Dynamics (동역학)

Lecture 5: Rigid Body Kinematics

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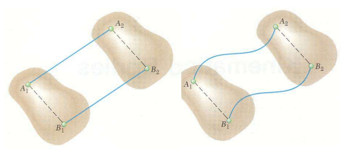
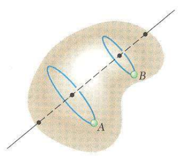
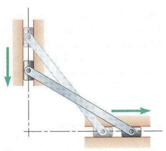
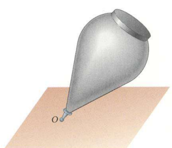
Introduction



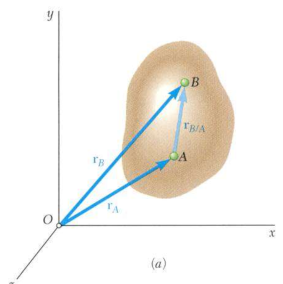


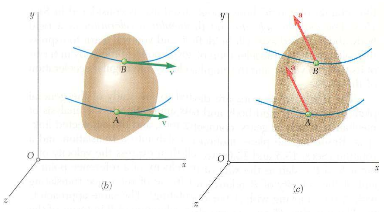




- Kinematics of rigid bodies: relations between time and the positions, velocities, and accelerations of the particles forming a rigid body.
- Classification of rigid body motions:
  - translation:
    - rectilinear translation
    - curvilinear translation
  - rotation about a fixed axis
  - general plane motion
  - motion about a fixed point
  - general 3D motion

Pure Translation





- Consider rigid body in translation:
  - direction of any straight line $\vec{r}_{B/A}$ inside the body is constant,
  - all particles forming the body move in parallel lines.
- For any two particles A and B in the body,

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

- Differentiating with respect to time,

$$\dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} = \dot{\vec{r}}_A$$
$$\vec{v}_B = \vec{v}_A$$

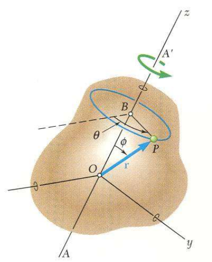
  All particles have the same velocity.

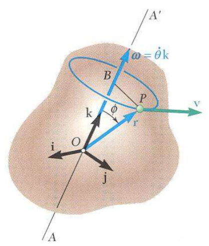
- Differentiating with respect to time again,

$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A$$
$$\vec{a}_B = \vec{a}_A$$

  All particles have the same acceleration.

Rotation about a Fixed Axis: Velocity





$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

- Consider rotation of rigid body about a fixed axis AA'
- Velocity vector $\vec{v} = d\vec{r}/dt$ of the particle P is tangent to the path with magnitude $v = ds/dt$

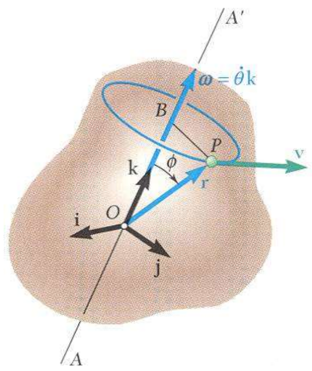
$$\Delta s = (BP)\Delta\theta = (r\sin\phi)\Delta\theta$$
$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0}(r\sin\phi)\frac{\Delta\theta}{\Delta t} = r\dot{\theta}\sin\phi$$

- The same result is obtained from

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$
$$\vec{\omega} = \omega\vec{k} = \dot{\theta}\vec{k} = angular\ velocity$$

- Angular velocity is a vector, whose direction is along the instantaneous rotation axis.

Rotation about a Fixed Axis: Acceleration



$v = w \times r$

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

- Differentiating to determine the acceleration,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

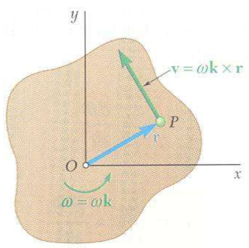
- $\dfrac{d\vec{\omega}}{dt} = \vec{\alpha} = angular\ acceleration$

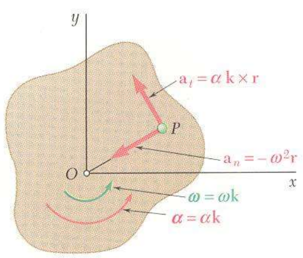
$$= \alpha\vec{k} = \dot{\omega}\vec{k} = \ddot{\theta}\vec{k}$$

- Acceleration of P is combination of two vectors,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$\vec{\alpha} \times \vec{r} = \text{tangential acceleration component}$$
$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \text{radial acceleration component}$$

Rotation about a Fixed Axis: in Plane





$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

- Consider the motion of a representative slab in a plane perpendicular to the axis of rotation.

- Velocity of any point P of the slab,

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega\vec{k} \times \vec{r}$$
$$v = r\omega$$

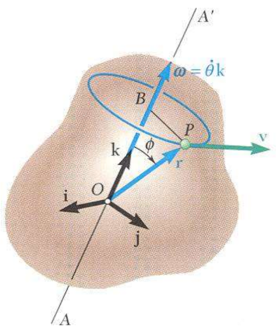
- Acceleration of any point P of the slab,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$= \alpha\vec{k} \times \vec{r} - \omega^2\vec{r}$$

- Resolving the acceleration into tangential and normal components,

$$\vec{a}_t = \alpha\vec{k} \times \vec{r} \qquad a_t = r\alpha$$
$$\vec{a}_n = -\omega^2\vec{r} \qquad a_n = r\omega^2$$

Rotation about a Fixed Axis w/ Constant α



- Motion of a rigid body rotating around a fixed axis is often specified by the type of angular acceleration.
- Recall $\omega = \frac{d\theta}{dt}$ or $dt = \frac{d\theta}{\omega}$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$$

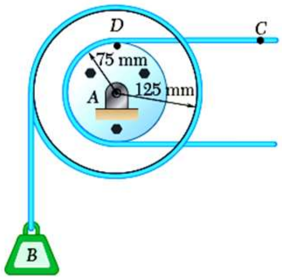
- *Uniform Rotation,* $\alpha = 0$:

$$\theta = \theta_0 + \omega t$$

- *Uniformly Accelerated Rotation,* α = constant:

$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Sample Problem 15.1



Cable *C* has a constant acceleration of 225 mm/s² and an initial velocity of 300 mm/s, both directed to the right.

Determine *(a)* the number of revolutions of the pulley in 2 s, *(b)* the velocity and change in position of the load *B* after 2 s, and *(c)* the acceleration of the point *D* on the rim of the inner pulley at t = 0.

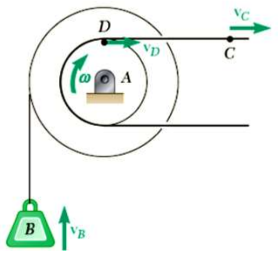
SOLUTION:

- Due to the action of the cable, the tangential velocity and acceleration of *D* are equal to the velocity and acceleration of *C*. Calculate the initial angular velocity and acceleration.
- Apply the relations for uniformly accelerated rotation to determine the velocity and angular position of the pulley after 2 s.
- Evaluate the initial tangential and normal acceleration components of *D*.

- *Uniformly Accelerated Rotation,* α = constant:

$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Sample Problem 5.1



SOLUTION:

- The tangential velocity and acceleration of D are equal to the velocity and acceleration of C.

$$(\vec{v}_D)_0 = (\vec{v}_C)_0 = 300\,\text{mm/s} \rightarrow$$
$$(v_D)_0 = r\omega_0$$
$$\omega_0 = \frac{(v_D)_0}{r} = \frac{300}{75} = 4\,\text{rad/s}$$

$$(\vec{a}_D)_t = \vec{a}_C = 9\,\text{in./s} \rightarrow$$
$$(a_D)_t = r\alpha$$
$$\alpha = \frac{(a_D)_t}{r} = \frac{9}{3} = 3\,\text{rad/s}^2$$

- Apply the relations for uniformly accelerated rotation to determine velocity and angular position of pulley after 2 s.

$$\omega = \omega_0 + \alpha t = 4\,\text{rad/s} + (3\,\text{rad/s}^2)(2\,\text{s}) = 10\,\text{rad/s}$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = (4\,\text{rad/s})(2\,\text{s}) + \tfrac{1}{2}(3\,\text{rad/s}^2)(2\,\text{s})^2$$
$$= 14\,\text{rad}$$

$$N = (14\,\text{rad})\left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right) = \text{number of revs} \qquad \boxed{N = 2.23\,\text{rev}}$$

$$v_B = r\omega = (125\,\text{mm})(10\,\text{rad/s})$$
$$\Delta y_B = r\theta = (125\,\text{mm})(14\,\text{rad})$$

$$\boxed{\vec{v}_B = 1.25\,\text{m/s} \uparrow \quad \Delta y_B = 1.75\,\text{m}}$$

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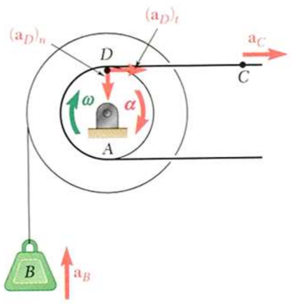
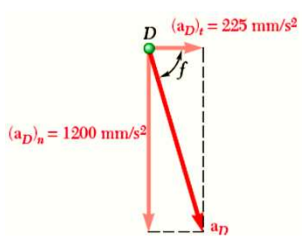
Sample Problem 5.1



- Evaluate the initial tangential and normal acceleration components of D.

$$(\vec{a}_D)_t = \vec{a}_C = 225\,\text{mm/s}^2 \rightarrow$$
$$(a_D)_n = r_D\omega_0^2 = (75\,\text{mm})(4\,\text{rad/s})^2 = 1200\,\text{mm/s}^2$$

$$\boxed{(\vec{a}_D)_t = 225\,\text{mm/s}^2 \rightarrow \qquad (\vec{a}_D)_n = 1200\,\text{mm/s}^2 \downarrow}$$



Magnitude and direction of the total acceleration,

$$a_D = \sqrt{(a_D)_t^2 + (a_D)_n^2}$$
$$= \sqrt{225^2 + 1200^2} \qquad \boxed{a_D = 1220\,\text{mm/s}^2}$$

$$\tan\phi = \frac{(a_D)_n}{(a_D)_t}$$
$$= \frac{1200}{225} \qquad \boxed{\phi = 79.4°}$$

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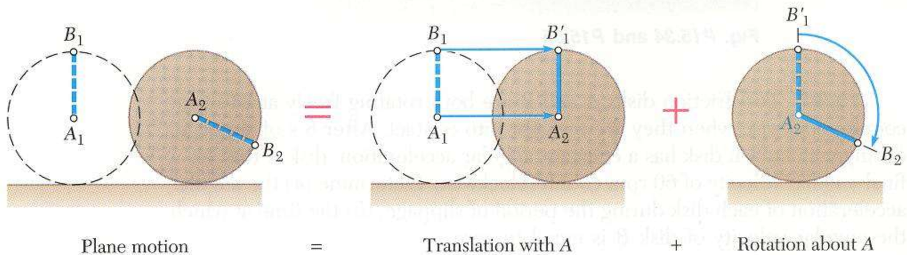
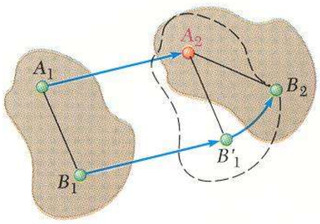
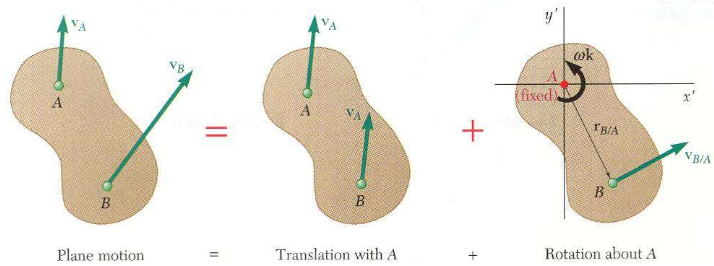
General Plane Motion

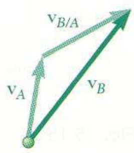


- *General plane motion* is neither a translation nor a rotation.
- General plane motion can be considered as the *sum* of translation (2-DOF) and rotation (1-DOF).
- Displacement of particles A and B to A_2 and B_2 can be divided into two parts:
  - translation to A_2 and B_1'
  - rotation of B_1' about A_2 to B_2



Absolute and Relative Velocity in Plane Motion





$v_B = v_A + v_{B/A}$

- Any plane motion (or rigid-body motion) can be replaced by a translation of an arbitrary point A and a simultaneous (relative) rotation about that point A.

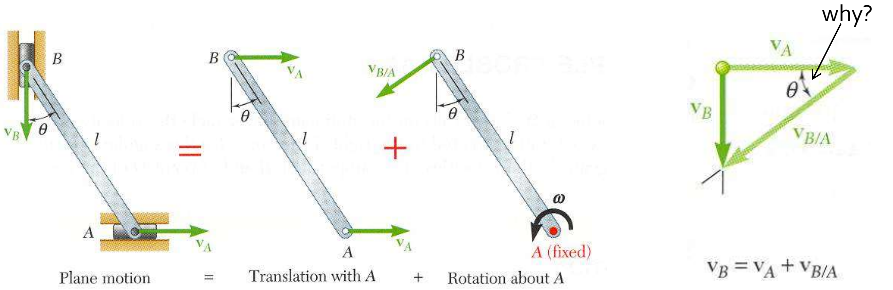
$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \qquad \vec{v}_{B/A} = \omega\vec{k} \times \vec{r}_{B/A} \quad v_{B/A} = r\omega$$

$$\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$

- Angular velocity of any point of a rigid body is the same: for the point B, with $\vec{v}_{A/B} = -\vec{v}_{B/A}$ and $\vec{r}_{A/B} = -\vec{r}_{B/A}$,

$$\vec{v}_A = \vec{v}_B + \vec{v}_{A/B} \qquad \vec{v}_{A/B} = -\vec{v}_{B/A} = -\omega\vec{k} \times \vec{r}_{B/A} = \omega\vec{k} \times \vec{r}_{A/B}$$

Absolute/Relative Velocity with Constraints



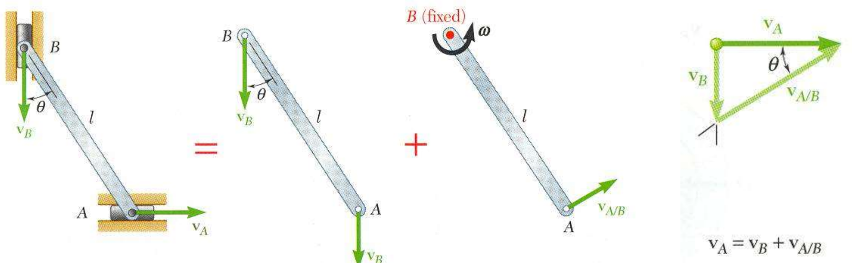
- Assuming that the velocity v_A of end A is known, wish to determine the velocity v_B of end B and the angular velocity ω in terms of v_A, l, and θ.
- The directions of v_B and $v_{B/A}$ are known. Complete the velocity diagram (cf. ICR).

$$\frac{v_B}{v_A} = \tan\theta \qquad \frac{v_A}{v_{B/A}} = \frac{v_A}{l\omega} = \cos\theta$$

$$v_B = v_A \tan\theta \qquad \omega = \frac{v_A}{l\cos\theta}$$

$$\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$

Absolute/Relative Velocity with Constraints

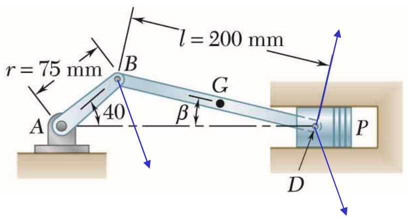


Plane motion = Translation with B + Rotation about B

$$\vec{v}_A = \vec{v}_B + \omega\vec{k} \times \vec{r}_{A/B}$$

- Selecting the point B and solving for the velocity v_A of end A as a combination of v_B and the relative velocity due to ω leads to an equivalent velocity triangle.
- $v_{A/B}$ has the same magnitude but opposite sense of $v_{B/A}$. The sign of the relative velocity is dependent on the choice of reference point.
- Angular velocity ω of the rod in its rotation about B is the same as its rotation about A. Angular velocity is not dependent on the choice of reference point.

Sample Problem 15.7 [1]



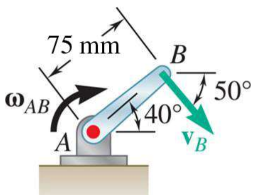
The crank *AB* has a constant clockwise angular velocity of 2000 rpm.

For the crank position indicated, determine *(a)* the angular velocity of the connecting rod *BD*, and *(b)* the velocity of the piston *P*.

Strategy:

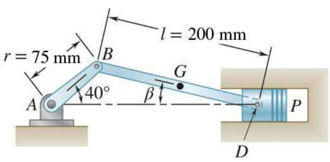
- Will determine the absolute velocity of point *D* with
$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$
- The velocity $\vec{v}_B$ is obtained from the given crank rotation data.
- The directions of the absolute velocity $\vec{v}_D$ and the relative velocity $\vec{v}_{D/B}$ are determined from the problem geometry.
- The unknowns in the vector expression are the velocity magnitudes v_D and $v_{D/B}$ which may be determined from the corresponding vector triangle.
- The angular velocity of the connecting rod is calculated from $v_{D/B}$.

Sample Problem 15.7 [2]



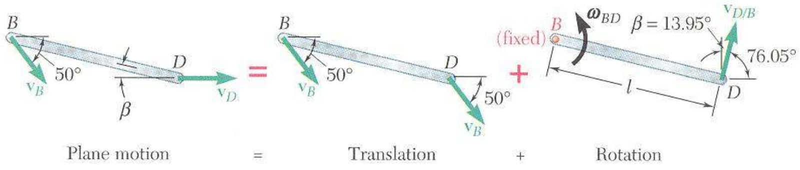
Modeling and Analysis:

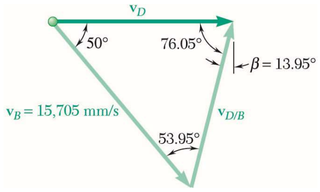
- Will determine the absolute velocity of point *D* with,
$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$
- The velocity $\vec{v}_B$ is obtained from the crank rotation data.
$$\omega_{AB} = \left(2000\frac{\text{rev}}{\text{min}}\right)\left(\frac{\text{min}}{60\,\text{s}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 209.4 \text{ rad/s}$$
$$v_B = (AB)\,\omega_{AB} = (75\,\text{mm})(209.4 \text{ rad/s}) = 15{,}705 \text{ mm/s}$$
The velocity direction is as shown.



- The direction of the absolute velocity $\vec{v}_D$ is horizontal. The direction of the relative velocity $\vec{v}_{D/B}$ is perpendicular to *BD*. Compute the angle between the horizontal and the connecting rod from the law of sines,
$$\frac{\sin 40°}{200\,\text{mm}} = \frac{\sin\beta}{75\,\text{mm}} \qquad \beta = 13.95°$$

Sample Problem 15.7 [3]





$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

- Determine the velocity magnitudes v_D and $v_{D/B}$ from the vector triangle.

$$\frac{v_D}{\sin 53.95\text{걑}} = \frac{v_{D/B}}{\sin 50} = \frac{15{,}705\,\text{mm/s}}{\sin 76.05?}$$

$$v_D = 13{,}083\,\text{mm/s} = 13.08\,\text{m/s} \qquad \boxed{v_P = v_D = 13.08\,\text{m/s}}$$

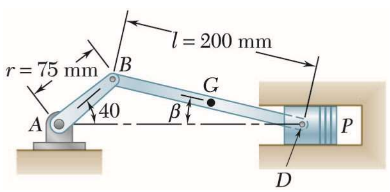
$$v_{D/B} = 12{,}400\,\text{mm/s} = 12.4\,\text{m/s}$$

$$v_{D/B} = l\omega_{BD}$$

$$\omega_{BD} = \frac{v_{D/B}}{l} = \frac{12{,}400\,\text{mm/s}}{200\ \text{mm}} = 62.0\ \text{rad/s} \circlearrowleft$$

$$\boxed{\vec{\omega}_{BD} = (62.0\,\text{rad/s})\vec{k}}$$

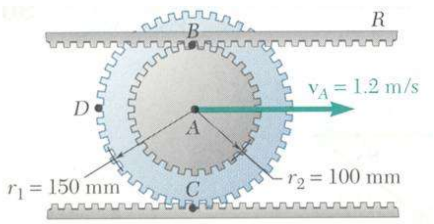
Sample Problem 15.7 [4]



Reflect and Think:
Note that as the crank continues to move clockwise below the center line, the piston changes direction and starts to move to the left.

Can you see what happens to the motion of the connecting rod at that point?

Sample Problem 15.2



The double gear rolls on the stationary lower rack: the velocity of its center is 1.2 m/s.

Determine *(a)* the angular velocity of the gear, and *(b)* the velocities of the upper rack *R* and point *D* of the gear.

$$\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$

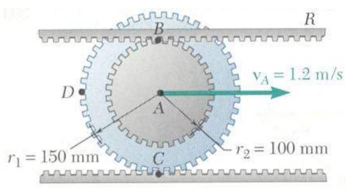
SOLUTION:

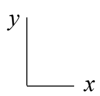
- The displacement of the gear center in one revolution is equal to the outer circumference. Relate the translational and angular displacements. Differentiate to relate the translational and angular velocities.
- The velocity for any point *P* on the gear may be written as

$$\vec{v}_P = \vec{v}_A + \vec{v}_{P/A} = \vec{v}_A + \omega\vec{k} \times \vec{r}_{P/A}$$

Evaluate the velocities of points *B* and *D*.

Sample Problem 15.2





SOLUTION:

- The displacement of the gear center in one revolution is equal to the outer circumference.

For $x_A > 0$ (moves to right), $\omega < 0$ (rotates clockwise).

$$\frac{x_A}{2\pi r} = -\frac{\theta}{2\pi} \qquad x_A = -r_1\theta \qquad \frac{2\pi r_1}{v_A} = -\frac{2\pi}{w}$$

Differentiate to relate the translational and angular velocities.

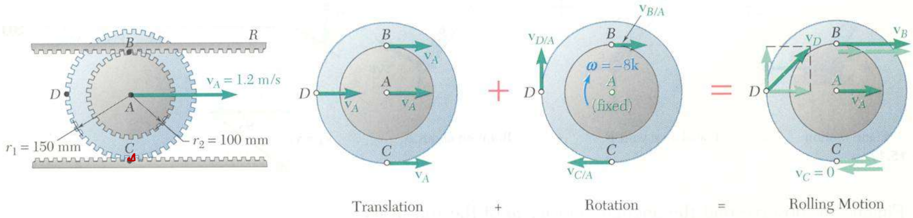
$$v_A = -r_1\omega$$

$$\omega = -\frac{v_A}{r_1} = -\frac{1.2\,\text{m/s}}{0.150\,\text{m}}$$

$$\vec{\omega} = \omega\vec{k} = -(8\,\text{rad/s})\vec{k}$$

Sample Problem 15.2

- For any point P on the gear, $\vec{v}_P = \vec{v}_A + \vec{v}_{P/A} = \vec{v}_A + \omega\vec{k} \times \vec{r}_{P/A}$



Velocity of the upper rack is equal to velocity of point B:

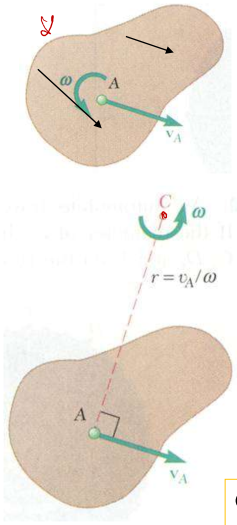
$$\vec{v}_R = \vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$
$$= (1.2\,\text{m/s})\vec{i} + (8\,\text{rad/s})\vec{k} \times (0.10\,\text{m})\vec{j}$$
$$= (1.2\,\text{m/s})\vec{i} + (0.8\,\text{m/s})\vec{i}$$
$$\vec{v}_R = (2\,\text{m/s})\vec{i}$$

Velocity of the point D:

$$\vec{v}_D = \vec{v}_A + \omega\vec{k} \times \vec{r}_{D/A}$$
$$= (1.2\,\text{m/s})\vec{i} + (8\,\text{rad/s})\vec{k} \times (-0.150\,\text{m})\vec{i}$$
$$\vec{v}_D = (1.2\,\text{m/s})\vec{i} + (1.2\,\text{m/s})\vec{j}$$
$$v_D = 1.697\,\text{m/s}$$

- analysis using the point C (i.e., instantaneous rotation center) is also possible

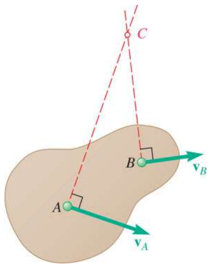
Instantaneous Center of Rotation in Plane Motion



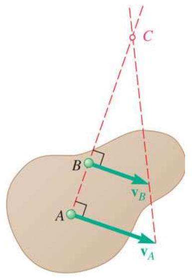
- Plane motion of a slab can always be represented by the translation velocity v_A of an arbitrary point A and angular velocity w about A.
- The translational and rotational velocities (v_A, w) at A can also be obtained by allowing the slab to rotate with the same angular velocity w about a point C on the line perpendicular to the velocity v_A at A.
- The velocity of all other particles in the slab rotating about this C will also be the same as originally given.
- As far as the velocities are concerned, the slab seems to rotate about the *instantaneous center of rotation C*.

Chasles Theorem: Any rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis ⇒ general plane motion = pure rotation about C

Instantaneous Center of Rotation in Plane Motion



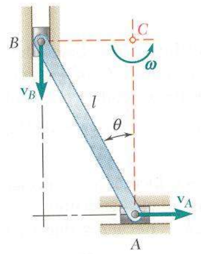
- If the velocity at two points A and B are known, the instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through A and B (e.g., car steering).
- If the velocity vectors are parallel, the instantaneous center of rotation is at infinity and the angular velocity is zero.

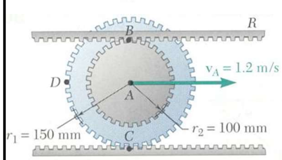


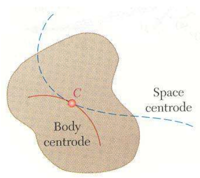
(b)

- If the velocity vectors at A and B are perpendicular to the line AB, the instantaneous center of rotation lies at the intersection of the line AB with the line joining the extremities of the velocity vectors at A and B.
- If the velocity magnitudes are equal, the instantaneous center of rotation is at infinity and the angular velocity is zero.

Instantaneous Center of Rotation





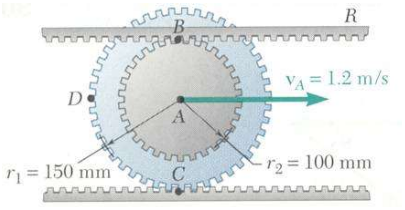


- The instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through A and B .

$$\omega = \frac{v_A}{AC} = \frac{v_A}{l\cos\theta} \qquad v_B = (BC)\omega = (l\sin\theta)\frac{v_A}{l\cos\theta} = v_A \tan\theta$$

- The velocities of all particles on the rod are as if they are rotating about the ICR C.
- The particle (or fictitious lab) at the ICR has zero (instantaneous) velocity.
- The ICR on the slab changes with time (i.e., instantaneous CR) with the acceleration at ICR is non-zero ⇒ the acceleration of the particles in the slab cannot be determined as if the slab were simply rotating about ICR C.
- The trace of the locus of the center of rotation on the body is the body centrode and in space is the space centrode.

Sample Problem 15.4



The double gear rolls on the stationary lower rack: the velocity of its center is 1.2 m/s.

Determine *(a)* the angular velocity of the gear, and *(b)* the velocities of the upper rack *R* and point *D* of the gear.

SOLUTION:

- The point *C* is in contact with the stationary lower rack and, instantaneously, has zero velocity. It must be the location of the instantaneous center of rotation.
- Determine the angular velocity about *C* based on the given velocity at *A*.
- Evaluate the velocities at *B* and *D* based on their rotation about *C*.

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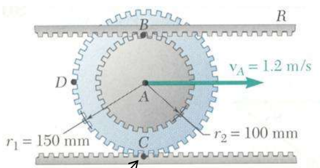
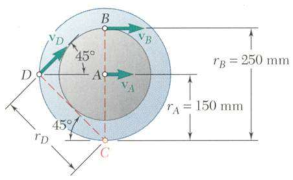
Sample Problem 15.4



$v_C = 0, yet, a_C \neq 0$



SOLUTION:

- The point *C* is in contact with the stationary lower rack and, instantaneously, has zero velocity. It must be the location of the instantaneous center of rotation.
- Determine the angular velocity about *C* based on the given velocity at *A*.

$$v_A = r_A\omega \qquad \omega = \frac{v_A}{r_A} = \frac{1.2\,\mathrm{m/s}}{0.15\,\mathrm{m}} = 8\,\mathrm{rad/s}$$

- Evaluate the velocities at *B* and *D* based on their rotation about *C*.

$$v_R = v_B = r_B\omega = (0.25\,\mathrm{m})(8\,\mathrm{rad/s})$$

$$\vec{v}_R = (2\,\mathrm{m/s})\vec{i}$$

$$r_D = (0.15\,\mathrm{m})\sqrt{2} = 0.2121\,\mathrm{m}$$

$$v_D = r_D\omega = (0.2121\,\mathrm{m})(8\,\mathrm{rad/s})$$

$$v_D = 1.697\,\mathrm{m/s}$$

$$\vec{v}_D = (1.2\vec{i} + 1.2\vec{j})(\mathrm{m/s})$$

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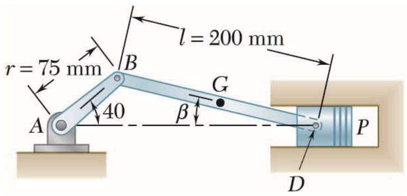
Sample Problem 15.10 [1]



The crank AB has a constant clockwise angular velocity of 2000 rpm.

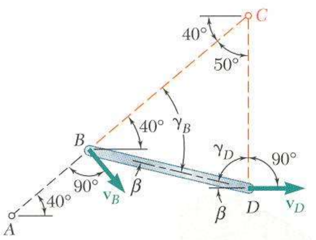
For the crank position indicated, determine *(a)* the angular velocity of the connecting rod BD, and *(b)* the velocity of the piston P.

Use method of instantaneous center of rotation

Strategy:

- Determine the velocity at B from the given crank rotation data.
- The direction of the velocity vectors at B and D are known. The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through B and D.
- Determine the angular velocity about the center of rotation based on the velocity at B.
- Calculate the velocity at D based on its rotation about the instantaneous center of rotation.

Sample Problem 15.10 [2]



$$\gamma_B = 40° + \beta = 53.95°$$
$$\gamma_D = 90° - \beta = 76.05°$$

$$\frac{BC}{\sin 76.05\text{갈}} = \frac{CD}{\sin 53.95} = \frac{200\text{ mm}}{\sin 50?}$$

$BC = 253.4\text{ mm} \quad CD = 211.1\text{ mm}$

Modeling and Analysis:

- From Sample Problem 15.3,

$$v_B = 15{,}705\,\text{mm/s}$$
$$\beta = 13.95°$$

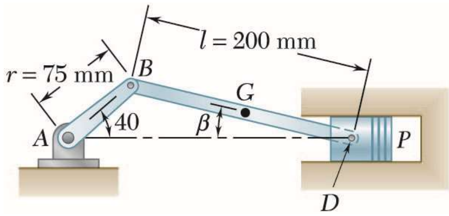
- The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through B and D.
- Determine the angular velocity about the center of rotation based on the velocity at B.

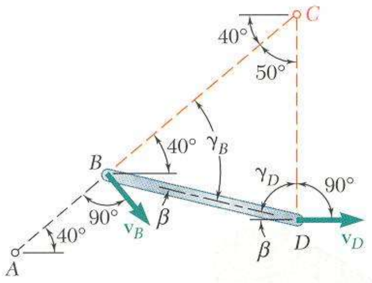
$$v_B = (BC)\omega_{BD}$$
$$\omega_{BD} = \frac{v_B}{BC} = \frac{15{,}705\,\text{mm/s}}{253.4\text{ mm}}$$
$$\boxed{\omega_{BD} = 62.0\,\text{rad/s}}$$

- Calculate the velocity at D based on its rotation about the instantaneous center of rotation.

$$v_D = (CD)\omega_{BD} = (211.1\text{ mm})(62.0\,\text{rad/s})$$
$$\boxed{v_P = v_D = 13{,}080\,\text{mm/s} = 13.08\,\text{m/s}}$$

Instantaneous Center of Zero Velocity



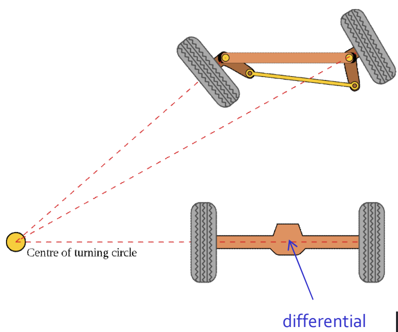


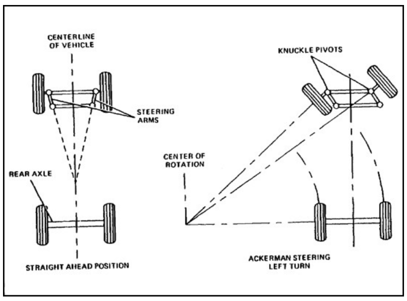
Reflect and Think:

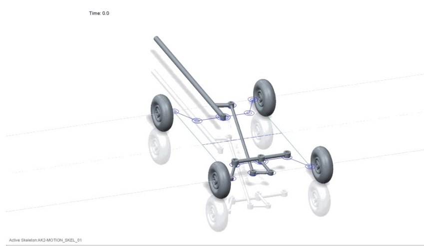
What happens to the location of the instantaneous center of velocity if the crankshaft angular velocity increases from 2000 rpm in the previous problem to 3000 rpm?

What happens to the location of the instantaneous center of velocity if the angle β is 0?

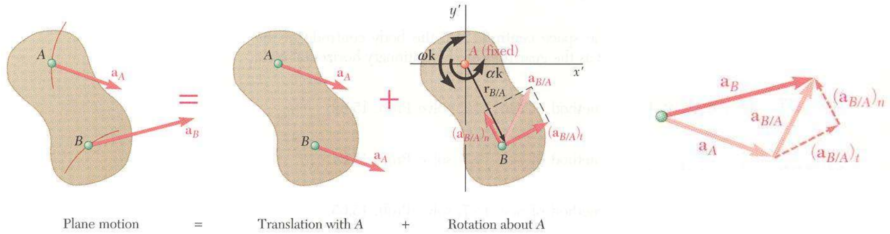
Application of ICR: Ackermann Steering







Absolute and Relative Acceleration in 2D



- Absolute acceleration of a particle of the slab: $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$
- Relative acceleration $\vec{a}_{B/A}$ associated with rotation about A includes tangential and normal components,

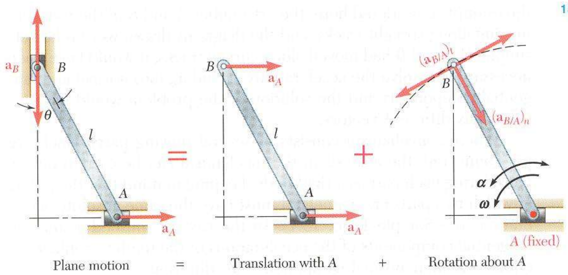
$$\left(\vec{a}_{B/A}\right)_t = \alpha\vec{k} \times \vec{r}_{B/A} \qquad \left(a_{B/A}\right)_t = r\alpha$$

$$\left(\vec{a}_{B/A}\right)_n = -\omega^2 \vec{r}_{B/A} \qquad \left(a_{B/A}\right)_n = r\omega^2$$

$$\vec{a}_B = \vec{a}_A + \alpha\vec{k} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$$
$$= \vec{a}_A + r\alpha\vec{e}_t - r\omega^2\vec{e}_r$$

- This can be thought similarly as: $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$
- Can also be derived from: $\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$

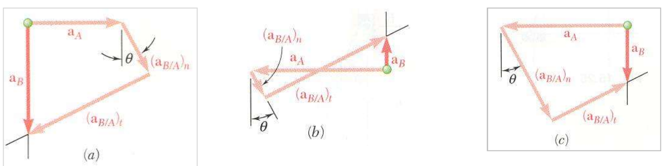
Absolute and Relative Acceleration in 2D

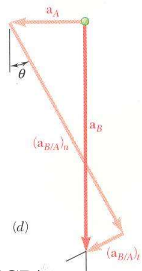


- Given $\vec{a}_A$ and $\vec{v}_A$, determine $\vec{a}_B$ and $\vec{\alpha}$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$
$$= \vec{a}_A + \left(\vec{a}_{B/A}\right)_n + \left(\vec{a}_{B/A}\right)_t$$

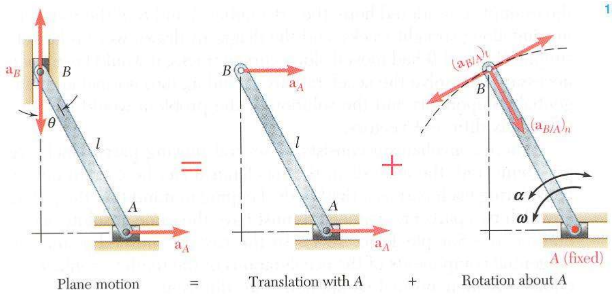
$$\vec{a}_B = \vec{a}_A + \alpha\vec{k} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$$
$$= \vec{a}_A + r\alpha\vec{e}_t - r\omega^2\vec{e}_r$$

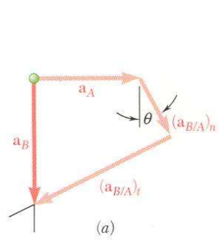




- Vector diagram result depends on sign of $\vec{a}_A$ and the relative magnitudes of a_A and $\left(a_{B/A}\right)_n$
- Angular velocity w (and $(a_{B/A})$_n) is specified by $\vec{v}_A$ (cf. ICR)

Absolute and Relative Acceleration in 2D





- Write $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ in terms of the two component equations,

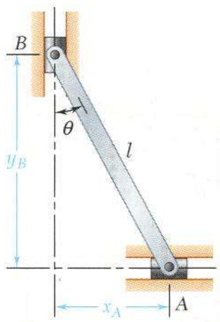
  $\overset{+}{\rightarrow}$ x components: $0 = a_A + l\omega^2 \sin\theta - l\alpha\cos\theta$

  $+\uparrow$ y components: $a_B = -l\omega^2 \cos\theta - l\alpha\sin\theta$

- Solve for a_B and α.

$$\vec{a}_B = \vec{a}_A + \alpha\vec{k}\times\vec{r}_{B/A} - \omega^2\vec{r}_{B/A}$$
$$= \vec{a}_A + r\alpha\vec{e}_t - r\omega^2\vec{e}_r$$

Analytical Expression of Kinematics



- In some cases, it is advantageous to determine the absolute velocity and acceleration of a mechanism directly.

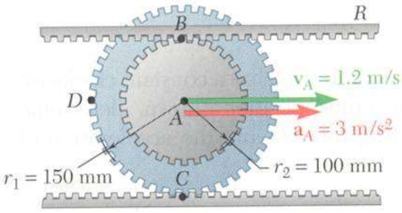
$$x_A = l\sin\theta \qquad y_B = l\cos\theta$$

$$v_A = \dot{x}_A = l\dot{\theta}\cos\theta = l\omega\cos\theta \qquad v_B = \dot{y}_B = -l\dot{\theta}\sin\theta = -l\omega\sin\theta$$

$$a_A = \ddot{x}_A = -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta = -l\omega^2\sin\theta + l\alpha\cos\theta$$

$$a_B = \ddot{y}_B = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta = -l\omega^2\cos\theta - l\alpha\sin\theta$$

Sample Problem 15.6



The center of the double gear has a velocity and acceleration to the right of 1.2 m/s and 3 m/s^2, respectively. The lower rack is stationary.

Determine *(a)* the angular acceleration of the gear, and *(b)* the acceleration of points *B*, *C*, and *D*.

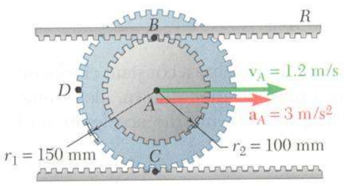
$$\vec{a}_C = \vec{a}_A + r\alpha\vec{e}_t - r\omega^2\vec{e}_r$$

SOLUTION:

- The expression of the gear position as a function of θ is differentiated twice to define the relationship between the translational and angular accelerations.
- The acceleration of each point on the gear is obtained by adding the acceleration of the gear center and the relative accelerations with respect to the center. The latter includes normal and tangential acceleration components.

A moves only horizontally -> constrained
B, C, D move along their circumferences

Sample Problem 15.6



SOLUTION:

- The expression of the gear position as a function of θ is differentiated twice to define the relationship between the translational and angular accelerations.

$$x_A = -r_1\theta$$

$$v_A = -r_1\dot{\theta} = -r_1\omega$$

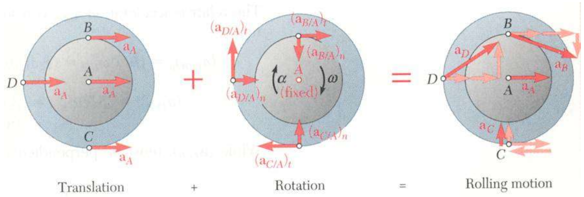
$$\omega = -\frac{v_A}{r_1} = -\frac{1.2\,\text{m/s}}{0.150\,\text{m}} = -8\ \text{rad/s}$$

$$a_A = -r_1\ddot{\theta} = -r_1\alpha$$

$$\alpha = -\frac{a_A}{r_1} = -\frac{3\,\text{m/s}^2}{0.150\,\text{m}}$$

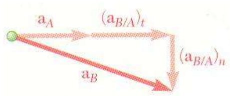
$$\vec{\alpha} = \alpha\vec{k} = -\left(20\,\text{rad/s}^2\right)\vec{k}$$

Sample Problem 15.6



- The acceleration of each point is obtained by adding the acceleration of the gear center and the relative accelerations from the center.

  The latter includes normal and tangential acceleration components.



$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = \vec{a}_A + \left(\vec{a}_{B/A}\right)_t + \left(\vec{a}_{B/A}\right)_n$$

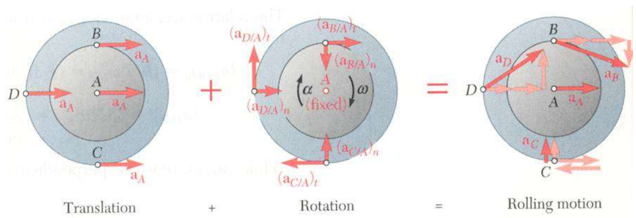
$$= \vec{a}_A + \alpha\vec{k}\times\vec{r}_{B/A} - \omega^2\vec{r}_{B/A}$$

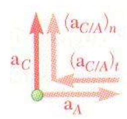
$$= \left(3\,\text{m/s}^2\right)\vec{i} - \left(20\,\text{rad/s}^2\right)\vec{k}\times(0.100\,\text{m})\vec{j} - (8\,\text{rad/s})^2(-0.100\,\text{m})\vec{j}$$

$$= \left(3\,\text{m/s}^2\right)\vec{i} + \left(2\,\text{m/s}^2\right)\vec{i} - \left(6.40\,\text{m/s}^2\right)\vec{j}$$

$$\vec{a}_B = \left(5\,\text{m/}s^2\right)\vec{i} - \left(6.40\,\text{m/s}^2\right)\vec{j} \qquad a_B = 8.12\,\text{m/s}^2$$

Sample Problem 15.6



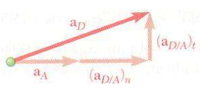


$$\vec{a}_C = \vec{a}_A + \vec{a}_{C/A} = \vec{a}_A + \alpha\vec{k}\times\vec{r}_{C/A} - \omega^2\vec{r}_{C/A}$$

$$= \left(3\,\text{m/s}^2\right)\vec{i} - \left(20\,\text{rad/s}^2\right)\vec{k}\times(-0.150\,\text{m})\vec{j} - (8\,\text{rad/s})^2(-0.150\,\text{m})\vec{j}$$

$$= \left(3\,\text{m/s}^2\right)\vec{i} - \left(3\,\text{m/s}^2\right)\vec{i} + \left(9.60\,\text{m/s}^2\right)\vec{j}$$

$$\vec{a}_c = \left(9.60\,\text{m/s}^2\right)\vec{j}$$



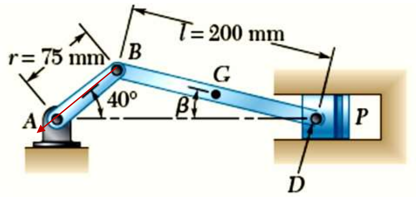
$$\vec{a}_D = \vec{a}_A + \vec{a}_{D/A} = \vec{a}_A + \alpha\vec{k}\times\vec{r}_{D/A} - \omega^2\vec{r}_{D/A}$$

$$= \left(3\,\text{m/s}^2\right)\vec{i} - \left(20\,\text{rad/s}^2\right)\vec{k}\times(-0.150\,\text{m})\vec{i} - (8\,\text{rad/s})^2(-0.150\text{m})\vec{i}$$

$$= \left(3\,\text{m/s}^2\right)\vec{i} + \left(3\,\text{m/s}^2\right)\vec{j} + \left(9.60\,\text{m/s}^2\right)\vec{i}$$

$$\vec{a}_D = \left(12.6\,\text{m/}s^2\right)\vec{i} + \left(3\,\text{m/s}^2\right)\vec{j} \qquad a_D = 12.95\,\text{m/s}^2$$

Sample Problem 15.7



Crank AG of the engine system has a constant clockwise angular velocity of 2000 rpm.

For the crank position shown, determine the angular acceleration of the connecting rod BD and the acceleration of point D.

Two unknowns: a_{Dx}, α_{BD}

$$\vec{a}_C = \vec{a}_A + \vec{a}_{C/A} = \vec{a}_A + r\alpha\vec{e}_t - r\omega^2\vec{e}_r$$

SOLUTION:

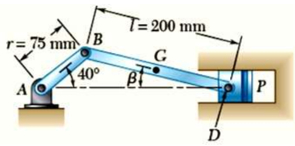
- The angular acceleration of the connecting rod BD and the acceleration of point D will be determined from

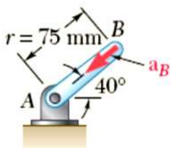
$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B} = \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

$a_{Dx}, (a_{Dy} = 0)$ $\quad w_{AB}$ $\quad \alpha_{BD}$ $\quad w_{BD}$

- The acceleration of B is determined from the given rotation speed of AB.
- The directions of the accelerations $\vec{a}_D, \left(\vec{a}_{D/B}\right)_t$, and $\left(\vec{a}_{D/B}\right)_n$ are determined from constraints
- Component equations for acceleration of point D are solved simultaneously for acceleration of D and angular acceleration of the connecting rod.

Sample Problem 15.7





SOLUTION:

- The angular acceleration of the connecting rod BD and the acceleration of point D will be determined from

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B} = \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

$a_{Dx}, (a_{Dy} = 0)$ $\quad w_{AB}$ $\quad \alpha_{BD}$ $\quad w_{BD}$

v_D

- The acceleration of B is determined from the given rotation speed of AB.

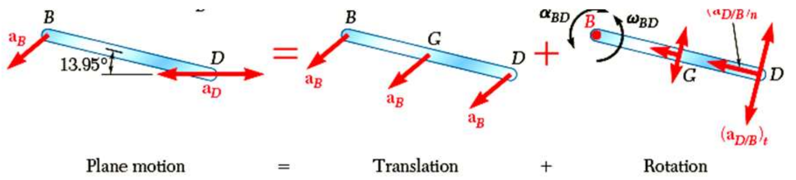
$$\omega_{AB} = 2000\,\text{rpm} = 209.4\,\text{rad/s} = \text{constant}$$

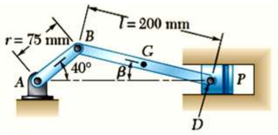
$$\alpha_{AB} = 0$$

$$a_B = r\omega_{AB}^2 = \left(\tfrac{7}{1000}\,\text{m}\right)(209.4\,\text{rad/s})^2 = 3289\,\text{m/s}^2$$

$$\vec{a}_B = \left(3289\,\text{m/s}^2\right)\left(-\cos 40°\vec{i} - \sin 40°\vec{j}\right)$$

Sample Problem 15.7





- The directions of the accelerations $\vec{a}_D$, $\left(\vec{a}_{D/B}\right)_t$, and $\left(\vec{a}_{D/B}\right)_n$ are determined from the geometry.

$$\vec{a}_D = \mp a_D \vec{i}$$

From Sample Problem 15.3, $\omega_{BD} = 62.0$ rad/s, $\beta = 13.95°$.

$$\left(a_{D/B}\right)_n = (BD)\omega_{BD}^2 = \left(\tfrac{200}{1000}\,\text{m}\right)(62.0\,\text{rad/s})^2 = 768.8\,\text{m/s}^2$$

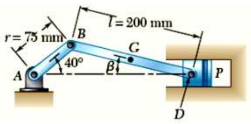
$$\left(\vec{a}_{D/B}\right)_n = \left(768.8\,\text{m/s}^2\right)\left(-\cos 13.95°\vec{i} + \sin 13.95°\vec{j}\right)$$

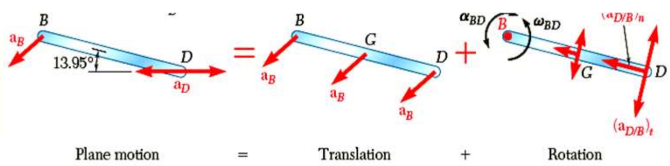
$$\left(a_{D/B}\right)_t = (BD)\alpha_{BD} = \left(\tfrac{200}{1000}\,\text{m}\right)\alpha_{BD} = 0.2\alpha_{BD}$$

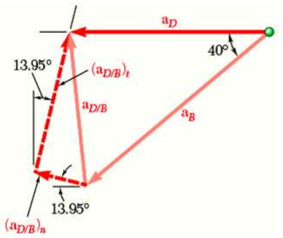
The direction of $(a_{D/B})_t$ is known but the sense is not known,

$$\left(\vec{a}_{D/B}\right)_t = \left(0.2\alpha_{BD}\right)\left(\pm \sin 76.05°\vec{i} \pm \cos 76.05°\vec{j}\right)$$

Sample Problem 15.7







- Component equations for acceleration of point D are solved simultaneously.

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B} = \vec{a}_B + \left(\vec{a}_{D/B}\right)_t + \left(\vec{a}_{D/B}\right)_n$$

x components:

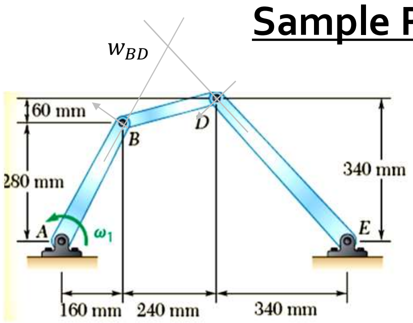
$$-a_D = -3289\cos 40° - 768.8\cos 13.95° + 0.2\alpha_{BD}\sin 13.95°$$

y components:

$$0 = -3289\sin 40° + 768.8\sin 13.95° + 0.2\alpha_{BD}\cos 13.95°$$

$$\vec{\alpha}_{BD} = \left(9940\,\text{rad/s}^2\right)\vec{k}$$
$$\vec{a}_D = -\left(2790\,\text{m/s}^2\right)\vec{i}$$

Sample Problem 15.8



In the position shown, crank *AB* has a constant angular velocity ω_1 = 20 rad/s counterclockwise.

Determine the angular velocities and angular accelerations of the connecting rod *BD* and crank *DE*.

Four unknowns.

SOLUTION:

- The angular velocities are determined by simultaneously solving the component equations for (or via ICR)

$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

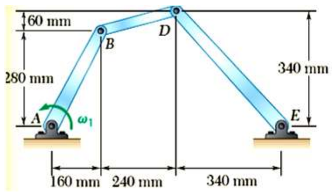
w_{DE} w_{BD}
(w/ dir)

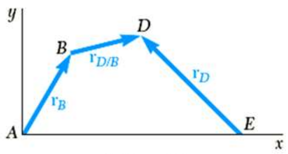
- The angular accelerations are determined by simultaneously solving the component equations for

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B}$$

w_{DE}, α_{DE} w_{BD}, α_{BD}

Sample Problem 15.8





$\mathbf{r}_B = 160\mathbf{i} + 280\mathbf{j}$
$\mathbf{r}_D = -340\mathbf{i} + 340\mathbf{j}$
$\mathbf{r}_{D/B} = 240\mathbf{i} + 60\mathbf{j}$

SOLUTION:

- The angular velocities are determined by simultaneously solving the component equations for

$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$

$$\vec{v}_D = \vec{\omega}_{DE} \times \vec{r}_D = \omega_{DE}\vec{k} \times \left(-340\vec{i} + 340\vec{j}\right) = -17\omega_{DE}\vec{i} - 17\omega_{DE}\vec{j}$$

$$\vec{v}_B = \vec{\omega}_{AB} \times \vec{r}_B = 20\vec{k} \times \left(160\vec{i} + 280\vec{j}\right) = -280\vec{i} + 160\vec{j}$$

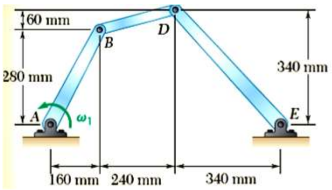
$$\vec{v}_{D/B} = \vec{\omega}_{BD} \times \vec{r}_{D/B} = \omega_{BD}\vec{k} \times \left(240\vec{i} + 60\vec{j}\right) = -3\omega_{BD}\vec{i} + 12\omega_{BD}\vec{j}$$

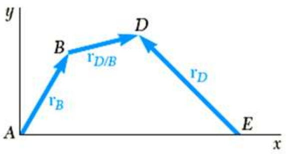
x components: $-17\omega_{DE} = -280 - 3\omega_{BD}$

y components: $-17\omega_{DE} = +160 + 12\omega_{BD}$

$$\vec{\omega}_{BD} = -(29.33\,\mathrm{rad/s})\vec{k} \qquad \vec{\omega}_{DE} = (11.29\,\mathrm{rad/s})\vec{k}$$

Sample Problem 15.8





$\mathbf{r}_B = 160\mathbf{i} + 280\mathbf{j}$
$\mathbf{r}_D = -340\mathbf{i} + 340\mathbf{j}$
$\mathbf{r}_{D/B} = 240\mathbf{i} + 60\mathbf{j}$

- The angular accelerations are determined by simultaneously solving the component equations for

$$\vec{a}_D = \vec{a}_B + \vec{a}_{D/B}$$

$$\vec{a}_D = \vec{\alpha}_{DE} \times \vec{r}_D - \omega_{DE}^2 \vec{r}_D$$
$$= \alpha_{DE}\vec{k} \times \left(-0.34\vec{i} + 0.34\vec{j}\right) - (11.29)^2\left(-0.34\vec{i} + 0.34\vec{j}\right)$$
$$= -0.34\alpha_{DE}\vec{i} - 0.34\alpha_{DE}\vec{j} + 43.33\vec{i} - 43.33\vec{j}$$

$$\vec{a}_B = \vec{\alpha}_{AB} \times \vec{r}_B - \omega_{AB}^2 \vec{r}_B = 0 - (20)^2\left(16\vec{i} + 0.28\vec{j}\right)$$
$$= -64\vec{i} + 112\vec{j}$$

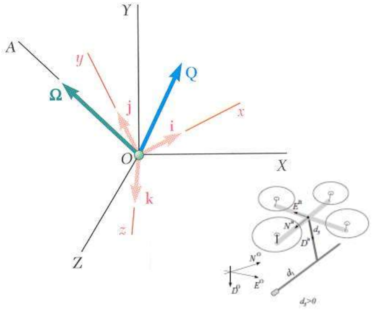
$$\vec{a}_{D/B} = \vec{\alpha}_{BD} \times \vec{r}_{B/D} - \omega_{BD}^2 \vec{r}_{B/D}$$
$$= \alpha_{B/D}\vec{k} \times \left(24\vec{i} + 0.06\vec{j}\right) - (29.33)^2\left(0.24\vec{i} + 0.06\vec{j}\right)$$
$$= 0.06\alpha_{B/D}\vec{i} + 0.24\alpha_{B/D}\vec{j} - 206.4\vec{i} - 51.61\vec{j}$$

x components: $-0.34\alpha_{DE} + 0.06\alpha_{BD} = -313.7$

y components: $-0.34\alpha_{DE} - 0.24\alpha_{BD} = -120.28$

$$\vec{\alpha}_{BD} = -\left(645\,\mathrm{rad/s^2}\right)\vec{k} \qquad \vec{\alpha}_{DE} = \left(809\,\mathrm{rad/s^2}\right)\vec{k}$$

Rate of Change with Rotating Frame



- Frame $OXYZ$ is fixed $\{O\}$.
- Frame $Oxyz$ $\{\mathfrak{F}\} = \{\vec{\imath}, \vec{\jmath}, \vec{k}\}$ rotates about a fixed axis OA with angular velocity $\vec{\Omega}$
- A vector $\vec{Q}(t)$ can be expressed in $Oxyz$ (or $\{\mathfrak{F}\}$)

recall $\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$

- If expressed by the rotating frame $Oxyz$ $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$:

$$\vec{Q} = Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}$$

- Rate of change (differentiation) of $\vec{Q}(t)$ in $\{O\}$:

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k} + Q_x\dot{\vec{i}} + Q_y\dot{\vec{j}} + Q_z\dot{\vec{k}}$$

- Rate of change of $\vec{Q}(t)$ as observed in rotating frame $\{\mathfrak{F}\}$ (or differentiation w.r.t. $\{\mathfrak{F}\}$):

$$\left(\dot{\vec{Q}}\right)_{Oxyz} = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k}$$

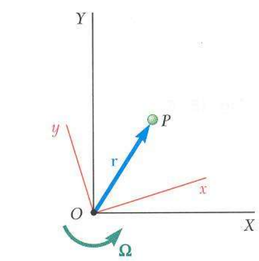
- If $\vec{Q}(t)$ is constant in $\{\mathfrak{F}\}$, then, $\left(\dot{\vec{Q}}\right)_{Oxyz} = 0$ and since $\left(\dot{\vec{Q}}\right)_{Oxyz} = 0$ is the velocity of a point $\vec{Q}(t)$ rigidly attached to $\{\mathfrak{F}\}$ rotating with $\vec{\Omega}$,

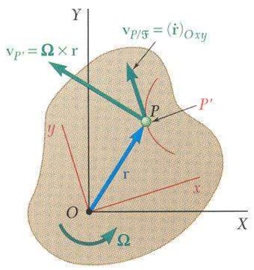
$$Q_x\dot{\vec{i}} + Q_y\dot{\vec{j}} + Q_z\dot{\vec{k}} = \vec{\Omega} \times \vec{Q}$$

- Combining these: $\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega} \times \vec{Q}$

rate of change in ***inertial frame {O}*** | rate of change as observed (or relative) ***in {𝔉}*** | rate of change due to rigid-motion ***with {𝔉}***

Plane Rotation with Rotating Frame





- Frame OXY is fixed $\{O\}$ and frame Oxy $\{\mathfrak{F}\}$ rotates with angular velocity $\vec{\Omega}$
- Position vector $\vec{r}_P$ for the particle P
- The absolute velocity of the particle P is

$$\vec{v}_P = (\dot{\vec{r}})_{OXY} = \vec{\Omega} \times \vec{r} + (\dot{r})_{Oxy}$$

- Imagine a rotating rigid slab, to which a frame Oxy or $\mathfrak{F}$ for short is rigidly-attached. Let P' be a point of the slab, which corresponds to P instantaneously.

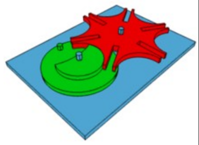
  $\vec{v}_{P'} =$ velocity of P' rigidly-attached on the slab

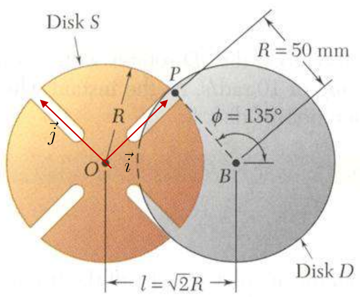
  $\vec{v}_{P/\mathbf{F}} = (\dot{\vec{r}})_{Oxy} =$ velocity of P as observed in (or relative to) the slab
- Absolute velocity for the particle P is given by:

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

rigid-motion ***with*** {F}   as observed ***in*** (relative to) {F}

Sample Problem 15.9





Disk D of the Geneva mechanism rotates with constant counterclockwise angular velocity ω_D = 10 rad/s.

At the instant when ϕ = 150°, determine (*a*) the angular velocity w_S of disk S, and (*b*) the velocity of pin P relative to disk S.

w.r.t. {D}, w.r.t. {S} $\quad \vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/S}$

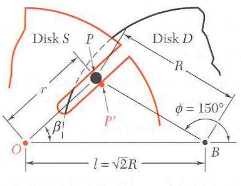
$$\vec{p} = r(t)\vec{i}(t) \;\Rightarrow\; \vec{v}_p = \dot{\vec{p}} = \dot{r}\vec{i} + r\dot{\vec{i}}$$

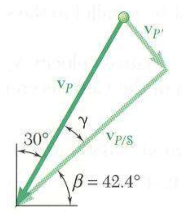
SOLUTION:

- The absolute velocity of the point P may be written as

  $\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/s}$
- Magnitude and direction of velocity $\vec{v}_P$ of pin P are calculated from the radius and angular velocity of disk D.
- Direction of velocity $\vec{v}_{P'}$ of point P' on S coinciding with P is perpendicular to radius OP.
- Direction of velocity $\vec{v}_{P/s}$ of P with respect to S is parallel to the slot.
- Solve the vector triangle for the angular velocity of S and relative velocity of P.

Sample Problem 15.9





SOLUTION:

- The absolute velocity of the point P may be written as

  $\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/s}$

- Magnitude and direction of absolute velocity of pin P are calculated from radius and angular velocity of disk D.

  $v_P = R\omega_D = (50\text{ mm})(10\text{ rad/s}) = 500\text{ mm/s}$

- Direction of velocity of P with respect to S is parallel to slot. From the law of cosines,

  $r^2 = R^2 + l^2 - 2Rl\cos 30° = 0.551R^2 \qquad r = 37.1\text{ mm}$

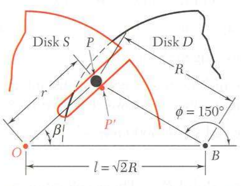
  From the law of cosines,

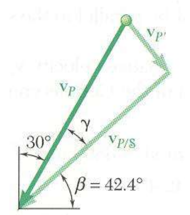
  $\dfrac{\sin\beta}{R} = \dfrac{\sin 30°}{r} \qquad \sin\beta = \dfrac{\sin 30°}{0.742} \qquad \beta = 42.4°$

  The interior angle of the vector triangle is

  $\gamma = 90° - 42.4° - 30° = 17.6°$

Sample Problem 15.9





$v_P = 500\text{ mm/s}$

- Direction of velocity of point P' on S coinciding with P is perpendicular to radius OP. From the velocity triangle,

  $v_{P'} = v_P \sin\gamma = (500\text{mm/s})\sin 17.6° = 151.2\text{mm/s}$

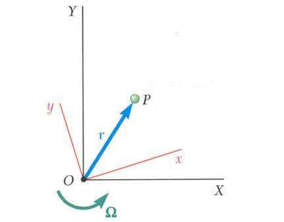
  $= r\omega_s \qquad \omega_s = \dfrac{151.2\text{mm/s}}{37.1\text{ mm}}$

  $\vec{\omega}_s = (-4.08\text{rad/s})\vec{k}$

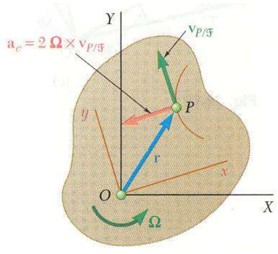
  $v_{P/s} = v_P\cos\gamma = (500\text{m/s})\cos 17.6°$

  $\vec{v}_{P/s} = (477\text{m/s})(-\cos 42.4°\vec{i} - \sin 42.4°\vec{j})$

Plane Rotation with Rotating Frame: Acceleration



$\vec{v}_P = \vec{\Omega}\times\vec{r} + (\dot{\vec{r}})_{Oxy} = \vec{v}_{P'} + \vec{v}_{P/\mathbf{F}}$



$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$

- Absolute acceleration for the particle P is

$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\dot{\vec{r}})_{OXY} + \frac{d}{dt}\left[(\dot{\vec{r}})_{Oxy}\right]$$

$$\frac{d}{dt}\vec{r} = \vec{\Omega}\times\vec{r} + (\dot{\vec{r}})_{Oxy} \rightarrow \frac{d}{dt}\left[(\dot{\vec{r}})_{Oxy}\right] = (\ddot{\vec{r}})_{Oxy} + \vec{\Omega}\times(\dot{\vec{r}})_{Oxy}$$

$$\boxed{\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy} + (\ddot{\vec{r}})_{Oxy}}$$

- Absolute acceleration for the particle P becomes

$$\boxed{\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/F} + \vec{a}_C}$$

acceleration due to rigid-motion rotating with {F}

acceleration of within motion relative (as observed) in {F}

something else! Coriolis acceleration (linear+angular velocities or linear motion in rotating frame)

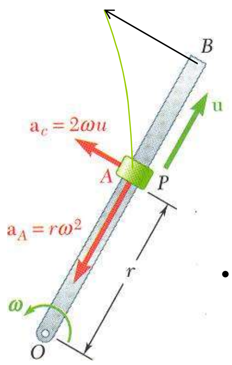
$$\vec{a}_{P'} = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\vec{a}_{P/\mathbf{F}} = (\ddot{\vec{r}})_{Oxy}$$

$$\vec{a}_C = 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy}$$

=> $\vec{a}_c$ *better seen when* $(\ddot{\vec{r}})_{Oxy} = \dot{\Omega} = 0$

Coriolis Acceleration



- Consider a collar P which is sliding at constant relative velocity u along rod OB *as observed in OB*. The rod is rotating at a constant angular velocity ω. The point A of the rod corresponds to the position of P instantaneously.

$$\boxed{\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy} + (\ddot{\vec{r}})_{Oxy}}$$

- Absolute acceleration of the collar is: $\vec{a}_P = \vec{a}_A + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$

$$\vec{a}_A = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \qquad a_A = r\omega^2$$

- centripetal: need to intentionally slow down to maintain constant speed u

$$\vec{a}_{P/\mathfrak{F}} = (\ddot{\vec{r}})_{Oxy} = 0$$

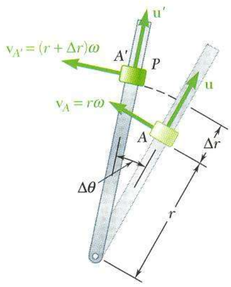
- no acceleration as observed in the rotating rod since u is constant

$$\vec{a}_c = 2\vec{\Omega}\times\vec{v}_{P/\mathfrak{F}} \qquad a_c = 2\omega u$$

- tangential: will experience being pushed to the left on the rod (if no rod (i.e., no force acting on it), the collar will drift to the right)

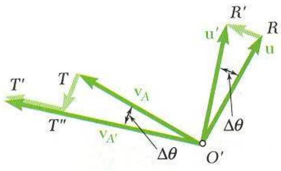
- Absolute acceleration consists of radial (rw^2) and tangential $(2wu)$ components

Coriolis Acceleration



at t, $\vec{v} = \vec{v}_A + \vec{u}$

at $t + \Delta t$, $\vec{v}' = \vec{v}_{A'} + \vec{u}'$



- Change in velocity over Δt is represented by the sum of three vectors

$$\Delta\vec{v} = \overline{RR'} + \overline{TT''} + \overline{T''T'}$$

- $\overline{TT''}$ is due to change in direction of the velocity of point A on the rod,

$$\lim_{\Delta t \to 0} \frac{\overline{TT''}}{\Delta t} = \lim_{\Delta t \to 0} v_A \frac{\Delta\theta}{\Delta t} = r\omega\omega = r\omega^2 = a_A$$

$$\text{recall, } \vec{a}_A = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) \qquad a_A = r\omega^2$$

- $\overline{RR'}$ and $\overline{T''T'}$ result from combined effects of relative motion of P and rotation of the rod

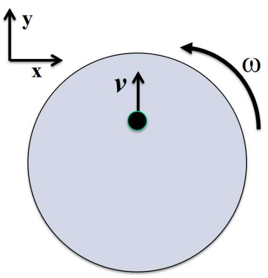
$$\lim_{\Delta t \to 0}\left(\frac{\overline{RR'}}{\Delta t} + \frac{\overline{T''T'}}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(u\frac{\Delta\theta}{\Delta t} + \omega\frac{\Delta r}{\Delta t}\right)$$

$$= u\omega + \omega u = 2\omega u$$

$$\text{recall, } \vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathfrak{F}} \qquad a_c = 2\omega u$$

Concept Question 5

You are walking with a constant velocity with respect to the platform, which rotates with a constant angular velocity w. At the instant shown, in which direction(s) will you experience an acceleration (choose all that apply)?



a) +x

b) −x

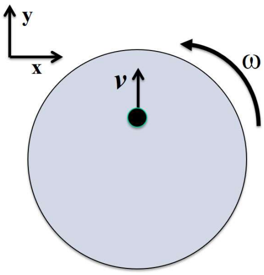
c) +y

d) −y

e) Acceleration = 0

Concept Question 6

You are walking with a constant velocity with respect to the platform, which rotates with a constant angular velocity w. At the instant shown, in which direction(s) will you experience an acceleration (choose all that apply)?



a) +x

b) −x

c) +y

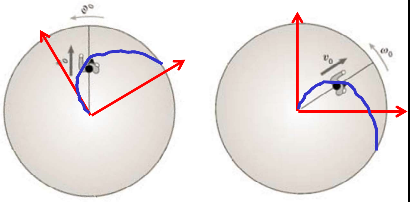
d) −y

e) Acceleration = 0

$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy} + (\ddot{\vec{r}})_{Oxy}$$

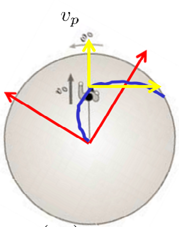
As Observed in a Rotating Frame

$$v_p = v_{p'} + v_{p/F}$$
$$= \omega\times r_p + (\dot{r}_p)_{Oxy}$$

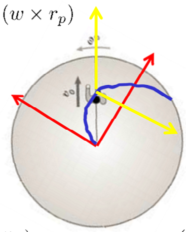


$$a_p = a_{p'} + a_{p/F} + a_c$$
$$= \alpha\times r_p + \omega\times(\omega\times r_p) + (\ddot{r}_p)_{Oxy} + 2\omega\times(\dot{r}_p)_{Oxy}$$

If you're sitting at the center of the rotating frame {F} (with constant w) and see the ball moving straight in the inertial frame {O} (with constant v), you will experience the relative motion (as observed in {F}) of the ball from you as follows:

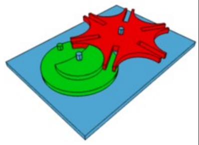


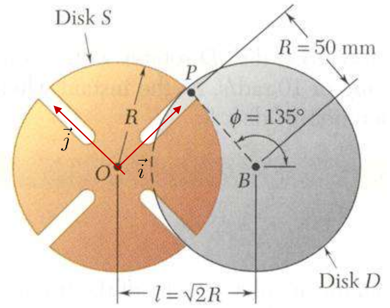
$$(\dot{r}_p)_{Oxy} = v_p - w\times r_p$$



$$(\ddot{r}_p)_{Oxy} = -w\times(w\times r_p) - 2w\times(\dot{r}_p)_{Oxy}$$

Sample Problem 15.10





SOLUTION:

- The absolute acceleration of the pin P may be expressed as

  $$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/s} + \vec{a}_c$$

- The instantaneous angular velocity of Disk S is determined as in Sample Problem 15.9.

- unknowns: $\alpha_S, a_{P/S}$ (directions known though)

- Resolve each acceleration term into the component parallel to the slot. Solve for the angular acceleration of Disk S.

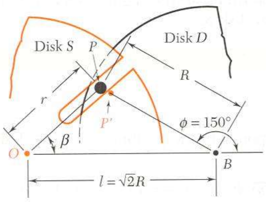
In the Geneva mechanism, disk D rotates with a constant counter-clockwise angular velocity of 10 rad/s. At the instant when φ = 150°, determine angular acceleration of disk S.

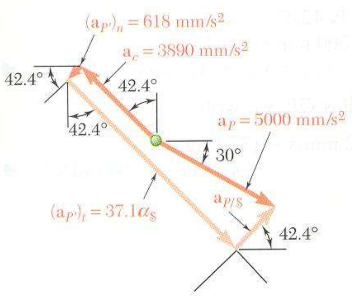
w.r.t. D, w.r.t. S

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/F} + \vec{a}_C$$

$$\vec{a}_P = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2\vec{\Omega} \times (\dot{\vec{r}})_{Oxy} + (\ddot{\vec{r}})_{Oxy}$$

Sample Problem 15.10





SOLUTION:

- Absolute acceleration of the pin P may be expressed as

  $$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/s} + \vec{a}_c$$

- From Sample Problem 15.9.

  $$\beta = 42.4^\circ \qquad \vec{\omega}_S = (-4.08\,\text{rad/s})\vec{k}$$

  $$\vec{v}_{P/s} = (477\,\text{mm/s})(-\cos 42.4^\circ \vec{i} - \sin 42.4^\circ \vec{j})$$

- Considering each term in the acceleration equation,

  $$a_P = R\omega_D^2 = (500\text{mm})(10\,\text{rad/s})^2 = 5000\,\text{mm/s}^2$$

  $$\vec{a}_P = (5000\,\text{mm/s}^2)(\cos 30^\circ \vec{i} - \sin 30^\circ \vec{j})$$

  $$\vec{a}_{P'} = (\vec{a}_{P'})_n + (\vec{a}_{P'})_t$$

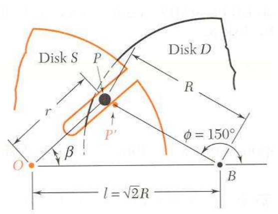
  $$(\vec{a}_{P'})_n = (r\omega_S^2)(-\cos 42.4^\circ \vec{i} - \sin 42.4^\circ \vec{j})$$

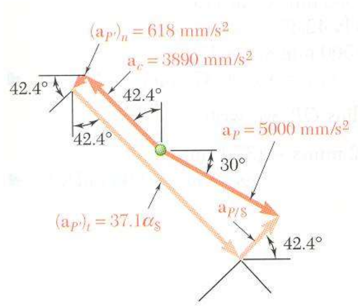
  $$(\vec{a}_{P'})_t = (r\alpha_S)(-\sin 42.4^\circ \vec{i} + \cos 42.4^\circ \vec{j})$$

  $$(\vec{a}_{P'})_t = (\alpha_S)(37.1\text{mm})(-\sin 42.4^\circ \vec{i} + \cos 42.4^\circ \vec{j})$$

  note: α_S may be positive or negative

Sample Problem 15.10





- The direction of the Coriolis acceleration is obtained by rotating the direction of the relative velocity $\vec{v}_{P/s}$ by 90° in the sense of ω_S.

$$\vec{a}_c = \left(2\omega_S v_{P/s}\right)\left(-\sin 42.4°\vec{i} + \cos 42.4\vec{j}\right)$$
$$= 2(4.08\,\text{rad/s})(477\,\text{mm/s})\left(-\sin 42.4°\vec{i} + \cos 42.4\vec{j}\right)$$
$$= \left(3890\,\text{mm/s}^2\right)\left(-\sin 42.4°\vec{i} + \cos 42.4\vec{j}\right)$$

- The relative acceleration $\vec{a}_{P/s}$ must be parallel to the slot.

- Equating components of the acceleration terms perpendicular to the slot,

$$37.1\alpha_S + 3890 - 5000\cos 17.7° = 0$$
$$\alpha_S = -233\,\text{rad/s}$$

$$\boxed{\vec{\alpha}_S = (-233\,\text{rad/s})\vec{k}}$$

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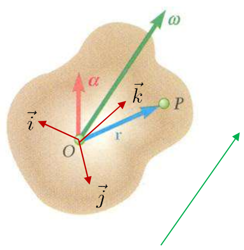
3D Rigid Rotation about a Fixed Point



$$(\dot{\vec{r}})_{Oxy} = 0$$

- The most general 3D motion of a rigid body with a fixed point O is equivalent to a pure rotation of the body about an axis through O (due to Chasles' or Euler's theorem)

Chasles' Theorem: Any rigid body motion can be produced by a rotation about an axis combined with a translation along that axis.
⇒ for O to be fixed, $\vec{w}$ should go through O with no translation.

Euler's Theorem: Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O.

- With instantaneous axis of rotation and angular velocity $\vec{w}$,

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r} \qquad \vec{a} = \vec{\alpha}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) \qquad \vec{\alpha} = \frac{d\vec{\omega}}{dt}.$$

vector

$$\vec{v}_P = (\dot{\vec{r}})_{OXY} = \vec{\Omega}\times\vec{r} + (\dot{r})_{Oxy}$$

$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + 2\vec{\Omega}\times(\dot{\vec{r}})_{Oxy} + (\ddot{\vec{r}})_{Oxy}$$

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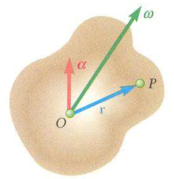
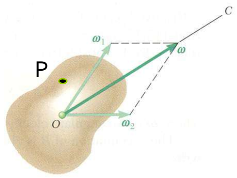
3D Rigid Rotation about a Fixed Point



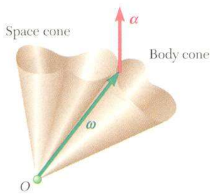
$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \qquad (\dot{\vec{r}})_{Oxy} = 0$$

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad \vec{\alpha} = \frac{d\vec{\omega}}{dt}.$$

- The angular acceleration $\vec{\alpha}$ represents the velocity of the tip of $\vec{\omega}$. Note that $\vec{\alpha}$ and $\vec{w}$ are not collinear!



- As the vector $\vec{\omega}$. moves within the body and in space, it generates a body cone and space cone which are tangent along the instantaneous axis of rotation.



- Angular velocities are vectors, obeying the law of addition. Rotation motion itself is not though.

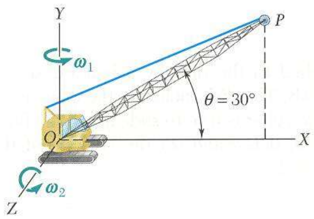
For the same point P, given velocities $v_1 = w_1 \times r$ and $v_2 = w_2 \times r$, we can add them s.t.,

$$\vec{v} = \vec{v}_1 + \vec{v}_2 = (\vec{w}_1 + \vec{w}_2) \times \vec{r} = \vec{w} \times \vec{r}$$

i.e., $\vec{w}$ combining w_1 and w_2 is given by $\vec{w} = \vec{w}_1 + \vec{w}_2$

(for more on motions ⇒ 로봇공학입문)

Sample Problem 15.11



The crane rotates with a constant angular velocity $\omega_1 = 0.30$ rad/s and the boom is being raised with a constant angular velocity $\omega_2 = 0.50$ rad/s. The length of the boom is $l = 12$ m.

Determine:
- angular velocity of the boom,
- angular acceleration of the boom,
- velocity of the boom tip, and
- acceleration of the boom tip.

SOLUTION:

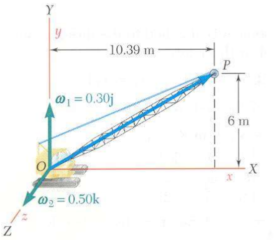
With $\vec{\omega}_1 = 0.30\vec{j} \quad \vec{\omega}_2 = 0.50\vec{k}$

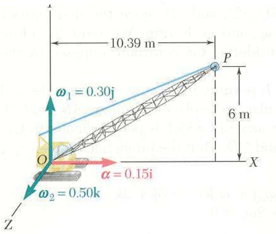
$$\vec{r} = 12(\cos 30°\vec{i} + \sin 30°\vec{j})$$
$$= 10.39\vec{i} + 6\vec{j}$$

- Angular velocity of the boom,
  $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$
- Angular acceleration of the boom,
  $\vec{\alpha} = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_2 = \dot{\vec{\omega}}_2 = (\dot{\vec{\omega}}_2)_{Oxyz} + \vec{\Omega} \times \vec{\omega}_2$
  $= \vec{\omega}_1 \times \vec{\omega}_2$
- Velocity of boom tip,
  $\vec{v} = \vec{\omega} \times \vec{r}$
- Acceleration of boom tip,
  $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$

$\vec{v}_P = (\dot{\vec{r}})_{OXY} = \vec{\Omega} \times \vec{r} + (\dot{r})_{Oxy}$ $\quad$ $\vec{a}_P = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2\vec{\Omega} \times (\dot{\vec{r}})_{Oxy} + (\ddot{\vec{r}})_{Oxy}$

Sample Problem 15.11





$\vec{\omega}_1 = 0.30\vec{j} \quad \vec{\omega}_2 = 0.50\vec{k}$

$\vec{r} = 10.39\vec{i} + 6\vec{j}$

SOLUTION:

- Angular velocity of the boom,

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

$$\boxed{\vec{\omega} = (0.30\,\mathrm{rad/s})\vec{j} + (0.50\,\mathrm{rad/s})\vec{k}}$$

- Angular acceleration of the boom,

$$\vec{\alpha} = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_2 = \dot{\vec{\omega}}_2 = (\dot{\vec{\omega}}_2)_{Oxyz} + \vec{\Omega} \times \vec{\omega}_2$$

$$= \vec{\omega}_1 \times \vec{\omega}_2 = (0.30\,\mathrm{rad/s})\vec{j} \times (0.50\,\mathrm{rad/s})\vec{k}$$

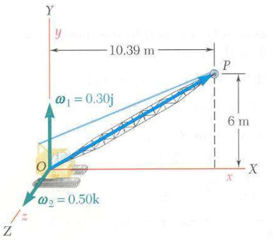
$$\boxed{\vec{\alpha} = \left(0.15\,\mathrm{rad/s^2}\right)\vec{i}}$$

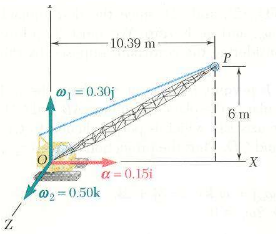
- Velocity of boom tip,

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0.3 & 0.5 \\ 10.39 & 6 & 0 \end{vmatrix}$$

$$\boxed{\vec{v} = -(3.54\,\mathrm{m/s})\vec{i} + (5.20\,\mathrm{m/s})\vec{j} - (3.12\,\mathrm{m/s})\vec{k}}$$

Sample Problem 15.11





$\vec{\omega}_1 = 0.30\vec{j} \quad \vec{\omega}_2 = 0.50\vec{k}$

$\vec{r} = 10.39\vec{i} + 6\vec{j}$

- Acceleration of boom tip,

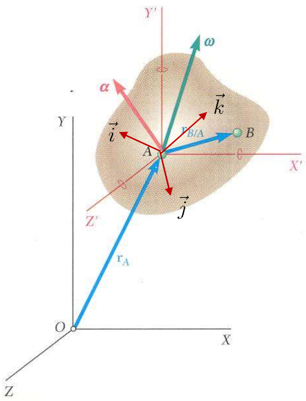
$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

$$\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.15 & 0 & 0 \\ 10.39 & 6 & 0 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0.30 & 0.50 \\ -3 & 5.20 & -3.12 \end{vmatrix}$$

$$= 0.90\vec{k} - 0.94\vec{i} - 2.60\vec{i} - 1.50\vec{j} + 0.90\vec{k}$$

$$\boxed{\vec{a} = -\left(3.54\,\mathrm{m/s^2}\right)\vec{i} - \left(1.50\,\mathrm{m/s^2}\right)\vec{j} + \left(1.80\,\mathrm{m/s^2}\right)\vec{k}}$$

General 3D Motion of Rigid Body



- For two points A and B of a rigid body,

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

$$\vec{r}_{B/A} = r^x_{B/A}\vec{i} + r^y_{B/A}\vec{j} + r^z_{B/A}\vec{k} \;\Rightarrow\; \left(\dot{\vec{r}}_{B/A}\right)_{\{\mathcal{F}\}} = 0$$

- Differentiating this, we obtain the velocity of B:

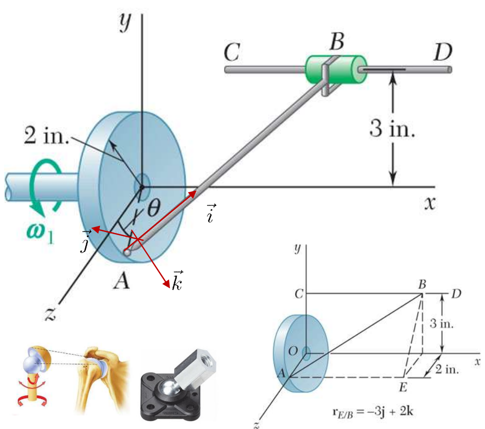
$$\vec{v}_B = \dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} \\ = \vec{v}_A + \vec{w} \times \vec{r}_{B/A}$$

- Similarly, the acceleration of the particle B is

$$\vec{a}_B = \dot{\vec{v}}_B = \dot{\vec{v}}_A + \frac{d}{dt}\left[\vec{w} \times \vec{r}_{B/A}\right] \\ = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{w} \times (\vec{w} \times \vec{r}_{B/A})$$

- Most general motion of a rigid body is equivalent to the combination of:
  - a translation in which all the particles are translating together with a velocity of a point of the rigid-body; and
  - a rotation in which all the particles are rotating about that point

Sample Problem 15.12



A: ball and socket connection
B: collar with clevis connection
$w_1 = 12 rad/s$: constant
determine: (a) v_B and (b) w_{AB}

SOLUTION:

- Position of the collar is ($\{\mathfrak{F}\}=\{i,j,k\}$):

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$
$$\vec{r}_{B/A} = r^x_{B/A}\vec{i} + r^y_{B/A}\vec{j} + r^z_{B/A}\vec{k}$$

- Velocity of the collar is:

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$
$$\vec{v}_A = \vec{w}_1 \times \vec{r}_A$$
$$\vec{v}_{B/A} = \vec{w}_{AB} \times \vec{r}_{B/A}$$

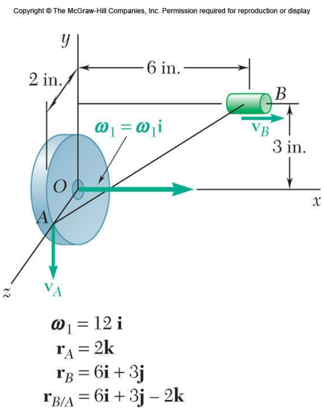
since ($\{\mathfrak{F}\}=\{i,j,k\}$) is rotating w/ $\vec{w}_{AB}$

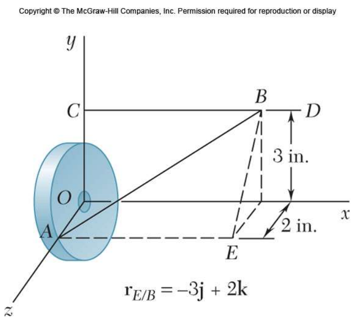
- 3 equations, four unknowns…

$$\vec{v}_B = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}, \; \vec{w}_{AB} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

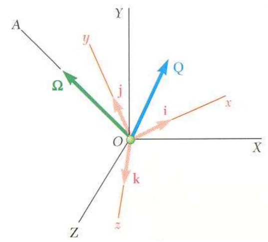
+ one constraint: $\vec{w}^T_{AB} = \vec{r}_{EB} = 0$

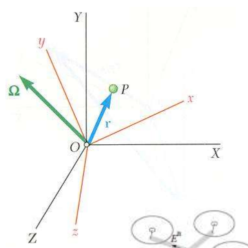
Sample Problem 15.12

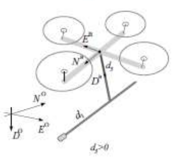




Recall: Particle Motion with Rotating Frame







- With respect to the fixed frame *OXYZ* and rotating frame *Oxyz* with Ω,

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega} \times \vec{Q}$$

- Consider motion of particle *P* relative to a rotating frame *Oxyz* or $\{\mathfrak{F}\}$ for short. The absolute velocity can be expressed as

$$\frac{d\vec{r}}{dt} = \vec{v}_P = \vec{\Omega} \times \vec{r} + \left(\dot{\vec{r}}\right)_{Oxyz}$$
$$= \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

rigid-motion with $Oxyz$ — as-observed (relative) in $Oxyz$ (= 0)

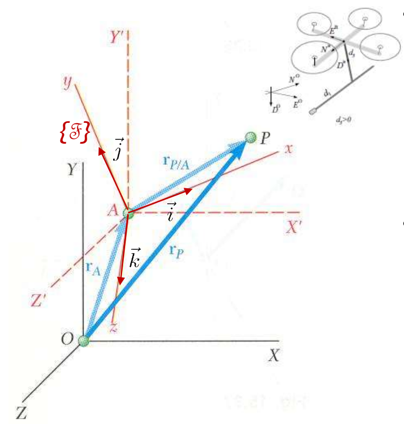
- The absolute acceleration can be expressed as

$$\vec{a}_P = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxyz} + \left(\ddot{\vec{r}}\right)_{Oxyz}$$
$$= \vec{a}_{p'} + \vec{a}_{P/\mathbf{F}} + \vec{a}_c$$

rigidly-motion with $Oxyz$ — as observed (relative) in $Oxyz$

$$\vec{a}_c = 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxyz} = 2\vec{\Omega} \times \vec{v}_{P/\mathbf{F}} = \text{Coriolis acceleration}$$

3D Motion with Moving/Rotating Frame



- Kinematics equation of P with relative motion (as observed) in rotating-frame $\{\mathfrak{F}\}$:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} = \vec{r}_A + r^x_{B/A}\vec{i} + r^y_{B/A}\vec{j} + r^z_{B/A}\vec{k}$$

where $\{\mathcal{F}\} = \{\vec{i}, \vec{j}, \vec{k}\}$ and $\dot{r}^{\star}_{B/A} \neq 0$

- Velocity and acceleration of P can still be found as a combination of rigid-motion with $\{\mathfrak{F}\}$ and as observe motion in $\{\mathfrak{F}\}$:

$$\vec{v}_P = \vec{v}_A + \vec{\Omega}\times\vec{r}_{P/A} + \left(\dot{\vec{r}}_{P/A}\right)_{Axyz}$$

$$= \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

rigid-motion with $\{\mathfrak{F}\}$ — as observed (relative) in $\{\mathfrak{F}\}$

$$\vec{a}_P = \vec{a}_A + \dot{\vec{\Omega}}\times\vec{r}_{P/A} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}_{P/A}\right) + 2\vec{\Omega}\times\left(\dot{\vec{r}}_{P/A}\right)_{Axyz} + \left(\ddot{\vec{r}}_{P/A}\right)_{Axyz}$$

$$= \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

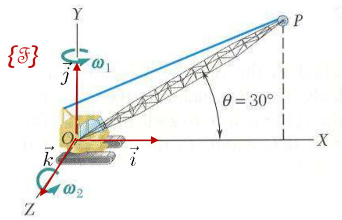
rigid-motion with $\{\mathfrak{F}\}$ — as observed (relative) in $\{\mathfrak{F}\}$ — Coriolis acceleration

With pure-rotating frame:

$$\vec{v}_P = \vec{\Omega}\times\vec{r} + \left(\dot{\vec{r}}\right)_{Oxyz} = \vec{v}_{P'} + \vec{v}_{P/\mathrm{F}}$$

$$\vec{a}_P = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) + 2\vec{\Omega}\times\left(\dot{\vec{r}}\right)_{Oxyz} + \left(\ddot{\vec{r}}\right)_{Oxyz} = \vec{a}_{P'} + \vec{a}_{P/\mathrm{F}} + \vec{a}_c$$

Sample Problem 15.11



The crane rotates with a constant angular velocity $\omega_1 = 0.30$ rad/s and the boom is being raised with a constant angular velocity $\omega_2 = 0.50$ rad/s. The length of the boom is $l = 12$ m.

Determine:
- angular velocity of the boom,
- angular acceleration of the boom,
- velocity of the boom tip, and
- acceleration of the boom tip.

$\Rightarrow$ using body frame $\{\mathfrak{F}\}$ attached on cab?

SOLUTION:

- Angular velocity of the boom,

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

- Angular acceleration of the boom,

$$\vec{\alpha} = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_2 = \vec{\omega}_1 \times \vec{\omega}_2$$

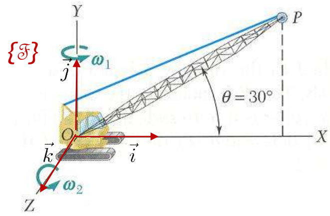
- Velocity of boom tip,

$$\vec{v} = \vec{\omega}\times\vec{r}$$

- Acceleration of boom tip,

$$\vec{a} = \vec{\alpha}\times\vec{r} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) = d\vec{v}_p / dt$$

Sample Problem 15.11



SOLUTION:

- Angular velocity of the boom,

  $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$

- Angular acceleration of the boom,

  $\vec{\alpha} = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_2 = \vec{\omega}_1 \times \vec{\omega}_2$

- Velocity of boom tip,

  $\vec{v} = \vec{\omega} \times \vec{r}$

- Acceleration of boom tip,

  $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = d\vec{v}_p / dt$

SOLUTION with body-frame {𝔉}:

- Body-frame {𝔉} is rigidly-attached to the cabin and rotating with constant w_1
- Velocity of the boom-tip:

  $\vec{v}_p = \vec{v}_{p'} + \vec{v}_{p/F}$

  $\vec{v}_{p'} = \vec{w}_1 \times \vec{r}$: rotating together with {𝔉}

  $\vec{v}_{p/F} = \vec{w}_2 \times \vec{r}$: motion as observed in {𝔉}

- Acceleration of the boom-tip:

  $\vec{a}_p = \vec{a}_{p'} + \vec{a}_{p/F} + \vec{a}_c$

  $\vec{a}_{p'} = \vec{\alpha}_1 \times \vec{r} + \vec{w}_1 \times (\vec{w}_1 \times \vec{r})$: with {𝔉}

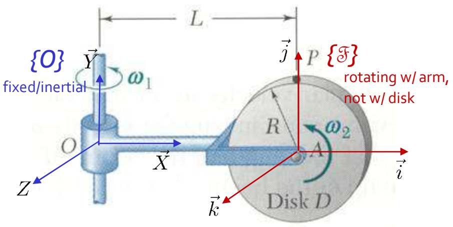
  $\vec{a}_{p/F} = \vec{w}_2 \times (\vec{w}_2 \times \vec{r})$: in {𝔉}

  $\vec{a}_c = 2\vec{w}_1 \times (\dot{\vec{r}})_{\{\mathcal{F}\}} = 2\vec{w}_1 \times (\vec{w}_2 \times \vec{r})$

- This is the same as before, which can be shown by using the Jacobi identity:

$$w_1 \times (w_2 \times r) + w_2 \times (r \times w_1) + r \times (w_1 \times w_2) = 0$$

Sample Problem 15.15



For the disk mounted on the arm, the indicated angular rotation rates are constant.

Determine:

- the velocity of the point P
- the acceleration of P; and
- angular velocity and angular acceleration of the disk.

where to put the rotating frame?

SOLUTION:

- Define a rotating frame $Axyz$ or 𝔉 rigidly-attached to the arm at A.
- Velocity kinematics:

  $\vec{v}_p = \vec{v}_{p'} + \vec{v}_{p/F}$

  $\vec{v}_{p'} = \vec{w}_1 \times \vec{r}_{P/O} = -Lw_1\vec{K}$: with {𝔉}

  $\vec{v}_{p/F} = \vec{w}_2 \times \vec{r}_{P/A} = -Rw_2\vec{X}$: in {𝔉}

- Acceleration kinematics:

  $\vec{a}_p = \vec{a}_{p'} + \vec{a}_{p/F} + \vec{a}_c$

  $\vec{a}_{p'} = \vec{w}_1 \times (\vec{w}_1 \times \vec{r}_{P/O}) = -Lw_1^2\vec{I}$: with {𝔉}

  $\vec{a}_{p/F} = \vec{w}_2 \times (\vec{w}_2 \times \vec{r}_{P/A}) = -Rw_2^2\vec{J}$: in {𝔉}

  $\vec{a}_c = 2\vec{w}_1 \times (\dot{\vec{r}})_{\{\mathcal{F}\}}$

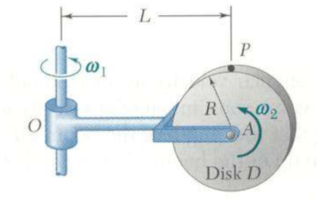
  $= 2\vec{w}_1 \times (\vec{w}_2 \times \vec{r}_{P/A}) = 2Rw_1w_2\vec{K}$

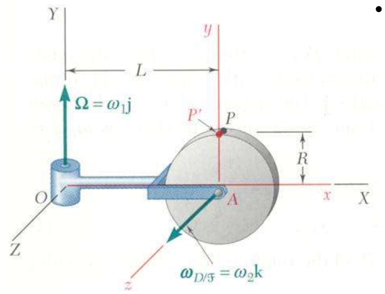
- Disk angular velocity and acceleration:

  $\vec{w} = \vec{w}_1 + \vec{w}_2 = w_1\vec{K} + w_1\vec{K}$

  $\dot{\vec{w}} = \dot{\vec{w}}_2 = \vec{w}_1 \times \vec{w}_2 = w_1w_2\vec{I}$

Sample Problem 15.15





SOLUTION:

- Define a fixed reference frame *OXYZ* at *O* and a moving reference frame *Axyz* or $\mathfrak{F}$ attached to the arm at *A*.

$$\vec{r} = L\vec{i} + R\vec{j} \qquad \vec{r}_{P/A} = R\vec{j}$$

$$\vec{\Omega} = \omega_1 \vec{j} \qquad \vec{\omega}_{D/\mathfrak{F}} = \omega_2 \vec{k}$$

- With *P'* of the moving reference frame coinciding with *P*, the velocity of the point *P* is found from

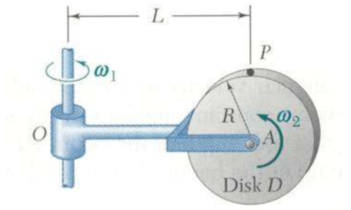
$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathfrak{F}}$$

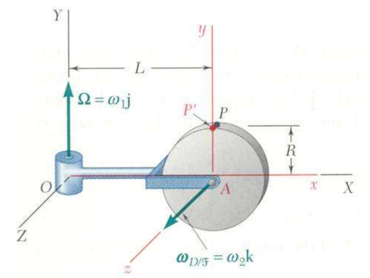
$$\vec{v}_{P'} = \vec{\Omega} \times \vec{r} = \omega_1 \vec{j} \times \left(L\vec{i} + R\vec{j}\right) = -\omega_1 L\vec{k}$$

$$\vec{v}_{P/\mathfrak{F}} = \vec{\omega}_{D/\mathfrak{F}} \times \vec{r}_{P/A} = \omega_2 \vec{k} \times R\vec{j} = -\omega_2 R\vec{i}$$

$$\boxed{\vec{v}_P = -\omega_2 R\vec{i} - \omega_1 L\vec{k}}$$

Sample Problem 15.15





- The acceleration of *P* is found from

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathfrak{F}} + \vec{a}_c$$

$$\vec{a}_{P'} = \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) = \omega_1 \vec{j} \times \left(-\omega_1 L\vec{k}\right) = -\omega_1^2 L\vec{i}$$

$$\vec{a}_{P/\mathfrak{F}} = \vec{\omega}_{D/\mathfrak{F}} \times \left(\vec{\omega}_{D/\mathfrak{F}} \times \vec{r}_{P/A}\right)$$

$$= \omega_2 \vec{k} \times \left(-\omega_2 R\vec{i}\right) = -\omega_2^2 R\vec{j}$$

$$\vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathfrak{F}}$$

$$= 2\omega_1 \vec{j} \times \left(-\omega_2 R\vec{i}\right) = 2\omega_1 \omega_2 R\vec{k}$$

$$\boxed{\vec{a}_P = -\omega_1^2 L\vec{i} - \omega_2^2 R\vec{j} + 2\omega_1 \omega_2 R\vec{k}}$$

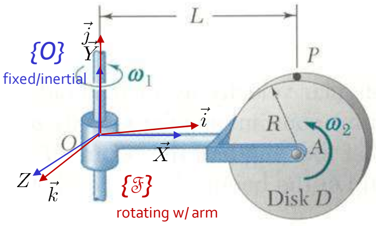
- Angular velocity and acceleration of the disk,

$$\vec{\omega} = \vec{\Omega} + \vec{\omega}_{D/\mathfrak{F}} \qquad \boxed{\vec{\omega} = \omega_1 \vec{j} + \omega_2 \vec{k}}$$

$$\vec{\alpha} = \left(\dot{\vec{\omega}}\right)_{\mathfrak{F}} + \vec{\Omega} \times \vec{\omega}$$

$$= \omega_1 \vec{j} \times \left(\omega_1 \vec{j} + \omega_2 \vec{k}\right) \qquad \boxed{\vec{\alpha} = \omega_1 \omega_2 \vec{i}}$$

Sample Problem 15.15



For the disk mounted on the arm, the indicated angular rotation rates are constant.

Determine:

- the velocity of the point P
- the acceleration of P; and
- angular velocity and angular acceleration of the disk.

where to put the rotating frame?

SOLUTION:

- Define a rotating frame $Axyz$ or $\mathfrak{F}$ rigidly-attached to the arm at O.
- Velocity kinematics:

$$\vec{v}_p = \vec{v}_{p'} + \vec{v}_{p/F}$$

$$\vec{v}_{p'} = \vec{w}_1 \times \vec{r}_{P/O} = -Lw_1\vec{K} \quad \text{: with } \{\mathfrak{F}\}$$

$$\vec{v}_{p/F} = \vec{w}_2 \times \vec{r}_{P/O} = -Rw_2\vec{X} \quad \text{: in } \{\mathfrak{F}\}$$

- Acceleration kinematics:

$$\vec{a}_p = \vec{a}_{p'} + \vec{a}_{p/F} + \vec{a}_c$$

$$\vec{a}_{p'} = \vec{w}_1 \times (\vec{w}_1 \times \vec{r}_{P/O}) = -Lw_1^2\vec{I} \quad \text{: with } \{\mathfrak{F}\}$$

$$\vec{a}_{p/F} = \vec{w}_2 \times (\vec{w}_2 \times \vec{r}_{P/A}) = -Rw_2^2\vec{J} \quad \text{: in } \{\mathfrak{F}\}$$

$$\vec{a}_c = 2\vec{w}_1 \times (\dot{\vec{r}})_{\{\mathcal{F}\}}$$

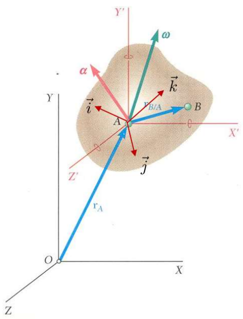
$$= 2\vec{w}_1 \times (\vec{w}_2 \times \vec{r}_{P/A}) = 2Rw_1w_2\vec{K}$$

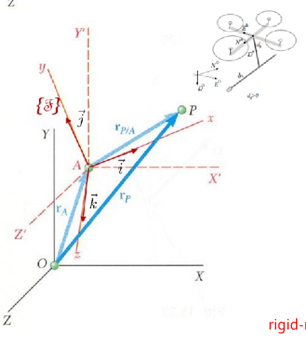
- Disk angular velocity and acceleration:

$$\vec{w} = \vec{w}_1 + \vec{w}_2 = w_1\vec{K} + w_1\vec{K}$$

$$\dot{\vec{w}} = \dot{\vec{w}}_2 = \vec{w}_1 \times \vec{w}_2 = w_1w_2\vec{I}$$

This Lecture





- Kinematics of rigid-body:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

$$\vec{r}_{B/A} = r^x_{B/A}\vec{i} + r^y_{B/A}\vec{j} + r^z_{B/A}\vec{k} \Rightarrow \left(\dot{\vec{r}}_{B/A}\right)_{\{\mathcal{F}\}} = 0$$

$$\vec{v}_B = \dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} = \vec{v}_A + \vec{w} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \dot{\vec{v}}_B = \dot{\vec{v}}_A + \frac{d}{dt}\left[\vec{w} \times \vec{r}_{B/A}\right]$$

$$= \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{w} \times (\vec{w} \times \vec{r}_{B/A})$$

- Kinematics of P w/ relative motion (as observed) in rotating-frame $\{\mathfrak{F}\}$:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} = \vec{r}_A + r^x_{B/A}\vec{i} + r^y_{B/A}\vec{j} + r^z_{B/A}\vec{k}$$

$$\vec{v}_P = \vec{v}_A + \vec{\Omega} \times \vec{r}_{P/A} + \left(\dot{\vec{r}}_{P/A}\right)_{\{F\}} = \vec{v}_{P'} + \vec{v}_{P/\mathbf{F}}$$

rigid-motion with $\{\mathfrak{F}\}$ — as observed (relative) in $\{\mathfrak{F}\}$

$$\vec{a}_P = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{P/A} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}_{P/A}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}_{P/A}\right)_{\{F\}} + \left(\ddot{\vec{r}}_{P/A}\right)_{\{F\}}$$

$$= \vec{a}_{P'} + \vec{a}_{P/\mathbf{F}} + \vec{a}_c$$

Coriolis acceleration

rigid-motion with $\{\mathfrak{F}\}$ — as observed (relative) in $\{\mathfrak{F}\}$