

Ch. 1 Interpolation

노트 제목

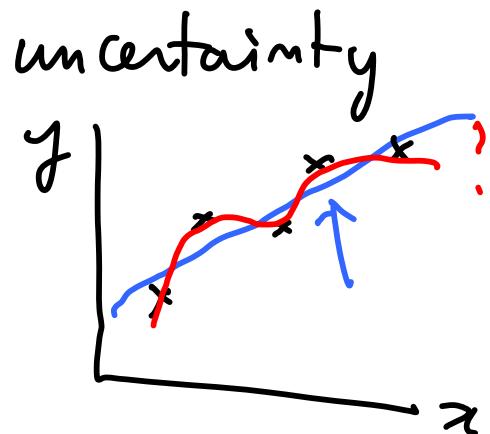
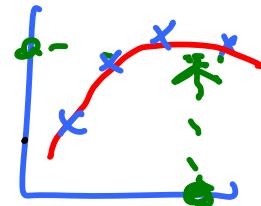
2019-09-16

Discrete data (x_i, y_i) , $i = 0, 1, 2, \dots, n$

→ Find value of y at a pt. between two data pts.

⇒ Fix a smooth curve through the data.

If a data is from a crude exp. w/ some uncertainty
it is best to use the method of
least square errors.



- Polynomial interpolation $(x_i, y_i), i=0, 1, 2, \dots, n$ $n+1$ pts.

Fix a polynomial of degree n through $n+1$ pts.

$$P = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

Find the coeffs. a_0, a_1, \dots, a_n by passing the polynomial going through the data pts.

$$y_{\bar{i}} = a_0 + a_1 x_{\bar{i}} + a_2 x_{\bar{i}}^2 + \dots + a_n x_{\bar{i}}^n, \bar{i} = 0, 1, 2, \dots, n$$

\uparrow
 $n+1$ eqs. $n+1$ unknowns

full matrix A

$$(n+1) \times (n+1)$$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \rightarrow O(n^3)$$

expensive!

For large n , the resulting system of eqs becomes ill-conditioned in general.

In practice, we define the polynomial in an explicit way as opposed to solving a matrix system.

- Lagrange polynomial

Define a polynomial of degree n associated w/ each pt. x_j

$$\begin{aligned}L_j(x) &= \alpha_j (x-x_0)(x-x_1) \cdots (x-x_{j-1})(x-x_{j+1}) \cdots (x-x_n) \\&= \alpha_j \prod_{\substack{i=0 \\ i \neq j}}^n (x-x_i)\end{aligned}$$

$$L_j(x_k) = 0 \quad \text{if } k \neq j \quad (k=0, 1, 2, \dots)$$

$$L_j(x_j) = \alpha_j \prod_{\substack{i=0 \\ i \neq j}}^n (x_j - x_i) \rightarrow \alpha_j = 1 / \prod_{\substack{i=0 \\ i \neq j}}^n (x_j - x_i)$$

$$\Rightarrow L_j(x_k) = \begin{cases} 0 & k \neq j \\ 1 & k=j \end{cases}$$

the desired polynomial

$$P(x) = \sum_{j=0}^n y_j L_j(x)$$

Lagrange
interpolation

$$P(x) = y_0 L_0(x) + y_1 L_1(x) + \dots + y_i L_i(x) + \dots + y_n L_n(x)$$

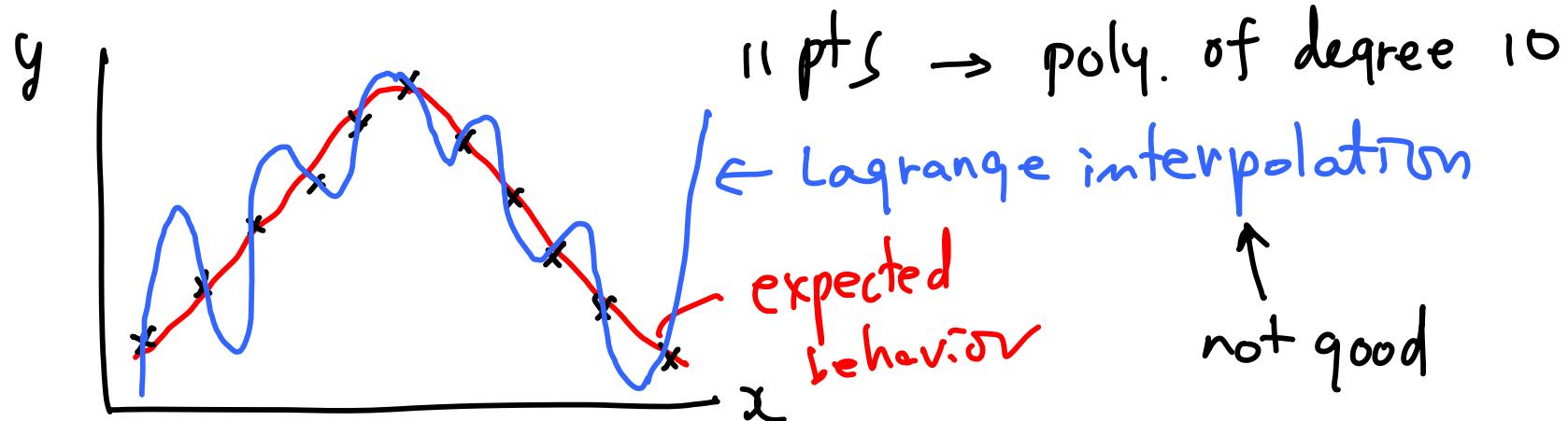
i	0	1	2	3
x_i	1	2	4	8
y_i	1	3	7	11

$$y(x=7) = ?$$

$$P(x) = 1 \cdot L_0(x) + 3 \cdot L_1(x) + 7 \cdot L_2(x) + 11 \cdot L_3(x)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \rightarrow L_0(7) = 0.714$$

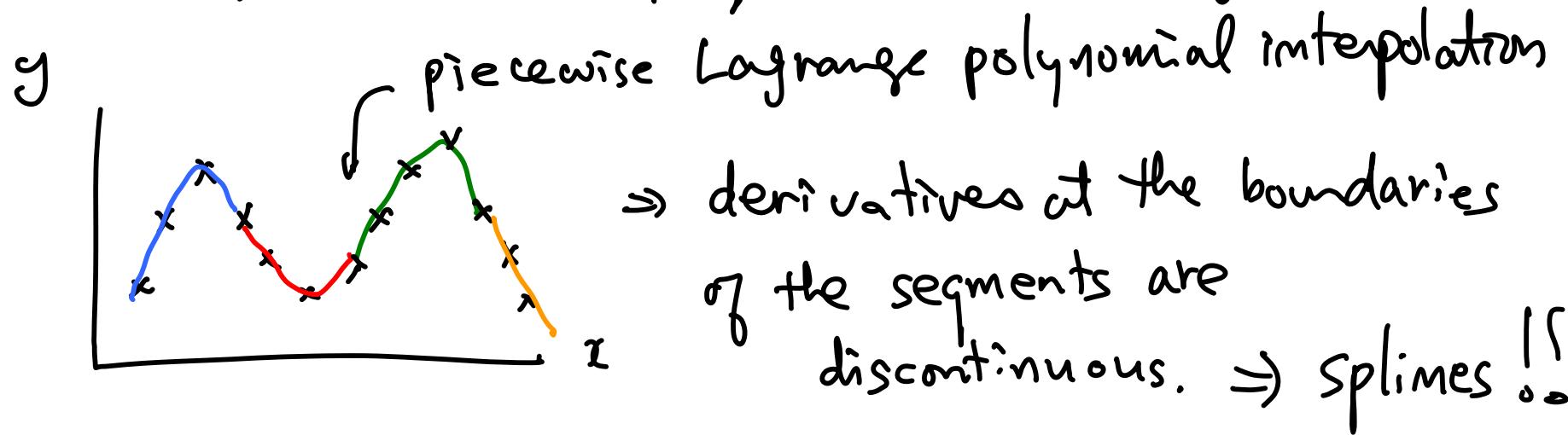
$$y(7) = P(7) = L_0(7) + 3L_1(7) + 7L_2(7) + 11L_3(7) = 10.857$$



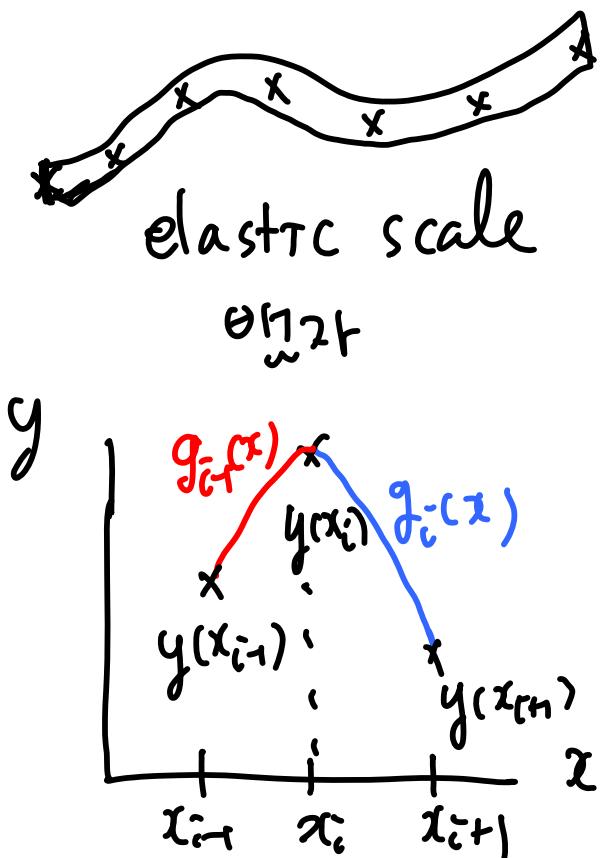
Although the polynomial is tied down at the data pts., it can wander in between. \rightarrow inaccurate.

\rightarrow Piecewise Lagrange polynomial interpolation instead of fitting one poly. to all the data.

\hookrightarrow fit lower-order polys. to sections of data



Spline interpolation



$$EIy^{(iv)} = f$$

between the data pts.,

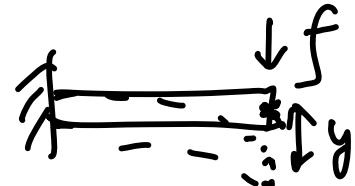
$$EIg^{(iv)} = 0$$

→ y is cubic in between $x_i < x < x_{i+1}$.

For i^{th} interval, ($x_i \leq x \leq x_{i+1}$)

$$g_i(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$g_i(x_i) = y(x_i), \quad g_i(x_{i+1}) = y(x_{i+1})$$



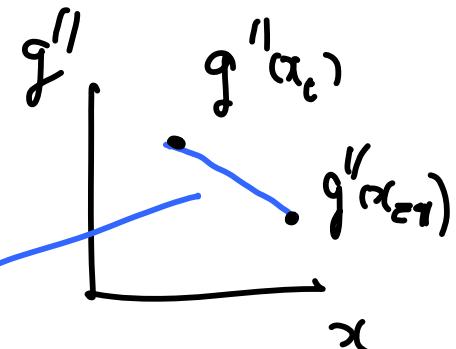
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Match the 1st & 2nd derivatives from the adjacent intervals.

$$\left(\begin{array}{l} g_i'(x_i) = g_{i-1}'(x_i) \\ g_i''(x_i) = g_{i-1}''(x_i) \end{array} \right. \quad g_i' = a_1 + 2a_2 x + 3a_3 x^2$$

$$\left. \begin{array}{l} g_i'' = 2a_2 + 6a_3 x \\ g_i''' = 6a_3 \end{array} \right. \quad \text{Graph: } \begin{array}{c} g'' \\ \downarrow \\ \text{---} \end{array}$$

$$\Rightarrow g_i''(x) = g''(x_i) \frac{x - x_{i+1}}{x_i - x_{i+1}} + g''(x_{i+1}) \frac{x - x_i}{x_{i+1} - x_i}$$



Integrate twice to get the cubic poly.

$$g_i'(x) = g''(x_i) \frac{(x - x_{i+1})^2}{2(x_i - x_{i+1})} + g''(x_{i+1}) \frac{(x - x_i)}{2(x_{i+1} - x_i)} + C_1$$

$$g_i(x) = \underbrace{g''(x_i) \frac{(x - x_{i+1})^3}{6(x_i - x_{i+1})}}_{C_1} + \underbrace{g''(x_{i+1}) \frac{(x - x_i)^3}{6(x_{i+1} - x_i)}}_{C_2} + C_1 x + C_2$$

Obtain c_1 & c_2 by $g_i'(x_c) = y(x_i)$ and $g_i'(x_{c+}) = y(x_{c+})$

$$\rightarrow g_i(x) = \frac{1}{6} g''(x_c) \left[\frac{(x_{c+}-x)^3}{x_{c+}-x_c} - (x_{c+}-x_c)(x_{c+}-x) \right] + \frac{1}{6} g''(x_{c+}) \left[\frac{(x-x_c)^3}{x_{c+}-x_c} - (x_{c+}-x_c)(x-x_c) \right] + y(x_c) \frac{x_{c+}-x}{x_{c+}-x_c} + y(x_{c+}) \frac{x-x_c}{x_{c+}-x_c}$$

Apply $g_i'(x_c) = g'_{i-1}(x_c)$

$\nwarrow n+1$ unknowns $g''(x_0), \dots, g''(x_n)$

$$\rightarrow \boxed{\frac{\Delta_{i-1}}{6} g''(x_c) + \frac{\Delta_{i-1} + \Delta_i}{3} g''(x_c) + \frac{\Delta_i}{6} g''(x_{c+}) = \frac{y(x_{c+}) - y(x_i)}{\Delta_i} + \frac{y(x_{c+}) - y(x_i)}{\Delta_{i-1}}}$$

for $i=1, 2, \dots, n-1$ ↪ $n-1$ eqs.

where $\alpha_i = \frac{\gamma_{i+1} - \gamma_i}{\Delta x}$, $\alpha_{i-1} = \frac{\gamma_i - \gamma_{i-1}}{\Delta x}$

$$\begin{pmatrix} & & & & \\ & \ddots & & & \\ & & \alpha_{i-1} & & \\ & & & \alpha_i & \\ & & & & \alpha_i \\ 0 & & & & \\ & & & & \end{pmatrix} \begin{pmatrix} g''_0 \\ g''_1 \\ g''_2 \\ \vdots \\ g''_{n-1} \\ g''_n \end{pmatrix} = \begin{pmatrix} & & & & \\ & \ddots & & & \\ & & g''_0 \\ & & & \ddots & \\ & & & & g''_n \end{pmatrix}$$

tri-diagonal matrix system

End conditions

$$\textcircled{1} \quad \begin{cases} g''_0 = g''_1 \\ g''_n = g''_{n-1} \end{cases}$$

$$g''_0 \text{ & } g''_n$$

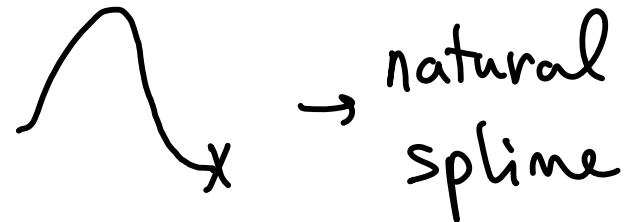
$$\xrightarrow{g''}$$

$$\gamma_0 \quad \gamma_1$$

→ parabolic runout

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & \\ & 0 & 1 & \ddots & \\ & & & \ddots & 0 \\ & & & & -1 \end{pmatrix} \begin{pmatrix} g''_0 \\ g''_1 \\ \vdots \\ g''_{n-1} \\ g''_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

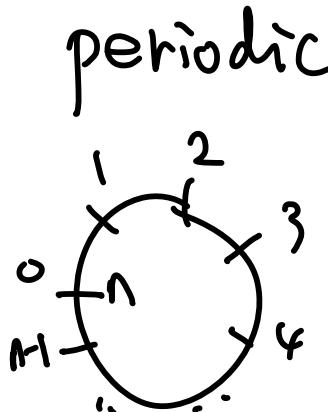
$$\textcircled{2} \quad \begin{cases} g_0'' = 0 \\ g_n'' = 0 \end{cases}$$



natural
spline

$$\textcircled{3} \quad \begin{cases} g_0'' = \lambda g_1'' \\ g_n'' = \lambda g_{n-1}'' \end{cases} \quad 0 \leq \lambda \leq 1$$

$$\textcircled{4} \quad \begin{cases} g_0'' = g_{n-1}'' \\ g_n'' = g_1'' \\ g_0'' = g_n'' \\ g_n'' = g_0'' \end{cases}$$



periodic

A diagram illustrating a periodic spline on a circle. The circle has points 0, 1, 2, 3, 4 labeled around its circumference. Two points on the circle are marked with blue asterisks. To the right, there is a wavy line representing the periodic spline. Below the circle, a matrix equation is shown:

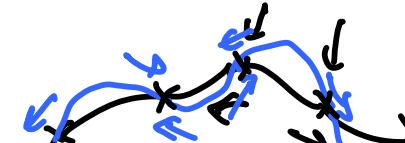
$$\begin{pmatrix} 1 & 0 & \dots & 0 & -1 & 0 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} g_0'' \\ g_1'' \\ \vdots \\ g_{n-1}'' \\ g_n'' \\ g_0'' \end{pmatrix} = \begin{pmatrix} q_0'' \\ q_1'' \\ \vdots \\ q_{n-1}'' \\ q_n'' \\ q_0'' \end{pmatrix}$$

For equally spaced pts.,

$$\frac{1}{6} [g''(x_{i-2}) + 4g''(x_i) + g''(x_{i+2})] = \frac{1}{\Delta x^2} (y_{i-1} - 2y_i + y_{i+1})$$

In some cases, you may get wiggles.

→ tension spline → pull on each ends ($y'' - \sigma^2 y''' = 0$)


$$\rightarrow g'' - \sigma^2 g = (g''_i - \sigma^2 y''_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} + (g''_{i+1} - \sigma^2 y''_{i+1}) \frac{x - x_i}{x_{i+1} - x_i}$$

$\sigma \rightarrow \infty$: straight line interpolation

$\sigma = 0$: usual spline

wiggles → increase σ !