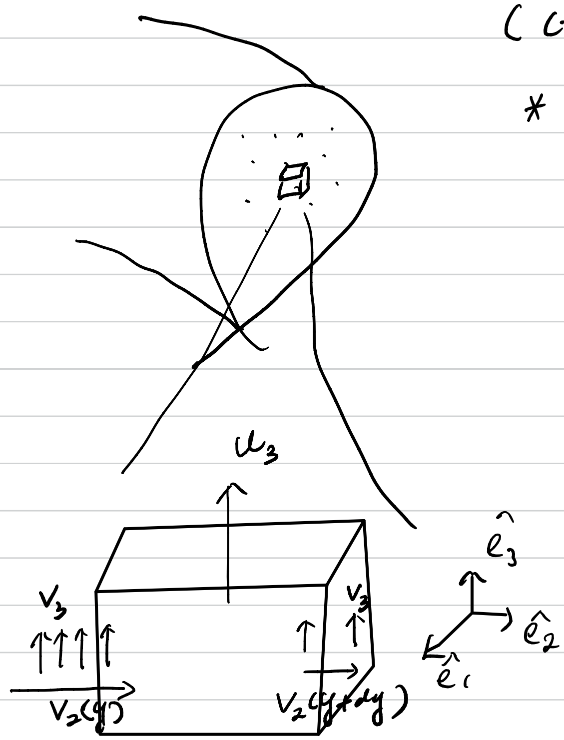
4/5/21 **Fluid description of plasmas / closure**
(Goldston Unit 3)



* note for individual particle approach

Governing Eq. $m_i \frac{d\vec{v}_i}{dt} = e_i(\vec{E} + \vec{v}_i \times \vec{B})$

for each $i = 1, \dots N$

\+ collisions

\+ source + sink + external field

\+ Maxwell Eq.

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 & \text{Gauss} \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} & \text{Faraday} \\ \vec{\nabla}\cdot\vec{B} = 0 & \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} & \text{Ampere} \end{cases}$$

infinitesimal volume element d^3x

$\rho = \sum_{i=1}^{N} e_i\,\delta(\vec{x}-\vec{x}_i)$, $\vec{J} = \sum_{i=1}^{N} e\vec{v}_i\,\delta(\vec{x}-\vec{x}_i)$

"Klimontovich"

e.g.) momentum evolution of u_3

$$\frac{\partial}{\partial t}(mnu_3) = -\frac{\partial}{\partial y}(mn\langle v_3 v_2\rangle) - \frac{\partial}{\partial x}(mn\langle v_3 v_1\rangle) - \frac{\partial}{\partial z}(mn\langle v_3 v_3\rangle)$$

Q) How big is N? $n \sim (10^{19} \sim 10^{20})\,m^{-3}$

$N \sim (10^{19} \sim 10^{22})\,m^{-3}$

Too big to follow every single particles

option 1) Statistically particle distribution fcn $f(\vec{x}, \vec{v}, t)$

Full, gyro, drift, bounce-averaged

option 2) Fluid approach (integral over velocity space, Ensemble of charged particles)

moment

$$\begin{cases} \text{density} & n(\vec{x},t) = \int d^3v\, f \\ \text{fluid velocity} & n\vec{u}(\vec{x},t) = \int d^3v\, \vec{v} f \\ \text{stress tensor} & \overleftrightarrow{P}(\vec{x},t) = \int d^3v\, m\vec{v}\vec{v} f \\ \text{pressure tensor} & \overleftrightarrow{p}(\vec{x},t) = \int d^3v\, m(\vec{v}-\vec{u})(\vec{v}-\vec{u}) f \\ \text{energy flux} & \vec{Q}(\vec{x},t) = \int d^3v\, \frac{1}{2}mv^2\vec{v} f \\ \text{heat flux} & \vec{q}(\vec{x},t) = \int d^3v\, \frac{1}{2}m(\vec{v}-\vec{u})^2(\vec{v}-\vec{u}) f \\ \text{energy stress} & \vec{R}(\vec{x},t) = \int d^3v\, \frac{1}{2}mv^2\vec{v}\vec{v} f \end{cases}$$

$$\rightarrow \quad \rho(\vec{x},t) = \sum_s e_s n_s$$
$$\vec{J}(\vec{x},t) = \sum_s e_s n_s \vec{u}_s$$

1. Fluid equations for infintesimal volume element

(Goldston Unit 3. chapter 6)

(1) particle conservation
continuity equation
$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{u}) = S_0$$
particle source
{ionization, recombination, fusion}

$$\rightarrow \frac{dn}{dt} + n(\vec{\nabla}\cdot\vec{u}) = S_0$$

(2) momentum conservation

$$m\frac{d\vec{v}}{dt} = e(\vec{E} + \vec{v}\times\vec{B})$$

$$m\frac{\partial}{\partial t}(n\vec{u}) = ne(\vec{E}+\vec{u}\times\vec{B}) - \vec{\nabla}\cdot\overleftrightarrow{P} + \vec{R}_1 + \vec{S}_1$$

momentum exchange
momentum source

$$\overleftrightarrow{P} = \overleftrightarrow{p} + mn\vec{u}\vec{u}$$

continuity eq.

$$m\frac{d}{dt}(n\vec{u}) = ne(\vec{E}+\vec{u}\times\vec{B}) - \vec{\nabla}\cdot\overleftrightarrow{p} + \vec{R}_1 + \vec{S}_1 - \vec{u}S_0$$

$$\rho, \vec{J} \leftarrow n \leftarrow \vec{u} \leftarrow \overleftrightarrow{P} \leftarrow \cdots \text{ higher moment}$$

"closure" problem

<u>For pressure tensor</u>

(a) Maxwellian $f_M = \frac{n}{(2\pi T/m)^{3/2}} e^{-\frac{1}{2}m(v-u)^2/T}$

$$(v_x - u_x)^2 = \frac{1}{3}(\vec{v}-\vec{u})^2$$

$$\overleftrightarrow{p} = \int d^3v \; m(\vec{v}-\vec{u})(\vec{v}-\vec{u}) f_M$$
$$= \overleftrightarrow{I}\left(\frac{1}{3}\int d^3v \; m(\vec{v}-\vec{u})^2 f_M\right) = p\overleftrightarrow{I} = nT\overleftrightarrow{I}$$

for any f. $\quad p \equiv \frac{1}{3}\int d^3v \; m(\vec{v}-\vec{u})^2 f = \frac{1}{3}\left(\mathrm{Tr}(\overleftrightarrow{p})\right)$

$$T \equiv \frac{p}{n}$$

(b) Gyrophase average $\bar{f}$

Assume $\vec{u}=0$ w/o loss of generality

let $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp$ (w. magnetic field)

$= v_\parallel \hat{b} + \vec{v}_\perp$ in $(\hat{e}_1, \hat{e}_2, \hat{b})$

$$\overleftrightarrow{P} = \int d^3v \, m (v_\parallel \hat{b} + \vec{v}_\perp)(v_\parallel \hat{b} + \vec{v}_\perp) \bar{f}$$

$$= \int d^3v \, m \left(v_\parallel^2 \hat{b}\hat{b} + \frac{1}{2} v_\perp^2 (\hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2) \right) \bar{f}$$

$$\equiv \hat{b}\hat{b} P_\parallel + (\overleftrightarrow{I} - \hat{b}\hat{b}) P_\perp = P_{CGL}$$

$$P_\parallel = \int d^3v \, m v_\parallel^2 \bar{f}, \quad P_\perp = \int d^3v \, \mu B \bar{f}$$

$$= n T_\parallel \qquad\qquad = n T_\perp$$

(c) general,

$$\overleftrightarrow{P} = p \overleftrightarrow{I} + \overleftrightarrow{\Pi}$$

$$= (p \overleftrightarrow{I})_0 + (\hat{b}\hat{b} P_\parallel + (\overleftrightarrow{I} - \hat{b}\hat{b}) P_\perp)_1 + (\overleftrightarrow{\Pi})_2$$

e.g) fast ion, suprathermal beam ion electrons

neoclassical viscosity

e.g) gyroviscosity viscous diffusion (Braginskii) turbulent viscosity

hybrid Kinetic-MHD

(3) Isotropic pressure evolution

Equation of state for closure.

$$\frac{d}{dt}\left(\frac{P}{\rho_m^\gamma}\right) = 0 \quad \rightarrow \quad P = (\text{const}) \cdot n^\gamma$$

$$PV^\gamma = \text{const}$$

(a) Isothermal $\gamma = 1$

when compression slower than heat conduction

(b) Adiabatic $\gamma = \frac{N+2}{N}$ (degree of freedom for collisional energy exchange)

when compression faster than heat conduction

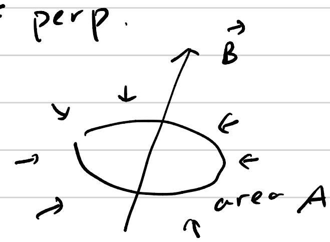
(b-1) Ideal gas isotropic $N=3$

$$\gamma = \frac{5}{3} \quad \frac{d}{dt}\left(\frac{p}{n^{5/3}}\right) = 0 \quad p \propto n^{5/3}$$

(b-2) Compression faster than collisions

$$\begin{cases} \text{perp.} \quad N=2, \ \gamma=2 \quad p_\perp \propto n^2 \\ \text{para.} \quad N=1, \ \gamma=3 \quad p_\parallel \propto n^3 \end{cases}$$

* perp.



$$p_\perp = \frac{1}{2} mn\langle v_\perp^2 \rangle = n\langle \mu \rangle B$$

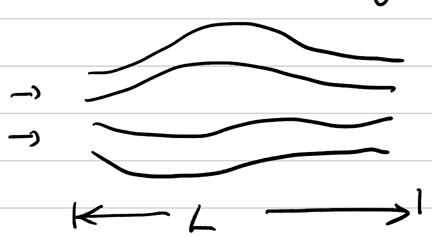
nA = const. for particle conservation

BA = const. for flux conservation

$$B \propto n$$

$$\frac{d}{dt}\left(\frac{p_\perp}{nB}\right) \propto \frac{d}{dt}\left(\frac{p_\perp}{n^2}\right) = 0$$

* para. comp by changing L, with const B



$$p_\parallel = mn\langle v_\parallel^2 \rangle$$

$$J = V_\parallel L \propto V_\parallel V \propto \frac{V_\parallel}{n}$$

$$\rightarrow \frac{d}{dt}\left(\frac{p_\parallel}{n^3}\right) = 0 \quad \because \frac{p_\parallel}{n^3} \propto \frac{\langle V_\parallel^2 \rangle}{n^2} \propto J$$

4/7/21 Moment / Closure - Continued

* For single particle motion
detail of expansion → Northrop & Rome 1978, PF.

2. Derivation from kinetic F-P equations

$$\left.\frac{df_s}{dt}\right|_{\text{phase-space}} = \frac{\partial f_s}{\partial t} + \vec{v}_s \cdot \vec{\nabla} f_s + \frac{\vec{F}_s}{m_s} \cdot \vec{\nabla}_v f_s = C_s[f_s] = C_{ss'}[f_s, f_{s'}]$$

"s" species omitted

General moment $\vec{M}(\vec{v})$ ×
& velocity space average

$$\int d^3v\, \vec{M}(\vec{v}) \frac{df}{dt} = \int d^3v\, \vec{M}(\vec{v})\, C = \vec{R}$$

$$\frac{\partial}{\partial t}\left(\int d^3v\, \vec{M} f\right) + \vec{\nabla}\cdot\left(\int d^3v\, \vec{M}\vec{v} f\right) - \frac{\vec{F}}{m} \int d^3v \frac{\partial \vec{M}}{\partial \vec{v}} f = \vec{R}$$

$$\because \int d^3v\, \frac{\vec{M}\vec{F}}{m} \cdot \vec{\nabla}_v f = -\int d^3v\, \frac{\vec{M}}{m} f \cancel{\left(\frac{\partial}{\partial \vec{v}} \cdot \vec{F}\right)} - \int d^3v\, \frac{\vec{F}}{m} \cdot \frac{\partial \vec{M}}{\partial \vec{v}} f$$

coupling to higher moment   to lower moment

"closure" problem

(1) $\vec{M} = 1$ (particle conservation)

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{u}) = S_0 \quad \text{← particle source}$$

(2) $\vec{M} = m\vec{v}$ (momentum conservation)   momentum source ↓

$$m\frac{\partial}{\partial t}(n\vec{u}) = ne(\vec{E} + \vec{u}\times\vec{B}) - \vec{\nabla}\cdot\overleftrightarrow{P} + \vec{R}_1 + \vec{S}_1$$

$\overleftrightarrow{P} = \overleftrightarrow{p} + mn\vec{u}\vec{u}$ ↓ − continuity eq

$$m\frac{d}{dt}(n\vec{u}) = ne(\vec{E} + \vec{u}\times\vec{B}) - \vec{\nabla}\cdot\overleftrightarrow{p} + \vec{R}_1 + \vec{S}_1 - \vec{u}S_0$$

(3) $\vec{M} = \frac{1}{2} m v^2$ (energy conservation)

$$\frac{\partial}{\partial t}\Big(\underbrace{\int d^3v \tfrac{1}{2} m v^2 f}_{(a)}\Big) + \underbrace{\vec{\nabla}\cdot\vec{Q}}_{(b)} = ne\vec{u}\cdot\vec{E} + R_2 + S_2$$

$$(a) = \int d^3v\, \tfrac{1}{2} m (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u}) f + \tfrac{1}{2} m n u^2$$

$$= \tfrac{3}{2} p + \tfrac{1}{2} m n u^2$$

$$(b)\ \vec{Q} = \vec{q} + \tfrac{3}{2} p\vec{u} + \overleftrightarrow{P}\cdot\vec{u} + \tfrac{1}{2} m n u^2 \vec{u}$$

$$= \vec{q} + \tfrac{5}{2} p\vec{u} + \overleftrightarrow{\Pi}\cdot\vec{u} + \tfrac{1}{2} m n u^2 \vec{u}$$

$-$ $m\vec{u}\cdot$ (momentum equation)

$+ \frac{1}{2} m u^2 \cdot$ Continuity eq

$$\frac{d}{dt}\left(\tfrac{3}{2}P\right) + \tfrac{5}{2} p(\vec{\nabla}\cdot\vec{u}) + \overleftrightarrow{\Pi} : \vec{\nabla}\vec{u} + \vec{\nabla}\cdot\vec{q}$$

$$= \vec{R}_2 - \vec{u}\cdot\vec{R}_1 + S_2 - \vec{u}\cdot\vec{S}_1 + \tfrac{1}{2} m u^2 S_0$$

If $\overleftrightarrow{P} = p\overleftrightarrow{I}$, $\overleftrightarrow{\Pi} = 0$, $(\vec{q} = 0)$ w/o. collision $\vec{R}_1 = 0$, $\vec{R}_2 = 0$

$$\left.\begin{aligned}&\frac{d}{dt}\left(\tfrac{3}{2}P\right) + \tfrac{5}{2} p(\vec{\nabla}\cdot\vec{u}) = 0\\ &\frac{dn}{dt} + n(\vec{\nabla}\cdot\vec{u}) = 0\end{aligned}\right\}\quad \begin{aligned}&\frac{3}{2}\frac{1}{p}\frac{dp}{dt} = \frac{5}{2}\frac{1}{n}\frac{dn}{dt}\\ &p = \text{const}\cdot n^{5/3}\end{aligned}$$

closed!

(4) $M = \frac{1}{2} m v^2 \vec{v}$

or * Asymptotic expansion for f

with $Kn = \frac{\lambda}{L}$ ← collisional mean free path

"Knudsen" L ← Length of system.

$Kn \propto \mathcal{O}(\epsilon)$

$f = f_M + \epsilon f_1 + \cdots$

calculate Π. q e.g.) $\vec{q} = \int d^3v \frac{1}{2}(\vec{v}-\vec{u})^2 (\vec{v}-\vec{u}) f_1$

$= -\kappa \vec{\nabla} T$ for gas

Hilbert infinite series

Chapman-Enskog (1916-1917) for gas

- first-order : Navier-Stokes
- second-order : Burnett. (1936)

Chapman-Cowling (1939) for plasma

Braginskii (1965) up to the first order

→ expansion is divergent
by case studies

* (Devil's invention)