# Lecture 5

# Basic Constitutive Laws

# (8 April 2020)

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# Outline

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- Importance
- Linear elasticity
- Elastic moduli and seismic wave velocity
- Elastic anisotropy
- Poroelasticity and effective stress
- Poroelasticity and dispersion
- Viscous deformation in uncemented sands
- Thermoporoelasticity

# Importance
# Constitutive Law(constitutive equation, 구성방정식)

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Porous media fluid flow (다공성매질의 유체유동)

Darcy's Law

Fluid Flux

Pressure gradient

$$q = -\frac{k}{\mu}\frac{dP}{dl}$$

Permeability

Time dependent

Conservation of mass

(Elastic) Geomechanics (암반역학/지오메카닉스)

Hooke's Law

Stress (응력)

strain (변형율)

$$\sigma = E\varepsilon = E\frac{du}{dx}$$

Elastic modulus & Poisson's ratio

Not time dependent (elastic)
time dependent → creep

Equilibrium Equation

(Poroelastic) Geomechanics

(Elastoplastic) Geomechanics

(Viscoelastic) Geomechanics

# Importance

- Constitutive law
  - Deformation of a rock in response to an applied stress
- Elastic
- Poroelastic
- Elastoplastic (Elastic-Plastic)
- Viscoelastic

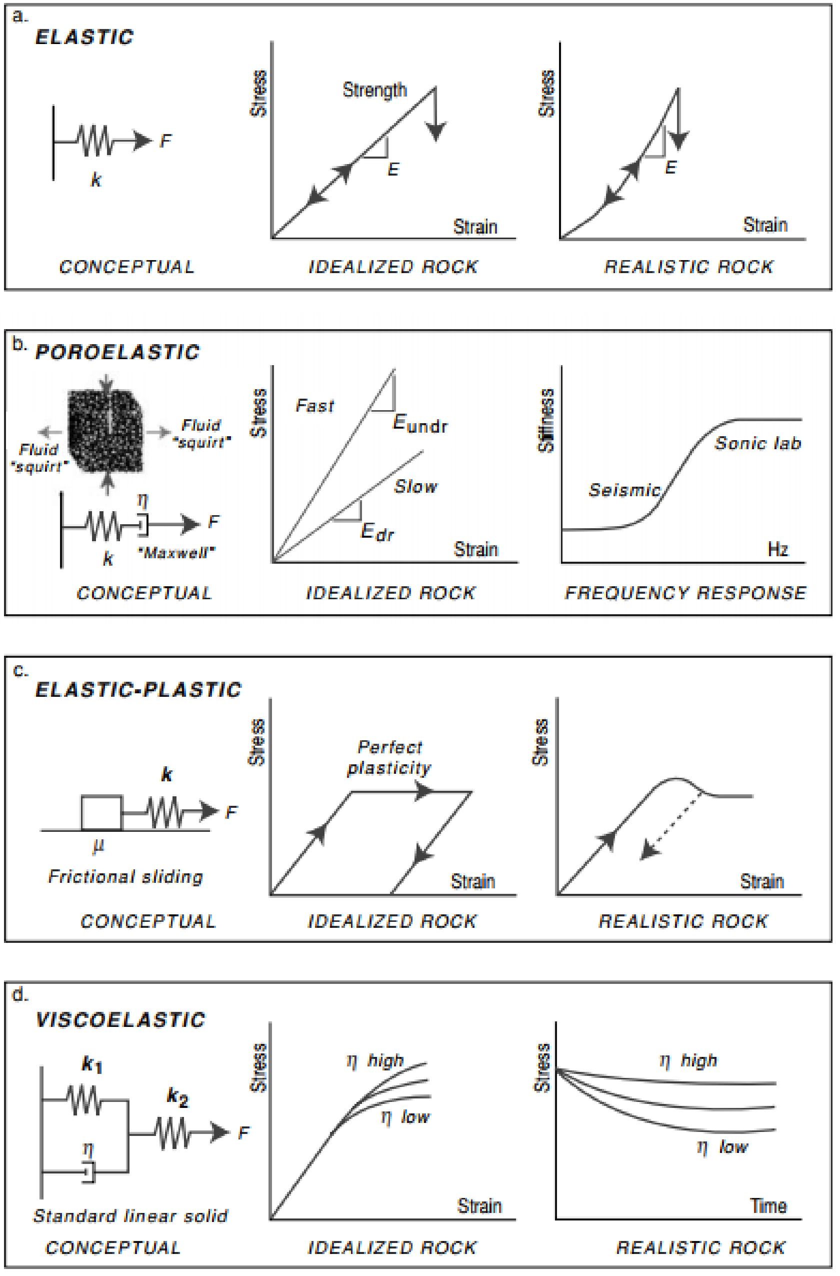


**Figure 3.1.** Schematic illustration of elastic, poroelastic, elastic–plastic and viscoelastic constitutive laws. In the left panels, analogous physical models are shown, in the center, idealized stress–strain curves, and in right panels, schematic diagrams representing more realistic rock behavior are shown.

Zoback MD, 2007, Reservoir Geomechanics, Cambridge University Press

# Linear elasticity
# Laboratory experiment

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- Typical laboratory stress-strain data for rock

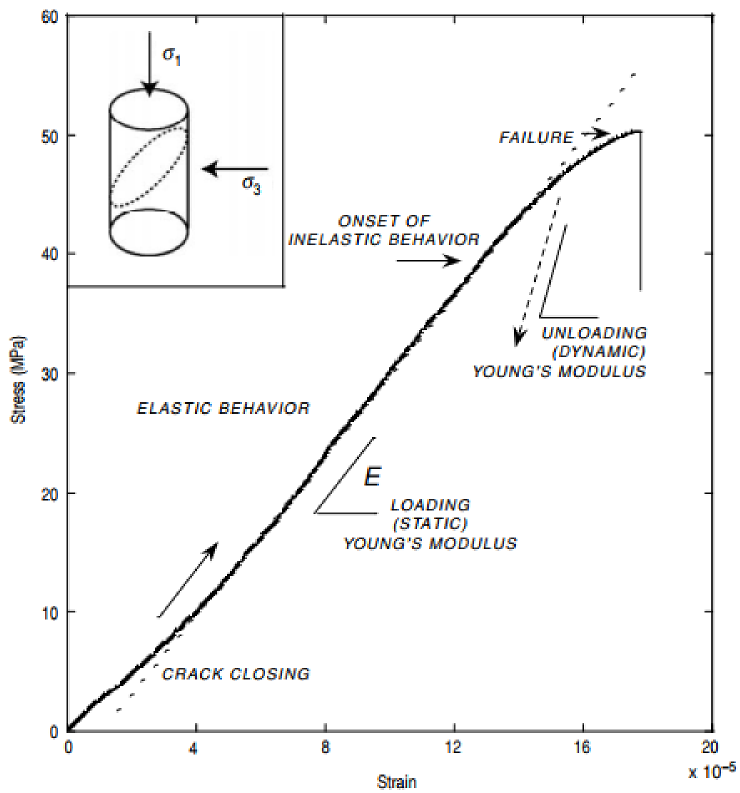


**Figure 3.2.** Typical laboratory stress–strain data for a well-cemented rock being deformed uniaxially. There is a small degree of crack closure upon initial application of stress followed by linear elastic behavior over a significant range of stresses. Inelastic deformation is seen again just before failure due to damage in the rock.

Zoback MD, 2007, Reservoir Geomechanics, Cambridge University Press

# Linear elasticity
# Definition of strain

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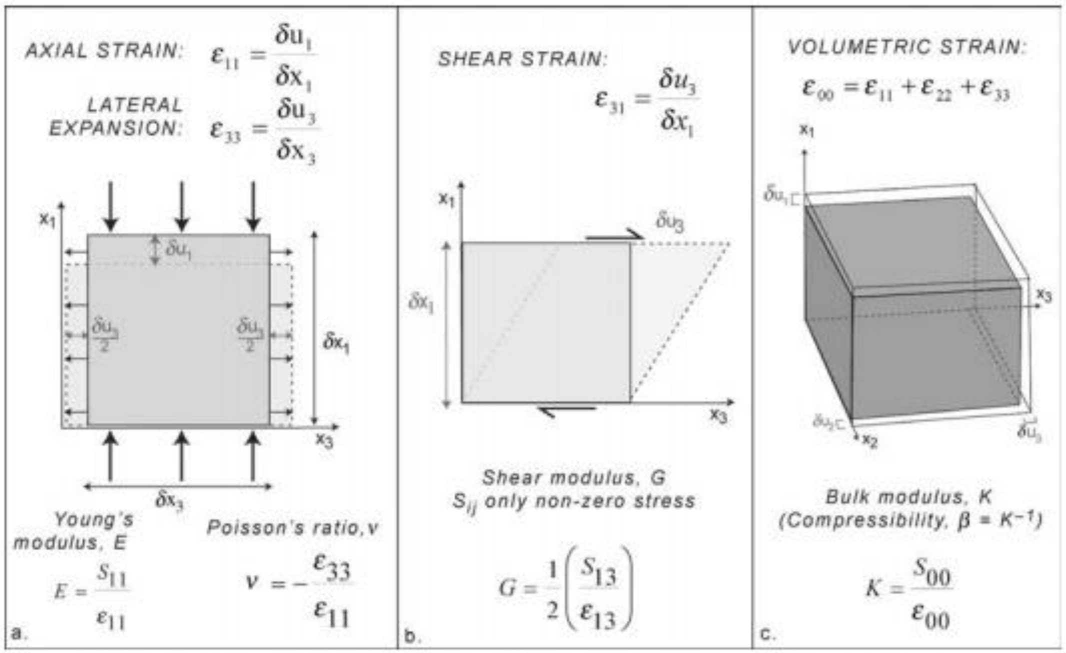


**Figure 3.3.** Schematic illustration of the relationships between stress, strain and the physical meaning of frequently used elastic moduli in different types of idealized deformation measurements.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i}\right) \tag{3.1}$$

$$\varepsilon_{00} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \qquad S_{00} = \frac{1}{3}(S_{11} + S_{22} + S_{33}) \tag{3.2}$$

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# Linear elasticity

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- Elastic constitutive equations
  - Various forms
  - Various elastic parameters (expressed only *two*)

$$
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}
$$

$\downarrow$ **invert**

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}
= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1 & \nu/(1-\nu) & \nu/(1-\nu) & 0 & 0 & 0 \\
\nu/(1-\nu) & 1 & \nu/(1-\nu) & 0 & 0 & 0 \\
\nu/(1-\nu) & \nu/(1-\nu) & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}
$$

$$\lambda = \frac{2\nu G}{(1-2\nu)} = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$S_1 = (\lambda + 2G)\,\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3 = \lambda\varepsilon_{00} + 2G\varepsilon_1$$
$$S_2 = \lambda\varepsilon_1 + (\lambda + 2G)\,\varepsilon_2 + \lambda\varepsilon_3 = \lambda\varepsilon_{00} + 2G\varepsilon_2$$
$$S_3 = \lambda\varepsilon_1 + \lambda\varepsilon_2 + (\lambda + 2G)\,\varepsilon_3 = \lambda\varepsilon_{00} + 2G\varepsilon_3$$

and $\lambda$ (Lame's constant), $K$ (bulk modulus) and $G$ (shear modulus) are all elastic moduli.

**Table 3.1.** *Relationships among elastic moduli in an isotropic material*

| $K$ | $E$ | $\lambda$ | $\nu$ | $G$ | $M$ |
|---|---|---|---|---|---|
| $\lambda + \frac{2G}{3}$ | $G\frac{3\lambda+2G}{\lambda+G}$ | – | $\frac{\lambda}{2(\lambda+G)}$ | – | $\lambda + 2G$ |
| – | $9K\frac{K-\lambda}{3K-\lambda}$ | – | $\frac{\lambda}{3K-\lambda}$ | $3\frac{K-\lambda}{2}$ | $3K - 2\lambda$ |
| – | $\frac{9K-G}{3K-G}$ | $K - \frac{2G}{3}$ | $\frac{3K-2G}{2(3K+G)}$ | – | $K + 4\frac{G}{3}$ |
| $\frac{\varepsilon G}{3(3G-E)}$ | – | $G\frac{E-2G}{3G-E}$ | $\frac{E}{2G} - 1$ | – | $G\frac{4G-E}{3G-E}$ |
| – | – | $3K\frac{3K-E}{9K-E}$ | $\frac{3K-E}{6K}$ | $\frac{3KE}{9K-E}$ | $3K\frac{3K+E}{9K-E}$ |
| $\lambda\frac{1+\nu}{3\nu}$ | $\lambda\frac{(1+\nu)(1-\nu)}{\nu}$ | – | – | $\lambda\frac{1-2\nu}{2\nu}$ | $\lambda\frac{1-\nu}{\nu}$ |
| $G\frac{2(1+\nu)}{3(1-2\nu)}$ | $2G(1+\nu)$ | $G\frac{2\nu}{1-2\nu}$ | – | – | $G\frac{2-2\nu}{1-2\nu}$ |
| – | $3K(1-2\nu)$ | $3K\frac{\nu}{1+\nu}$ | – | $3K\frac{1-2\nu}{2+2\nu}$ | $3K\frac{1-\nu}{1+\nu}$ |
| $\frac{E}{3(1-2\nu)}$ | – | $\frac{E\nu}{(1+\nu)(1-2\nu)}$ | – | $\frac{E}{2+2\nu}$ | $\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$ |

$$S_{ij} = \lambda\delta_{ij}\varepsilon_{00} + 2G\varepsilon_{ij} \tag{3.3}$$

where the Kronecker delta, $\delta_{ij}$, is given by

$\delta_{ij} = 1 \quad i = j$

$\delta_{ij} = 0 \quad i \neq j$

# Linear elasticity

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- Typical values
  - Elastic modulus
  - Poisson's ratio
  - Porosity

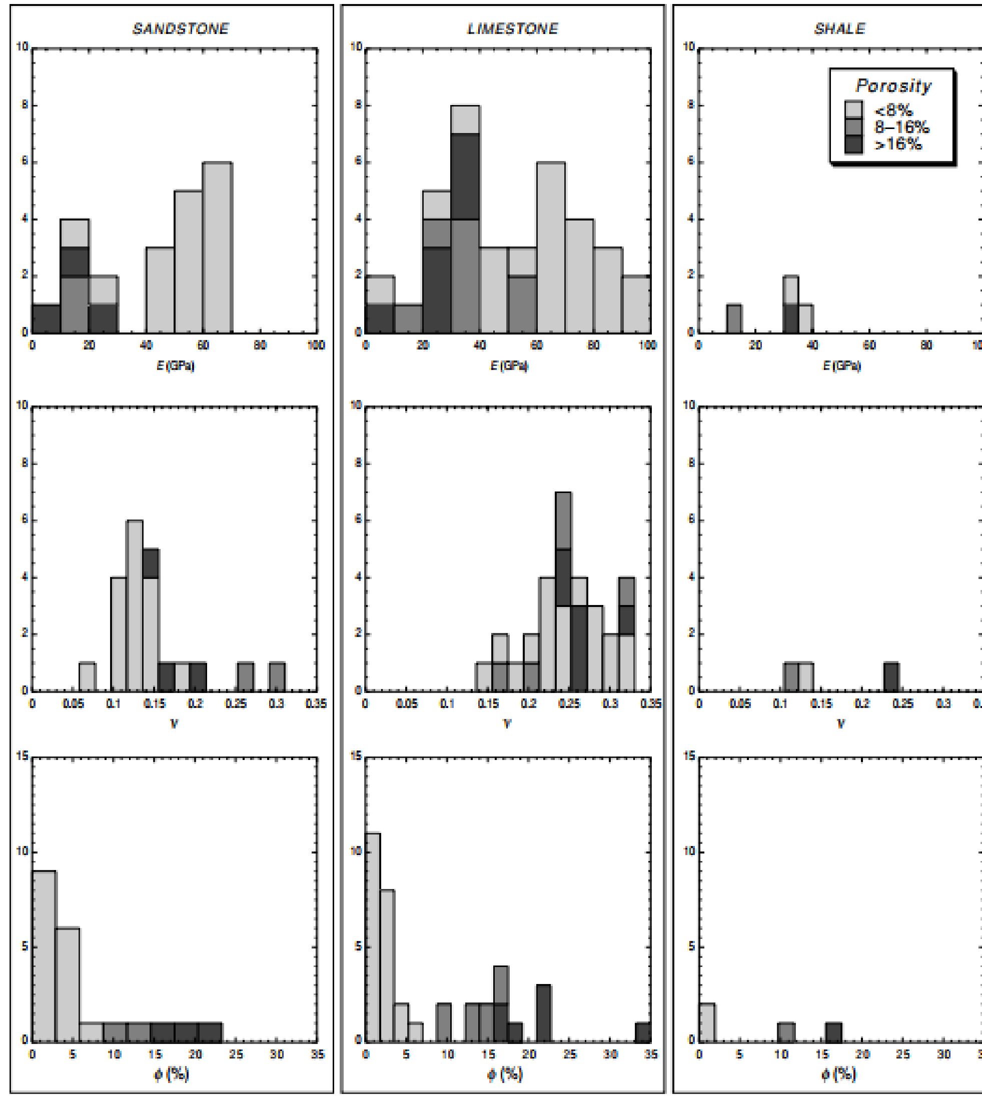


**Figure 3.4.** Typical values of static measurements of Young's modulus, $E$, and Poisson's ratio, $\nu$, for sandstone and limestone and porosity, $\phi$ (from Lama and Vutukuri 1978).

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# Poroelasticity and effective stress
## Definition of effective stress

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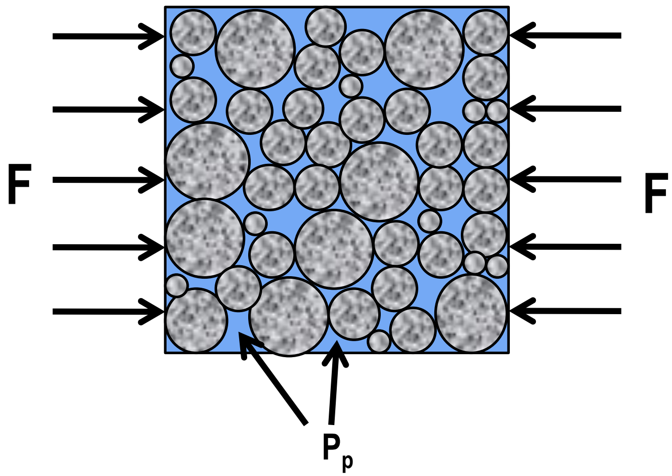


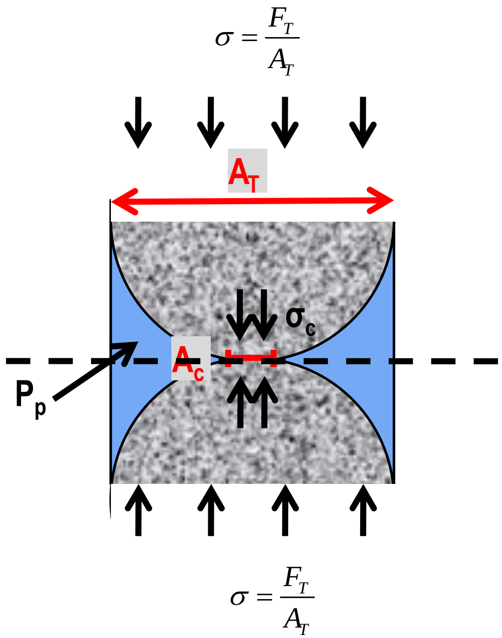


- $A_c$: contact area of grain
- $A_T$: diameter (area) of grain
- $\sigma_c$: normal stress acting on the grain contact
- $\sigma_g$: intergranular stress acting on the grain contact = σ´
- $p_p$: pore pressure

$$\sigma' = \sigma - p_p$$

$$\sigma_{ij} = S_{ij} - \delta_{ij} P_p \tag{3.8}$$

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# Poroelasticity and effective stress
## Definition of effective stress

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- "Exact" effective stress law by using Biot coefficient, α

$$\sigma_{ij} = S_{ij} - \delta_{ij}\alpha P_p \quad (3.10)$$

where $\alpha$ is the Biot parameter

$$\alpha = 1 - K_b/K_g$$

and $K_b$ is drained bulk modulus of the rock or aggregate and $K_g$ is the bulk modulus of the rock's individual solid grains. It is obvious that $0 \leq \alpha \leq 1$.

  - Range of alpha
    - Soil?
    - Hard rock?

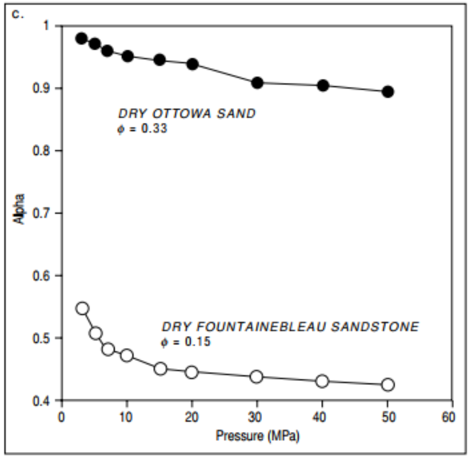


**Figure 3.5.** (a) Schematic illustration of a porous solid with external stress applied outside an impermeable boundary and pore pressure acting within the pores. (b) Considered at the grain scale, the force acting at the grain contact is a function of the difference between the applied force and the pore pressure. As $A_c/A_T$ goes to zero, the stress acting on the grain contacts is given by the Terzaghi effective stress law (see text). (c) Laboratory measurements of the Biot coefficient, $\alpha$, for a porous sand and well-cemented sandstone courtesy J. Dvorkin.

- Constitutive law considering exact effective stress

$$S_{ij} = \lambda\delta_{ij}\varepsilon_{00} + 2G\varepsilon_{ij} - \alpha\delta_{ij}P_0 \quad (3.11)$$

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# Poroelasticity and effective stress
# Importance of effective stress - deformation

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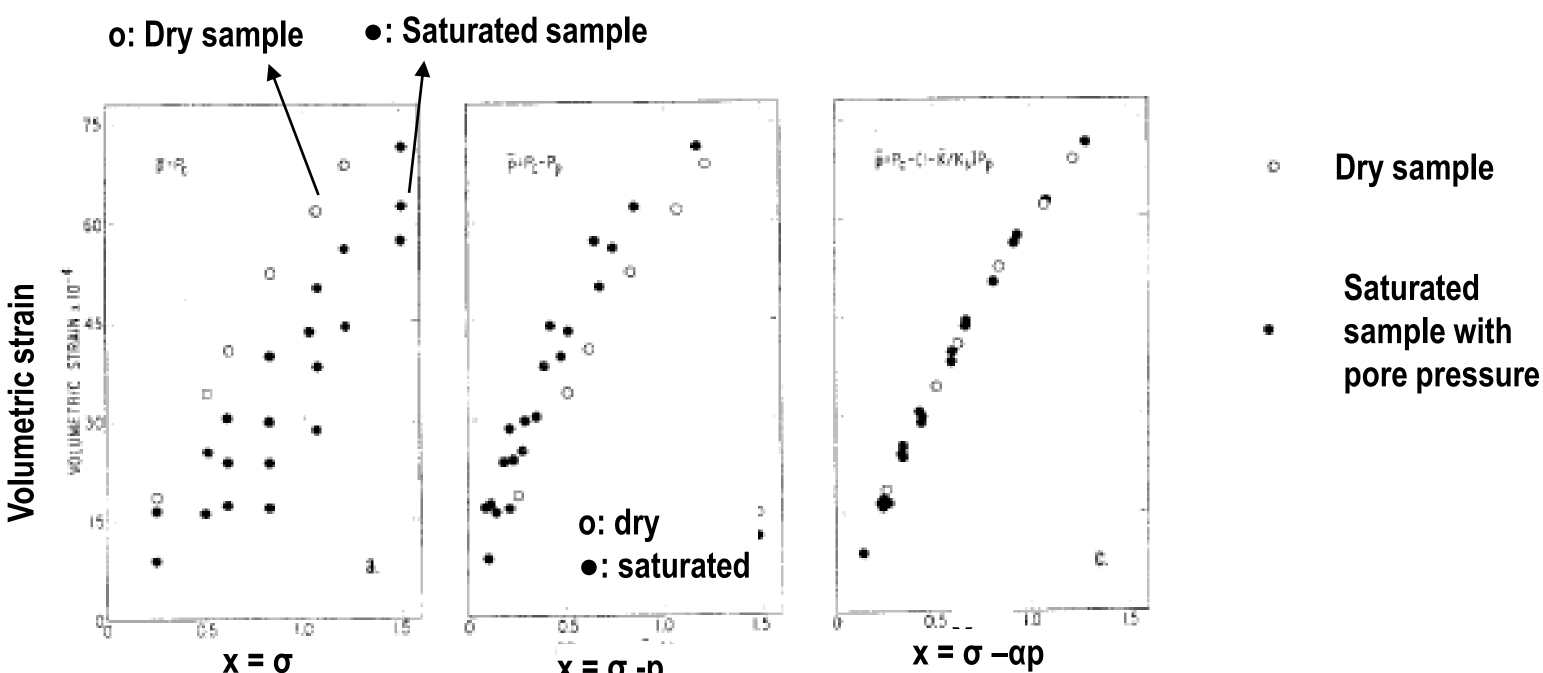


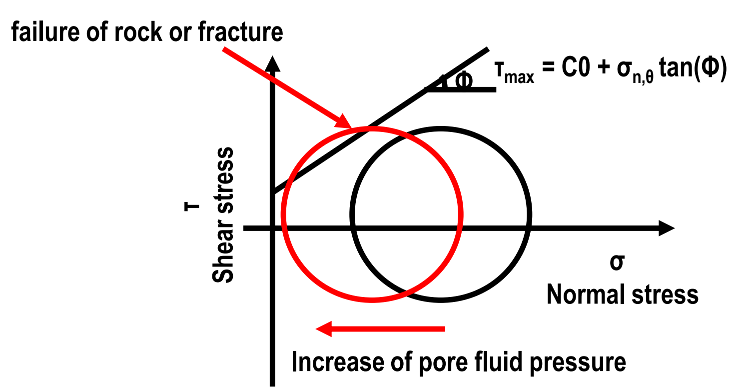

Fig. 2. Volumetric strain versus effective stress in porous Weber sandstone. (*a*) Strain versus confining pressure. (*b*) Strain versus the difference between confining and pore pressures. (*c*) Strain versus theoretical effective pressure. The open circles show the strain versus confining pressure in a dry confined sample.

- Volumetric strain versus effective stress in porous Weber sandstone

Nur, A. and J. D. Byerlee (1971). "Exact Effective Stress Law for Elastic Deformation of Rock with Fluids." Journal of Geophysical Research 76(26): 6414-6419.

# Poroelasticity and effective stress
# Importance of effective stress – rock failure

$$\sigma_{ij} = S_{ij} - \delta_{ij} P_{\mathrm{p}} \tag{3.8}$$




- Increase of pore pressure induce failure of intact rock

# Elastic anisotropy



- Constitutive equations for fully anisotropic elasticity

$$S_{ij} = c_{ijkl}\varepsilon_{kl} \tag{3.7}$$

$$\Big\downarrow \textbf{ Contracted form}$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix}$$

# Elastic anisotropy
## Transversely Isotropic

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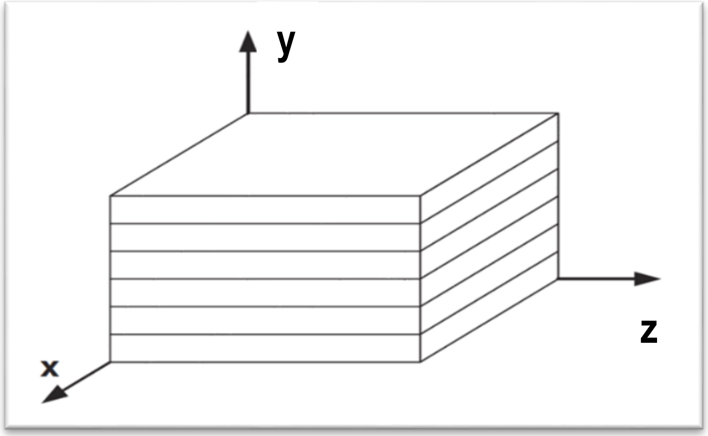
- Most common anisotropic model: Transversely isotropic
  - One axis of symmetry
  - 5 independent parameters

$$E_x = E_z = E$$
$$E_y = E'$$
$$\nu_{xz} = \nu_{zx} = \nu$$
$$\nu_{yx} = \nu_{yz} = \nu'$$
$$G_{xy} = G_{yz} = G'$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu'}{E'} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu'}{E'} & \frac{1}{E'} & -\frac{\nu'}{E'} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu'}{E'} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G'} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G'} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$




(H. Gercek, 2006)

# Elastic anisotropy
# Transversely Isotropic rock

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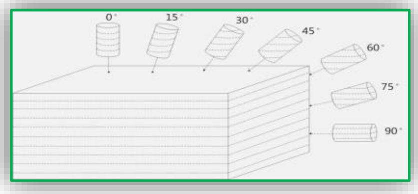
- Laboratory experiments for measuring anisotropic elastic parameters (Cho et al., 2012)

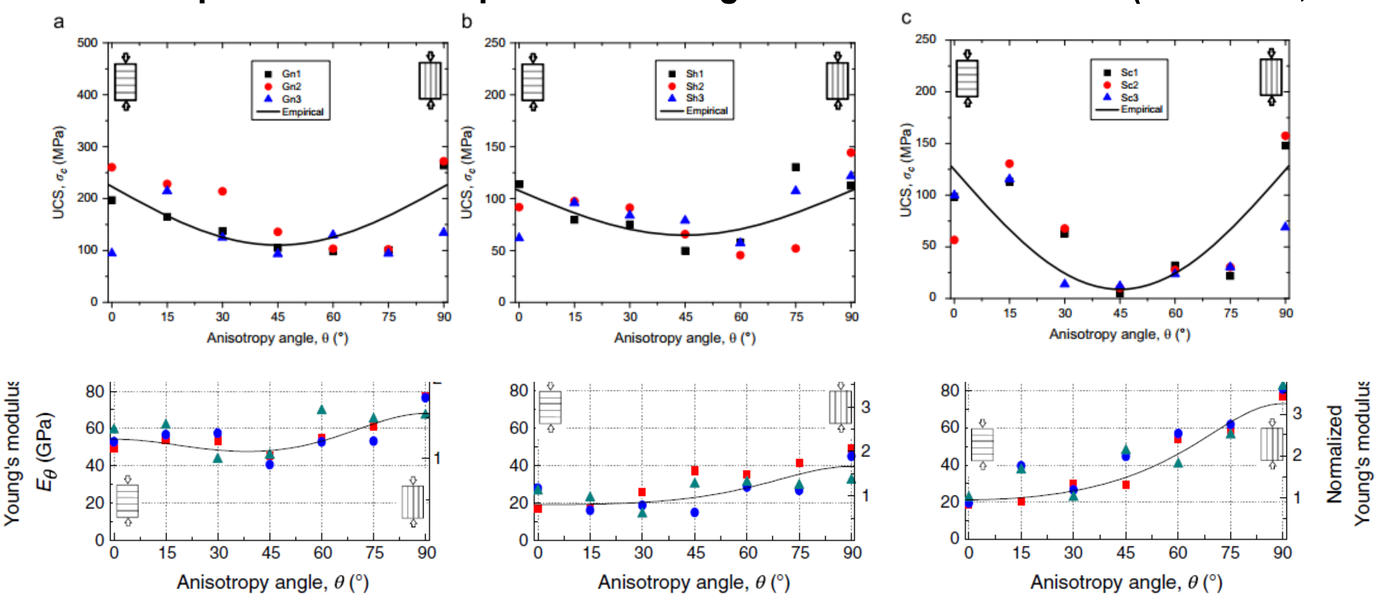



**Directional coring system**

**- Strength anisotropy is more important than elastic anisotropy. Why?**

**- Shear wave velocity anisotropy is used for determining principal stress direction (stress-induced anisotropy: faster shear wave velocity aligned with maximum principal stress)**

**Anisotropic Uniaxial Compressive Strength and Elastic Modulus (Cho et al., 2012)**




(a) Asan gneiss (b) Boryeong shale (c) Yeoncheon schist

Cho JW, Kim H, Jeon S, Min KB, Deformation and strength anisotropy of Asan gneiss Boryeong shale, and Yeoncheon schist, IJRMMS, 2012;50:158-169.

# Elastic moduli and seismic wave velocity



- Compressional and shear wave velocity in terms of elastic constants

$$V_p = \sqrt{\frac{K + 4G/3}{\rho}} \qquad V_s = \sqrt{\frac{G}{\rho}} \tag{3.5}$$

- Elastic constants in terms of wave velocities

$$E = \rho v_s^2 \frac{3v_p^2 - 4v_s^2}{v_p^2 - v_s^2} \qquad M = V_p^2 \rho = K + \frac{4G}{3}$$

$$\nu = \frac{V_p^2 - 2V_s^2}{2\left(V_p^2 - V_s^2\right)} \tag{3.6}$$

- Vp/Vs?

$$\frac{V_p}{V_s} = \frac{\sqrt{\frac{K+\frac{4}{3}G}{\rho}}}{\sqrt{\frac{G}{\rho}}} = \sqrt{\frac{K+\frac{4}{3}G}{G}} = \sqrt{\frac{\frac{\cancel{E}}{3(1-2\nu)} + \frac{2\cancel{E}}{3(1+\nu)}}{\frac{\cancel{E}}{2(1+\nu)}}}$$

$$= \sqrt{\frac{2(1-\nu)}{1-2\nu}} \qquad \text{if } \nu = 0.25 \quad \frac{V_p}{V_s} = \sqrt{3}$$

# Elastic moduli and seismic wave velocity
# Static and dynamic moduli

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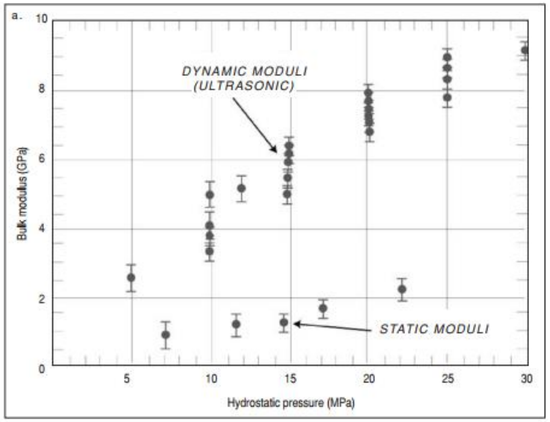
- Static vs. Dynamic moduli
  - Loading: Static moduli << dynamic moduli,
  - Unloading path: Static moduli ~ dynamic moduli
- Factors affecting seismic wave propagation
  - Waves with Reflection profiling, well logging and laboratory are different
  - In static measurement, amount of strains are different
  - Pore fluid play a role.



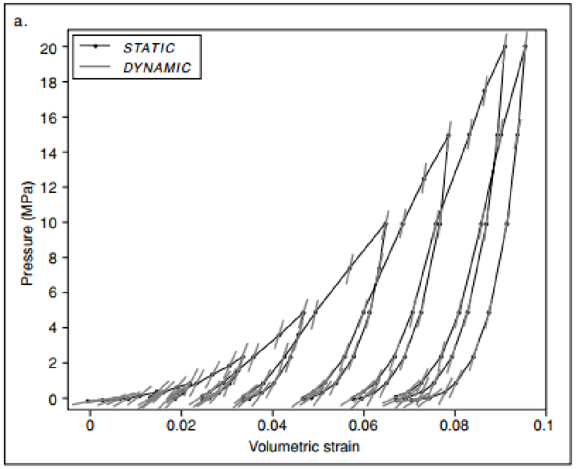



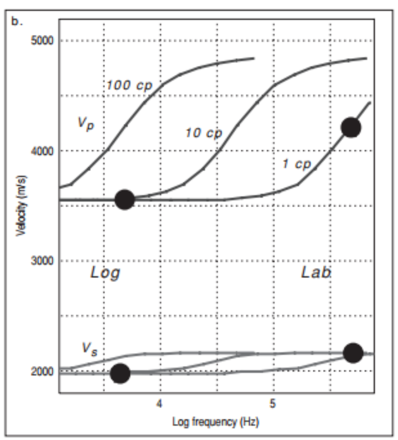

**Figure 3.7.** (a) Hydrostatic loading and unloading cycles of a saturated, uncemented Gulf of Mexico sand that shows the clear difference between static and dynamic stiffness, especially for loading cycles where both elastic and inelastic deformation is occurring (after Zimmer 2004). (b) An expanded view of several of the cycles shown in (a).

Zoback MD, 2007, Reservoir Geomechanics, Cambridge University Press

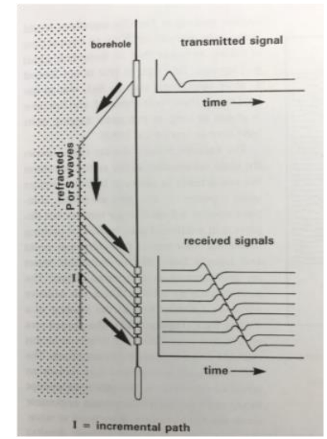
# Poroelasticity and dispersion
## Dispersion



- P-wave and S-wave velocities are frequency dependent → dispersion
  - Lab: ultrasonic (~ MHz)
  - Sonic logs (~10 kHz)
  - Reflection seismic survey (~10-50 Hz)
- Larger velocity with higher frequency, and higher viscosity
  - Low frequency ~ drained, high frequency ~ undrained




**Comparison between P- and S- wave velocity in a water saturated at frequencies corresponding to geophysical log and lab measurement.**




**Array sonic sampling system (Rider & Kennedy, 2011)**

Rider M and Kennedy M, 2011, The geological interpretation of well logs, 3rd ed., Rider French

# Viscous deformation in uncemented sands

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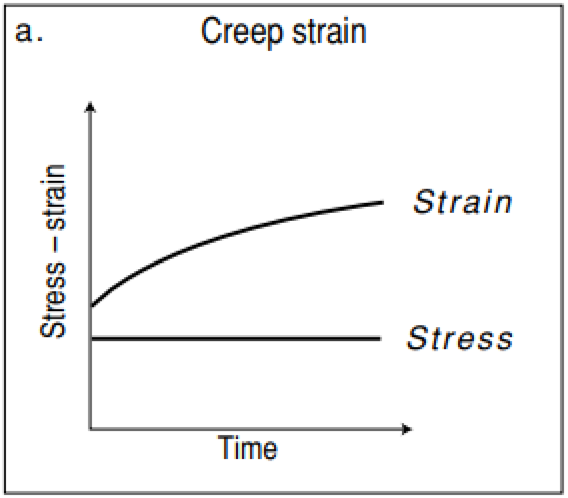
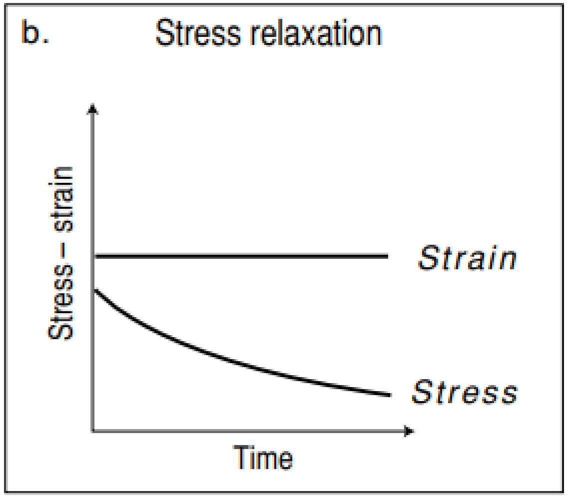
- Time dependent deformation
  - (Uncemented) sands and immature* shales tends to behavior viscously




* immature: Pertaining to a hydrocarbon source rock that has not fully entered optimal conditions for generation.

# Viscous deformation in uncemented sands

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- Example at Wilmington field, California
  - Viscous compaction will only be important when depletion results in an effective stress in situ
  - Amount of creep increase with clay content

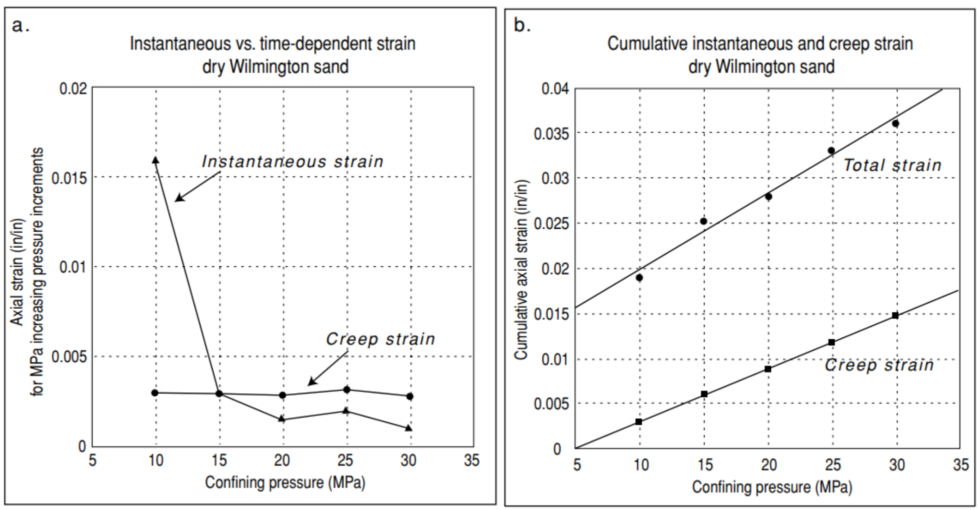


**Figure 3.9.** (a) Incremental instantaneous and creep strains corresponding to 5 MPa incremental increases in pressure. The data plotted at each pressure reflect the increases in strain that occurred during each increase in pressure. Note that above 15 MPa the incremental creep strain is the same magnitude as the incremental instantaneous strain (from Hagin and Zoback 2004b). The cumulative instantaneous and total (instantaneous plus creep) volumetric strain as a function of pressure. Note that above 10 MPa, both increase by the same amount with each increment of pressure application.

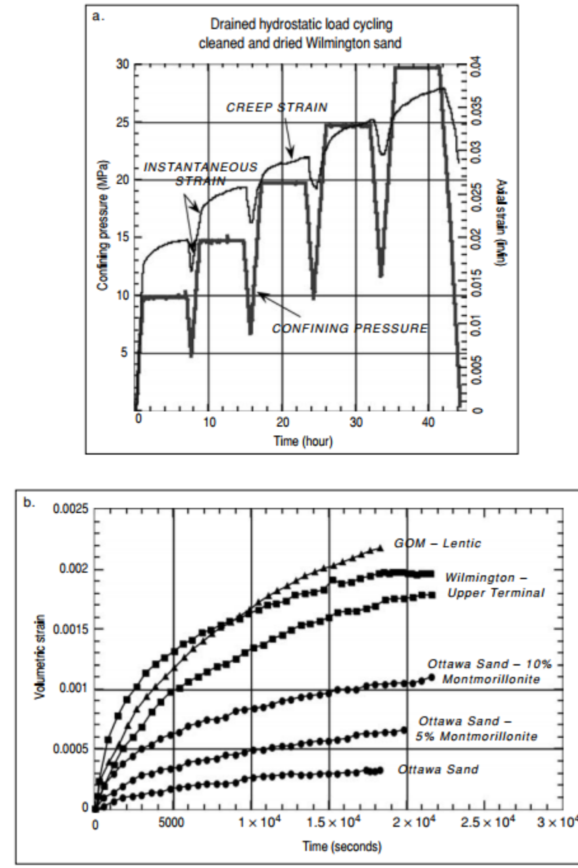


**Figure 3.8.** (a) Incremental instantaneous and creep strains corresponding to 5 MPa incremental increases in pressure. The data plotted at each pressure reflect the increases in strain that occurred during each increase in pressure. Note that above 15 MPa the incremental creep strain is the same magnitude as the incremental instantaneous strain from Hagin and Zoback (2004b). (b) Creep strain is dependent upon the presence of clay minerals and mica. In the synthetic samples, the amount of creep is seen to increase with clay content. The clay content of the Wilmington sand samples is approximately 15% (modified from Chang, Moos *et al.* 1997). *Reprinted with permission of Elsevier.*

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# Thermoporoelasticity

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- Linear thermal expansion coefficient (unit: /K)

$$\frac{\Delta l}{l} = \alpha\left(T - T_0\right)$$

  - Thermal expansion coefficient of Rock
    - Berea Sandstone: 1.5×10-5,
    - Boom clay: 3.3×10-6
    - Water: 6.6×10-5

- Thermal stress← thermal expansion + mechanical restraint
  - Thermal stress in 1D

$$\sigma_T = \alpha E\left(T - T_0\right)$$

  - Thermal stress when a rock is completely (in all directions) restrained

$$\sigma_T = 3\alpha K\left(T - T_0\right) = \frac{E}{1-2\nu}\alpha\left(T - T_0\right)$$

$$S_{ij} = \lambda\delta_{ij}\varepsilon_{00} + 2G\varepsilon_{ij} - \alpha_{\mathrm{T}}\delta_{ij}P_0 - K\alpha_{\mathrm{T}}\delta_{ij}\Delta T \qquad (3.21)$$

# Thermoporoelasticity
# Thermal expansion coefficient

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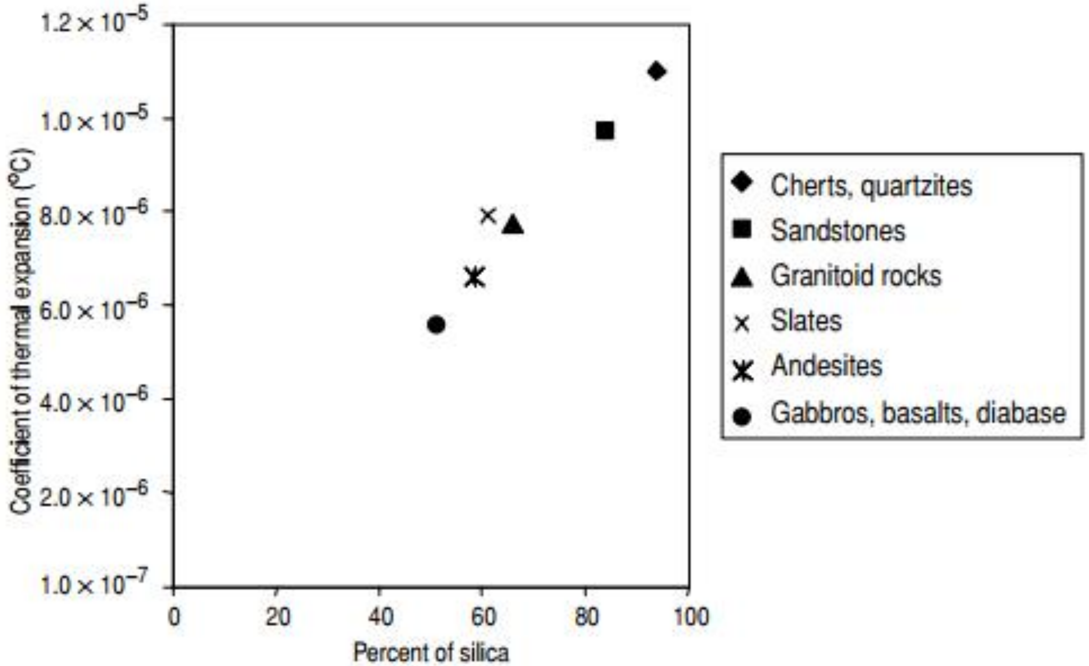


**Figure 3.14.** Measurements of the coefficient of linear thermal expansion for a variety of rocks as a function of the percentage of silica (data from Griffith 1936). As the coefficient of thermal expansion of silica ($\sim 10^{-5}$ °C$^{-1}$) is an order of magnitude higher than that of most other rock forming minerals ($\sim 10^{-6}$ °C$^{-1}$), the coefficient of thermal expansion ranges between those two amounts, depending on the percentage of silica.

Zoback MD, 2007, Reservoir Geomechanics, Cambridge University Press

# Thermoporoelasticity
# Thermal conductivity

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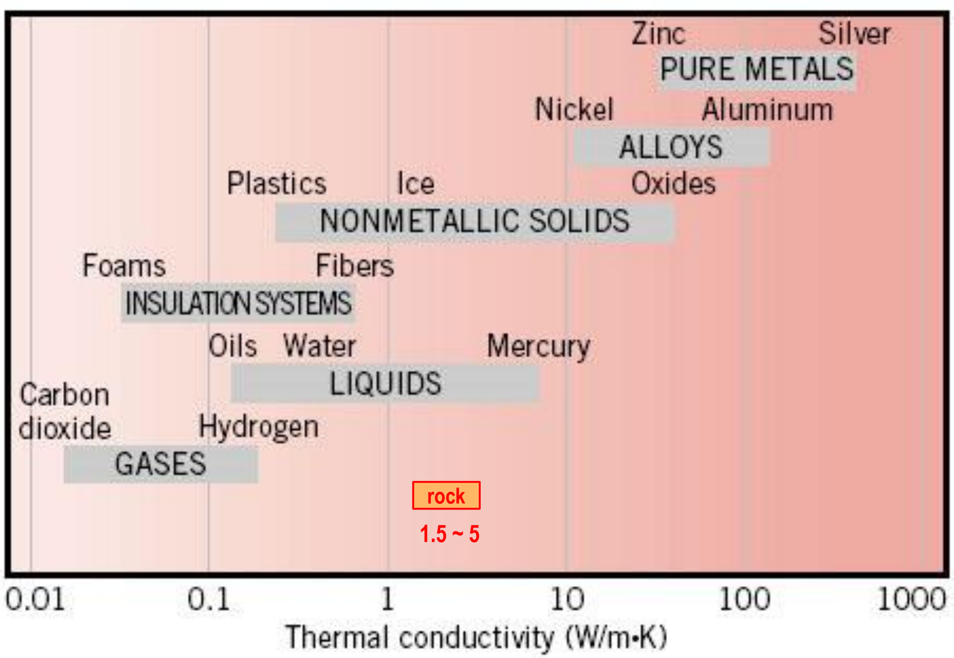
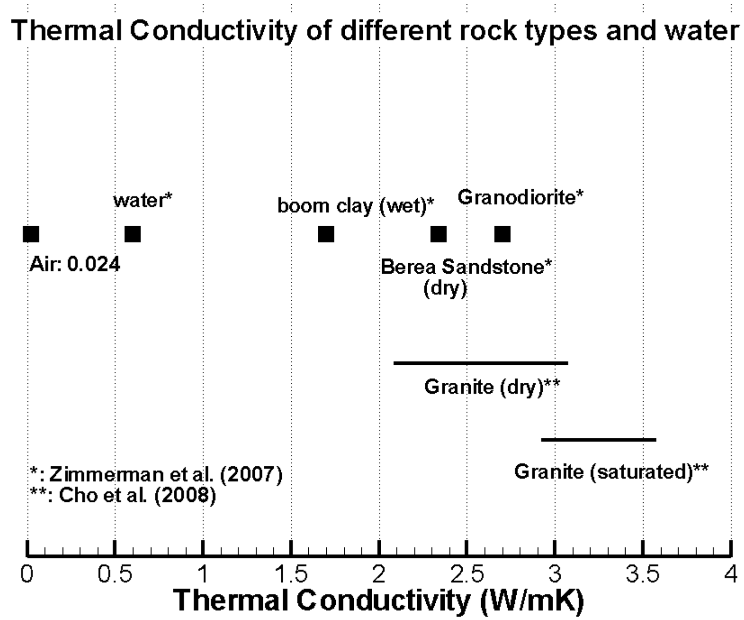
- Fourier's law

$$q_x'' = -k\frac{dT}{dx}$$

  - q'' : heat flux (W/m$^2$), rate of heat transfer per unit area (in the x direction, perpendicular to the direction of transfer)
  - k: thermal conductivity (W/m·K)




Incropera, 2007