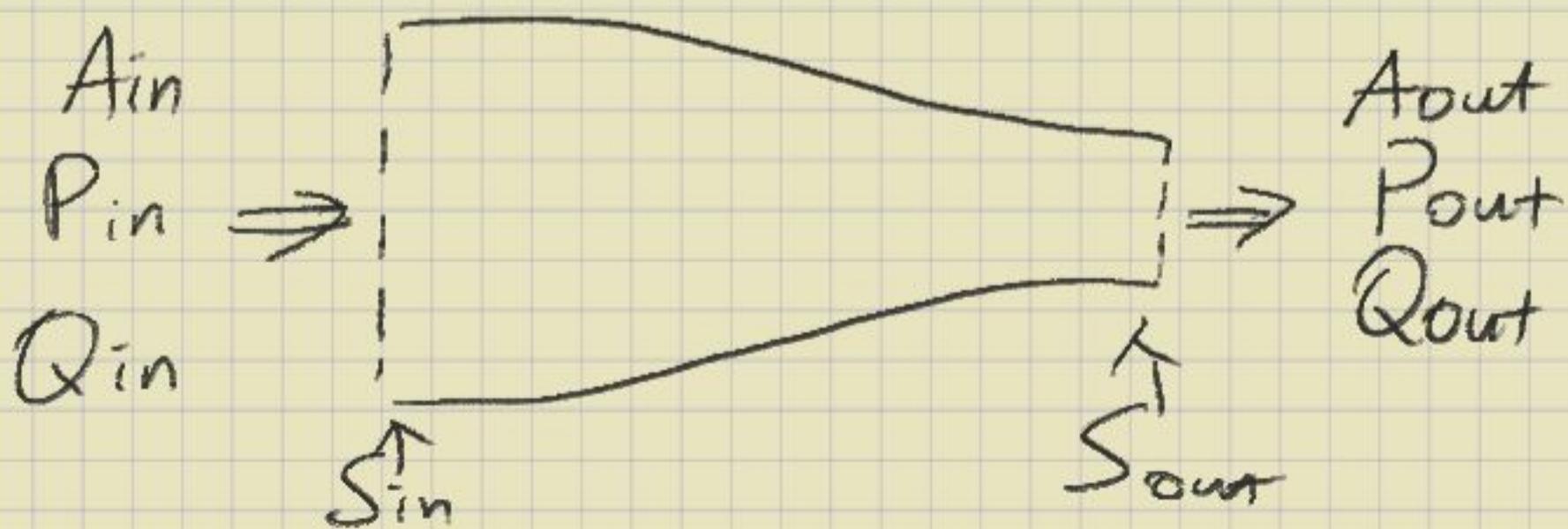
Work through the open surface

at this stage, slight SIMPLIFICATION

1. we take the control surface to be perpendicular to the inflow velocity

2. We assumed that pressure is the only force working at those surfaces.

This is often (but not always) a good approximation.



$$\dot{W}_{in}^{flow} = (\text{force} \times \text{velocity})_{in}$$

$$= (PA)_{in} \left(\frac{Q}{A}\right)_{in} = (PQ)_{in}$$

Similarly

$$\dot{W}_{out}^{flow} = (PQ)_{out}$$

Putting everything together

$$\frac{dE}{dt} = E_{in}^c - E_{out}^c + \dot{Q} + \dot{W}_{shaft} + \dot{W}_{in}^{flow} - \dot{W}_{out}^{flow}$$

$$\frac{d}{dt} M \left(e + gz + \frac{u^2}{2} \right) =$$

$$(PQ)_{in} \left(e + gz + \frac{u^2}{2} \right)_{in}$$
$$- (PQ)_{out} \left(e + gz + \frac{u^2}{2} \right)_{out}$$

$$+ \dot{Q}$$

$$+ \dot{W}^{shaft}$$

$$+ (PQ)_{in}$$

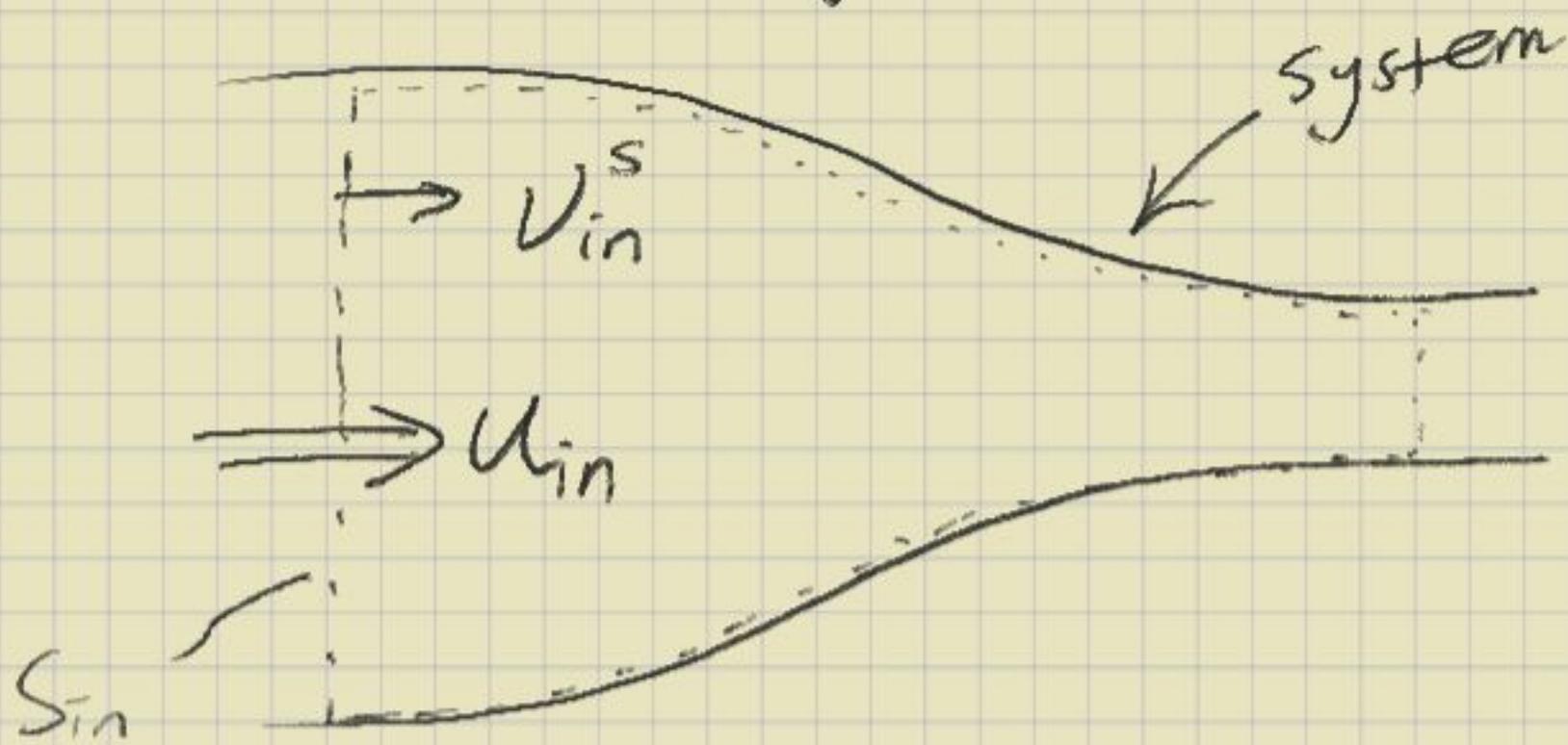
$$- (PQ)_{out}$$

→ can be rearranged

(also in Wilkes eq. 2.6)

$$\frac{d}{dt} M \left(e + gz + \frac{u^2}{2} \right) = (PQ)_{in} \left(e + gz + \frac{u^2}{2} + \frac{P}{\rho} \right)_{in}$$
$$- (PQ)_{out} \left(e + gz + \frac{u^2}{2} + \frac{P}{\rho} \right)_{out}$$
$$+ \dot{Q} + \dot{W}^{shaft}$$

Effect of moving surfaces



↑
surface moves at V_{in}^s ;
fluid moves at U_{in}

Convection of mass: $\rho(U_{in} - V_{in}^s)$

$$Q = (U_{in} - V_{in}^s)A$$

Work done on S_{in} : force \times velocity of fluid.

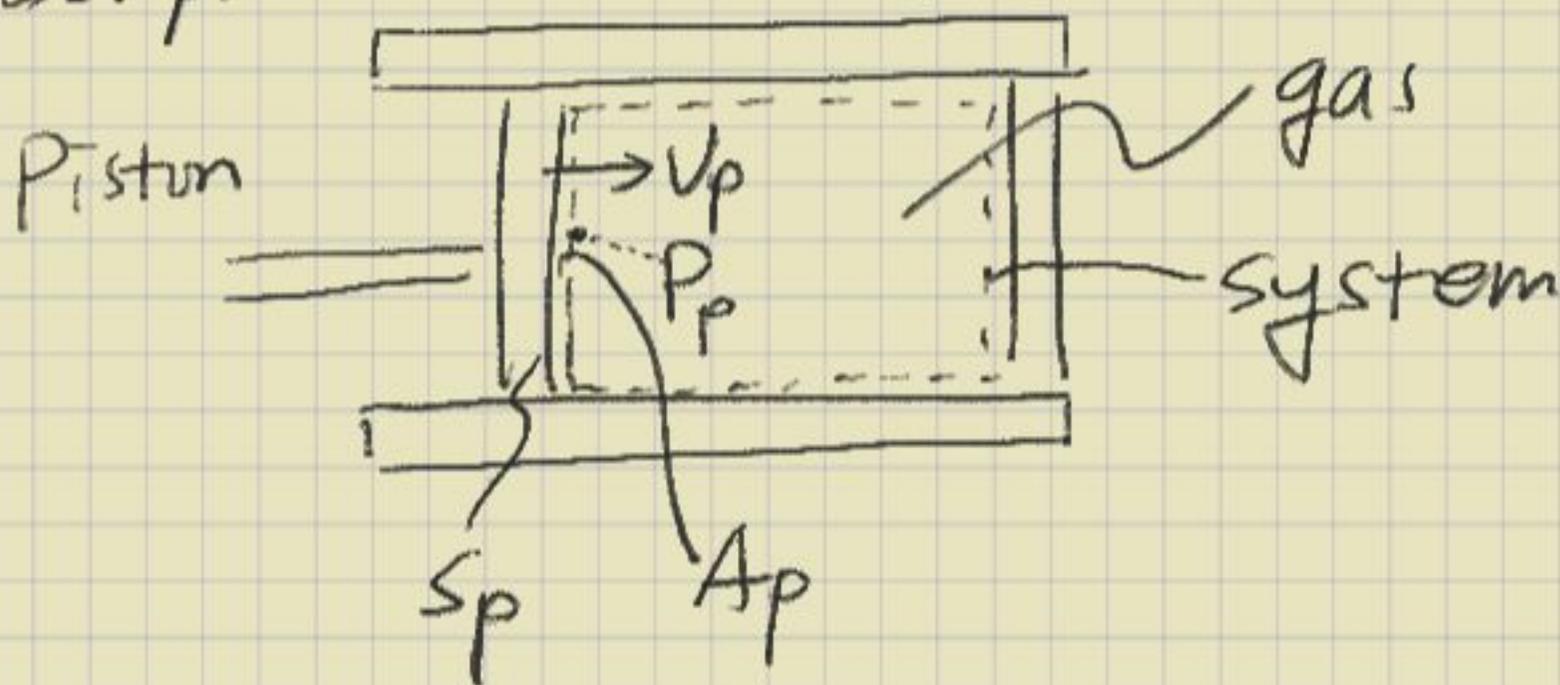
$$\dot{W}_{in}^{flow} = (PAU)_{in} \neq \underbrace{(PQ)_{in}}_{\text{fixed C.V.}}$$

unless $V_{in}^s = 0$

fixed C.V.

There is also another way of doing work if we allow moving surfaces: work done on impermeable moving surfaces.

Example



v_p : piston velocity = gas velocity at piston face.

P_p : gas pressure at piston face.

A_p : piston area

This is called compression or expansion work.

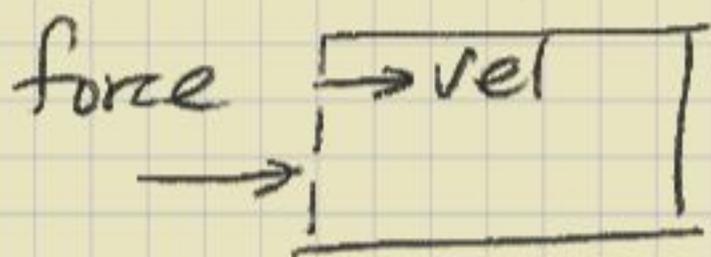
$$\dot{W}^{qe} = \text{force} \times \text{velocity}$$

$$= P_p A_p v_p$$

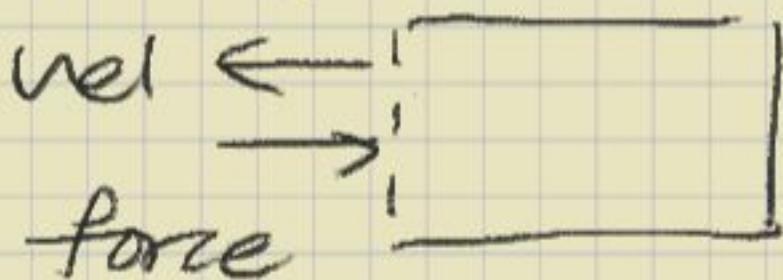
c.f.) Positive for compression

(force & velocity

point in the same dir)



Negative for expansion

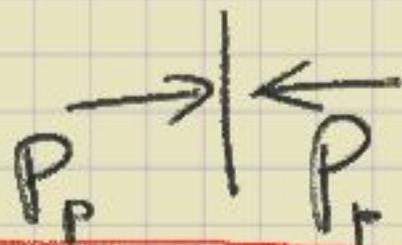


slight assumption :

the normal force on the two side

of S_p is equal and

its value is P_p .



Underlying the total energy balances are two separate between:

• mechanical $E : U + K$

• internal $E : E$

$$\frac{d}{dt}(Me) = (PQ)_{in} e_{in} - (PQ)_{out} e_{out}$$

$$+ \dot{Q} + \dot{C}$$

They are not conserved when they are treated separately

↳ rate of conversion of mechanical E into internal E

$$\begin{aligned} \frac{d}{dt} \left(gz + \frac{u^2}{2} \right) = & (PQ)_{in} \left(gz + \frac{u^2}{2} + \frac{P}{\rho} \right)_{in} \\ & - (PQ)_{out} \left(gz + \frac{u^2}{2} + \frac{P}{\rho} \right)_{out} \\ & + W_{shaft} + W_{c/e} \\ & - \dot{C} \end{aligned}$$

open surface not moving

What is \dot{C} ?

It is associated with

how the internal energy

(including pressure) combine with

local change of shape and

density to interchange

mechanical and thermal energy.

- It has two parts

$$\dot{C} = \dot{F} + \dot{R}$$

..... dissipation

* irreversible $\dot{F} \geq 0$ (but this is not precise...)

..... So called "work"

* reversible $\dot{R} \geq 0$ → Compress gas
 $\dot{R} < 0$ on cylinder

① Dissipation \mathcal{F} is associated w/
friction (viscosity);

and it is irreversible, i. e. $\mathcal{F} \geq 0$

and it always converts
mechanical E into thermal E .

It can be associated with
changes in shape or density.

② Reversible conversion of mechanical
 E into internal E .

For most fluids, this is associated
with changes in density.

It is the famous " $-pdv$ " term
in some embodiment of the 1st
law of thermodynamics, where it is
some times (inappropriately !!) called work.

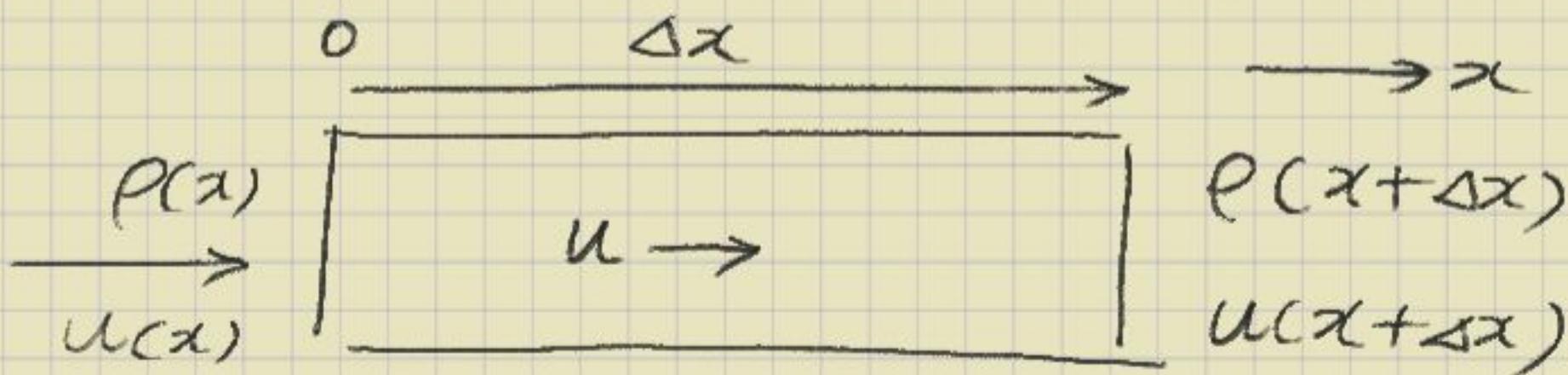
Here p is pressure, $v = 1/\rho$ is specific volume, and d denotes differential amount.

Let's call this whole term R .

R can be > 0 or < 0 .

- In a simple system, where p & v (or ρ) depend only on coordinate x

$$R = \int_0^{\Delta x} -P \left(\frac{dv}{dx} \right) dx$$



$u(x)$ is velocity

If the density is constant, $R=0$

$\frac{du}{dx}$ is related to changes in density $\frac{d\rho}{dx}$.

$$0 = \rho(x)Au(x) - \rho(x+\Delta x)Au(x+\Delta x)$$

$$0 = \rho(x)u(x) - \rho(x+\Delta x)u(x+\Delta x)$$

$$\Rightarrow \frac{d}{dx}(\rho(x)u(x)) = 0$$

$$u \frac{d\rho}{dx} + \rho \frac{du}{dx} = 0$$

$$\Rightarrow \frac{du}{dx} = -\frac{u}{\rho} \frac{d\rho}{dx}$$

• Let's take a look at steady ($\frac{d}{dt}=0$)

internal & mechanical E balance

$$0 = (\rho Q)_{in} e_{in} - (\rho Q)_{out} e_{out} + \dot{Q} + \mathcal{F}_p$$

$$0 = (\rho Q)_{in} \left(gz + \frac{u^2}{2} + \frac{p}{\rho} \right)_{in} - (\rho Q)_{out} \left(gz + \frac{u^2}{2} + \frac{p}{\rho} \right)_{out} + \dot{W}^{shaft} + \dot{W}^{c/e} - \dot{F}$$

A mass balance gives

$$0 = (\rho Q)_{in} - (\rho Q)_{out}$$

$$\Rightarrow (\rho Q)_{in} = (\rho Q)_{out} = \rho Q$$

Therefore, it is convenient to define

$$w \equiv \frac{\dot{W}^{shaft} + \dot{W}^{c/e}}{\rho Q} \quad \text{work per unit mass of fluid}$$

$$f \equiv \frac{\dot{F}}{\rho Q} \quad \text{dissipation per unit mass of fluid}$$

And

$$0 = \left(gz + \frac{u^2}{2} + \frac{p}{\rho} \right)_{in} - \left(gz + \frac{u^2}{2} + \frac{p}{\rho} \right)_{out} + w - f$$

If we denote Δ the difference between values at the outflow minus the values at the inflow, we can write

$$\Delta(gz) + \Delta\left(\frac{u^2}{2}\right) + \Delta\left(\frac{p}{\rho}\right) = w - f$$

or for g, ρ constants.

$$g\Delta z + \Delta\left(\frac{u^2}{2}\right) + \frac{\Delta p}{\rho} = w - f$$

positive when
work is done onto
the system

This is called

"generalized Bernoulli's equation"

why generalized? He made two assumptions

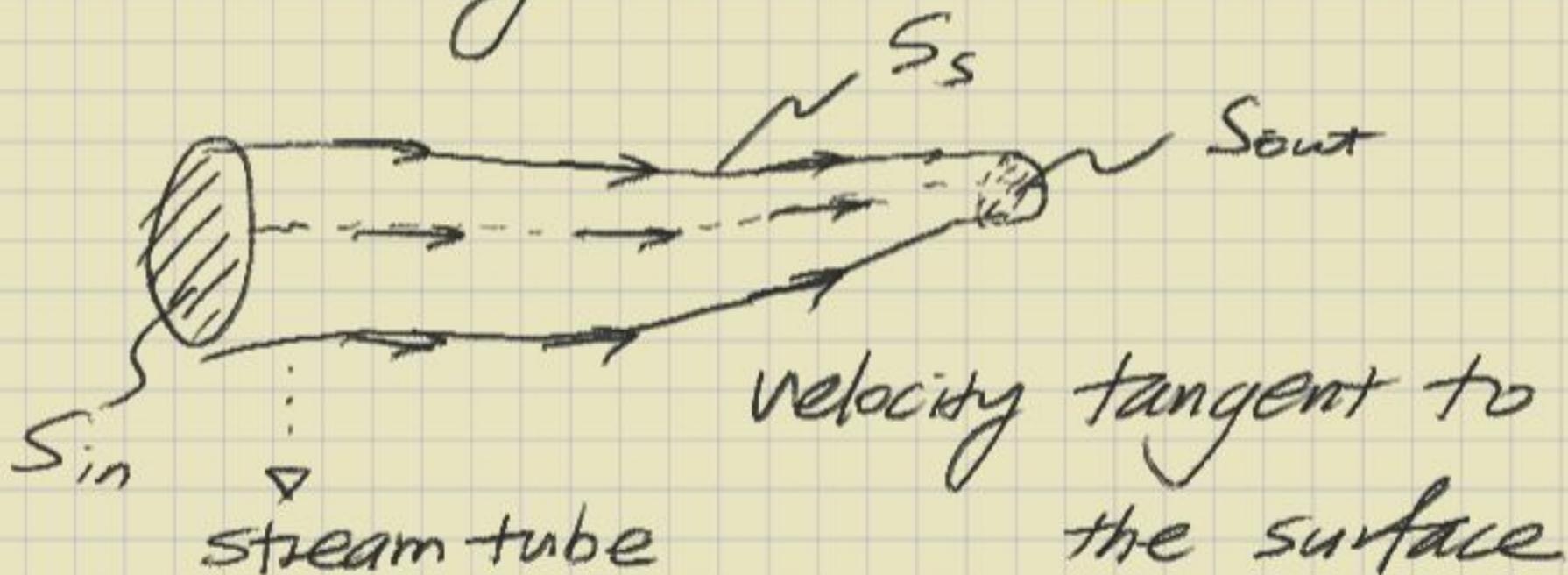
- ① no work is done on the fluid, $w=0$
- ② no dissipation (ideal fluid $\mu=0$), $f=0$

Then

$$g\Delta z + \Delta\left(\frac{u^2}{2}\right) + \frac{\Delta p}{\rho} = 0$$

→ Bernoulli's eqn

Note both version of B. eqns hold along a streamline.



All assumption satisfied as long as there is no flow across S_s

(i.e. S_s is delimited by stream tubes)

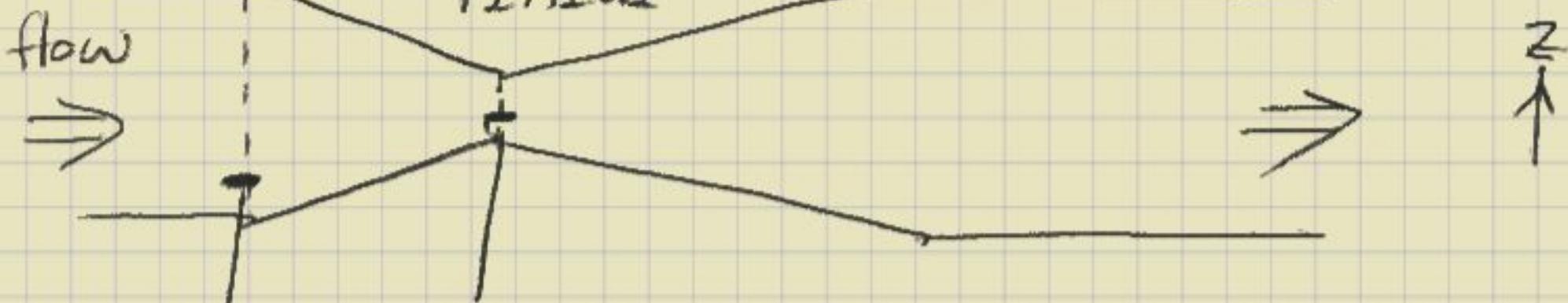
We can shrink S_{in} & S_{out} to a point

⇒ The B eqns hold along streamlines (under the stated assumptions)

Example) Venturi flow meter

P_1, A_1, u_1

P_2, A_2, u_2



pressure
measurement

P_1

P_2

dissipation
neglected

$$g\Delta z + \frac{\Delta u^2}{2} + \frac{\Delta p}{\rho} = 0 + \Delta z = 0$$

$$\rightarrow \frac{u_2^2}{2} - \frac{u_1^2}{2} + \frac{P_2 - P_1}{\rho} = 0$$

$$u_2^2 - u_1^2 = \frac{2(P_2 - P_1)}{\rho}$$

$$\text{or } u_1^2 \left(\left(\frac{u_2}{u_1} \right)^2 - 1 \right) = \frac{2(P_2 - P_1)}{\rho}$$

However

$$u_1 A_1 = u_2 A_2 \Rightarrow \frac{u_2}{u_1} = \frac{A_1}{A_2}$$

$$u_1^2 \left(\left(\frac{A_1}{A_2} \right)^2 - 1 \right) = 2 \frac{P_1 - P_2}{\rho}$$

$$u_1 = \sqrt{\frac{2}{\left(\frac{A_1}{A_2} \right)^2 - 1} \frac{P_1 - P_2}{\rho}}$$

Approximately,

losses can be taken into

account through a correction factor