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소성재료역학
(Metal Plasticity)

Chapter 6: Plasticity in pure bending and beam theory

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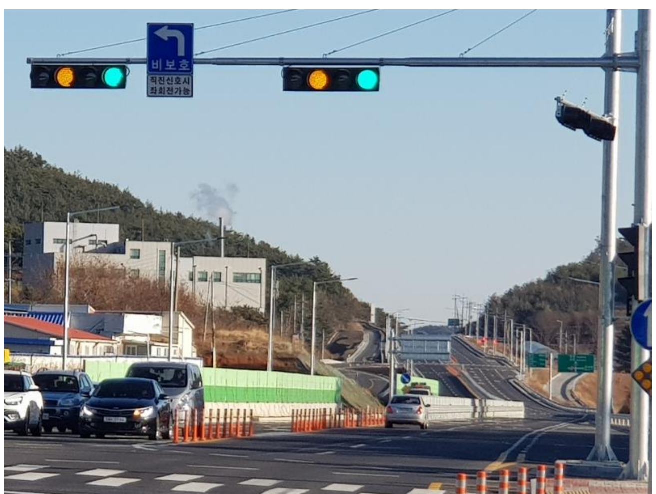
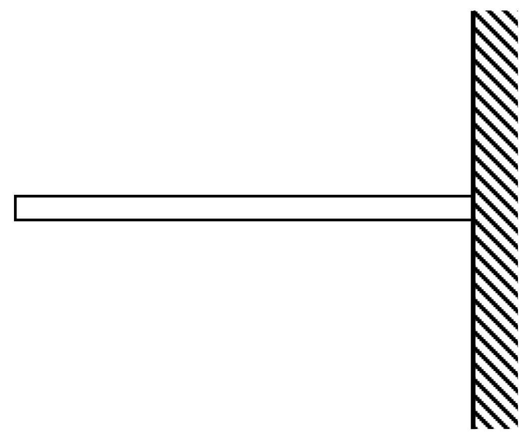
Beam theory

- Study on the deflection of a beam
- Beam: a long slender straight object under transverse loading
- Assumption: small deformation, uniform symmetric cross-section shape
- Some of geometric assumptions (on straightness and the cross-sectional shape) can be released with added complexity in the solution procedure

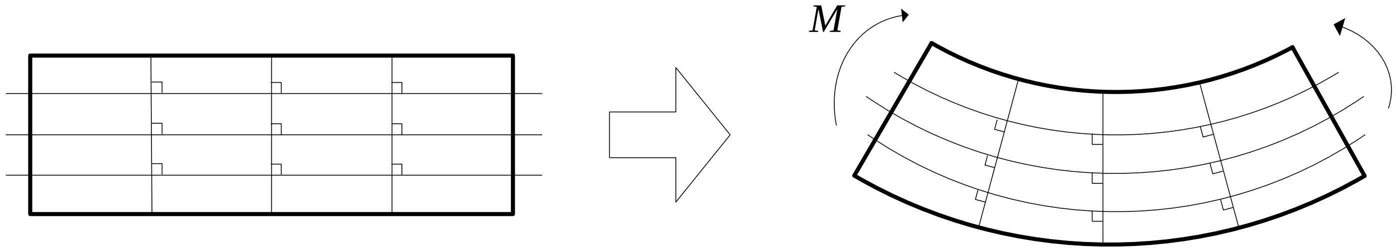


Cantilever beam (외팔보)

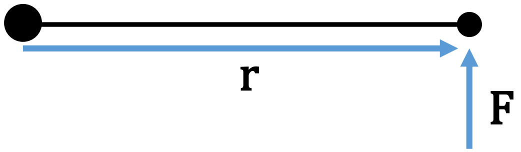


Pure bending theory

- Deformation of a straight object under bending moment only
- Small deformation and a uniform symmetric cross-section shape are assumed
- An accurate solution procedure supports that vertical planes remain vertical to curved outer surfaces and inner planes (parallel to outer upper and bottom surfaces) after deformation, *regardless of material properties*

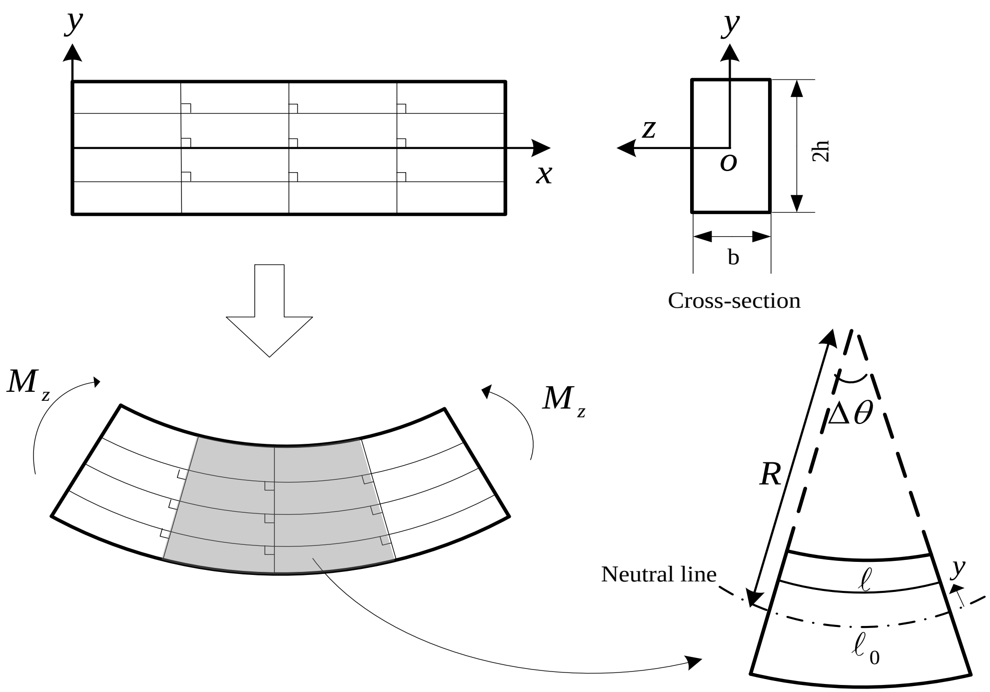


Moment $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$



Pure bending theory

- Consider the rectangular cross-section, for simplicity.



R: radius of curvature, K: curvature

$$l_0 = R\Delta\theta$$

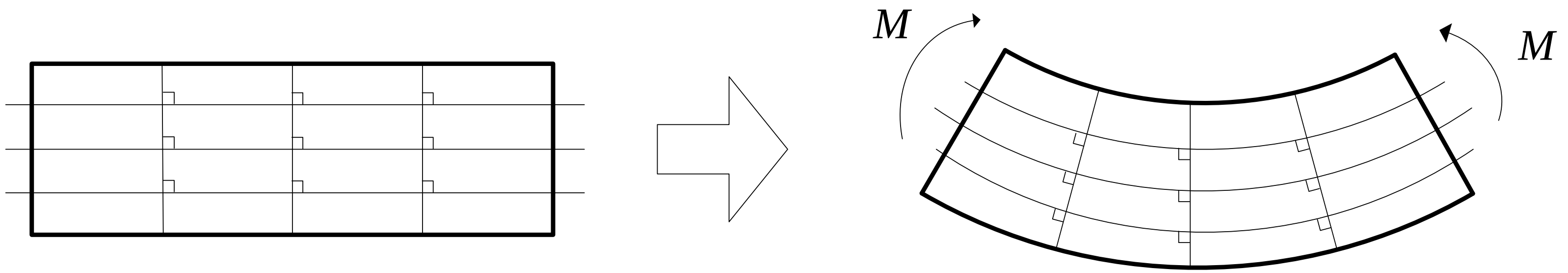
$$l = (R - y)\Delta\theta$$

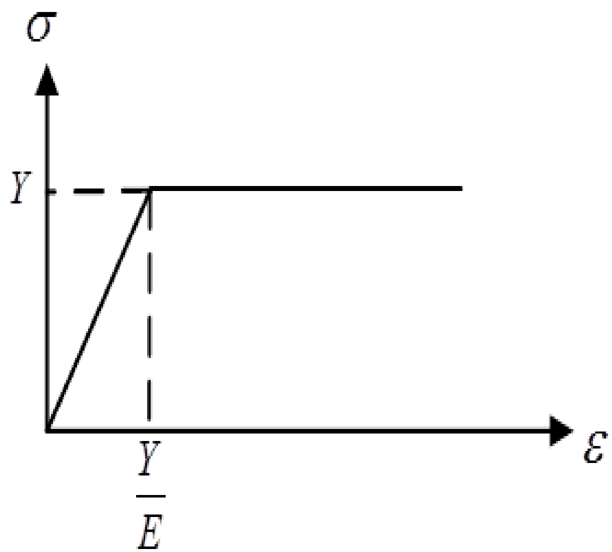
$$\varepsilon_x = \frac{\Delta l}{l_0} = -\frac{y\Delta\theta}{R\Delta\theta} = -\frac{y}{R} = -yK$$

- Following relations are valid regardless of material properties
- The origin of the coordinate system is located at the middle of the neutral plane. However, the vertical position of the neutral plane in the cross-section is unknown.

Pure bending theory

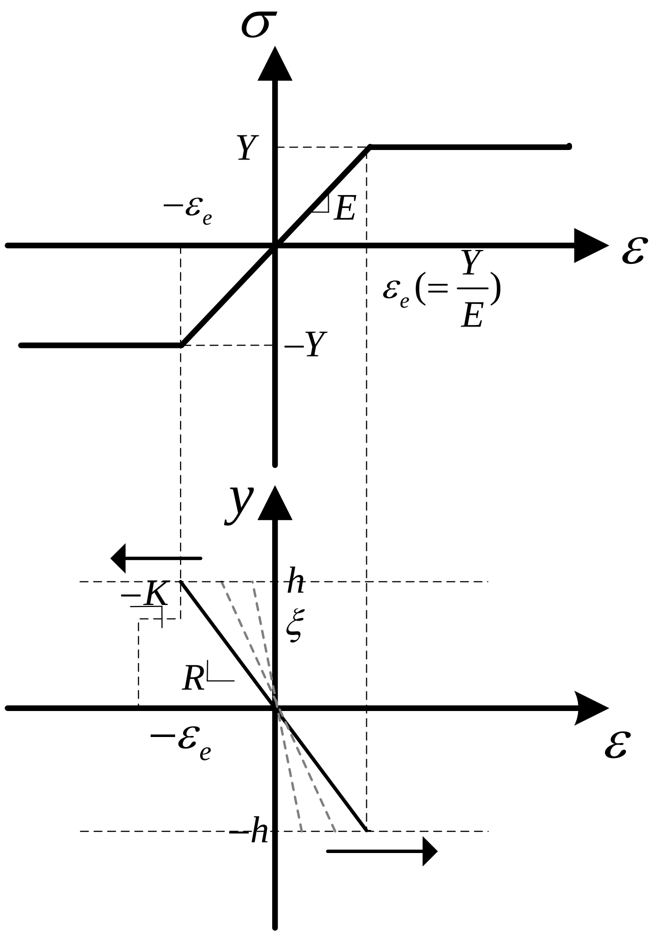
Pure bending of elasto-perfect plastic material

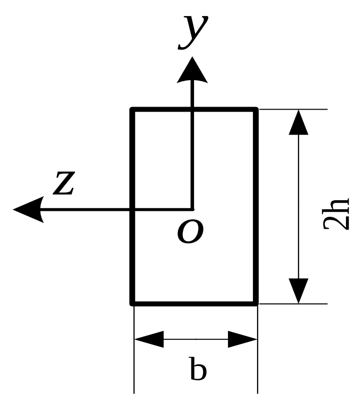




Pure bending theory

- **Elastic range**





Cross-section

Kinematics $\varepsilon = -\dfrac{y}{R} = -yK$

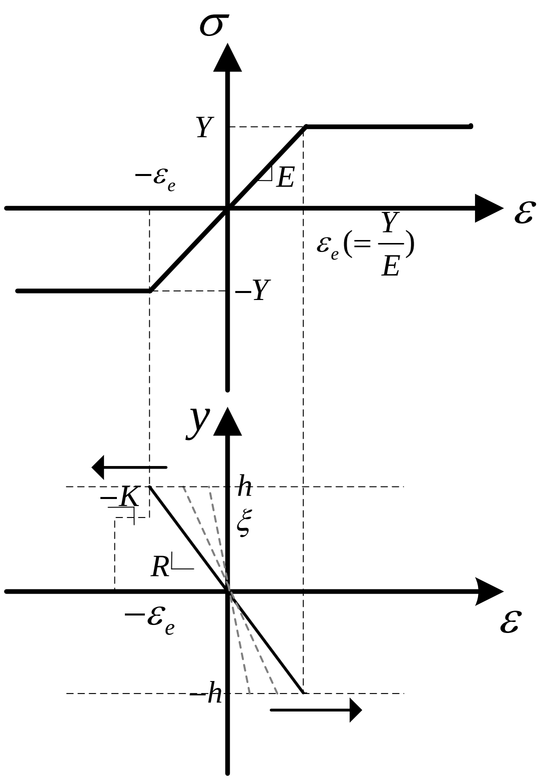
$\varepsilon = 0$ at neutral line (y=0)

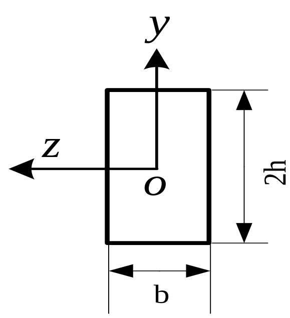
Material property (constitutive law) – linear elasticity

$$\sigma = E\varepsilon = -E\frac{y}{R} = -EyK$$

Pure bending theory

- **<u>Elastic range</u>**





Cross-section

Equilibrium equation

$$F = 0, M \neq 0$$

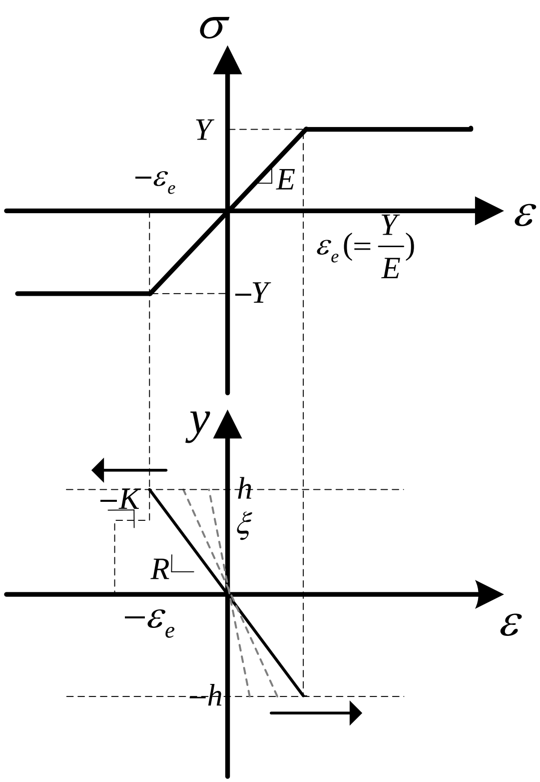
$$F = \int \sigma dA = -\int E \frac{y}{R} dA = -\frac{E}{R} \int y dA = 0$$

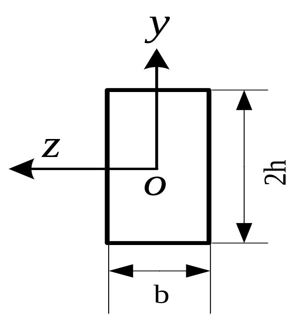
$$\rightarrow \int y dA = 0 \text{ (neutral plane=central plane)}$$

$$((F \neq 0) \leftrightarrow \text{neutral plane} \neq \text{central plane})$$

Pure bending theory

- **Elastic range**





Cross-section

Moment acting on beam

$$M = -\int \sigma y dA = \frac{E}{R}\int y^2 dA$$

$$= \frac{EI}{R} = EIK = \boxed{\frac{2Ebh^3K}{3}}$$

I: second moment of inertia

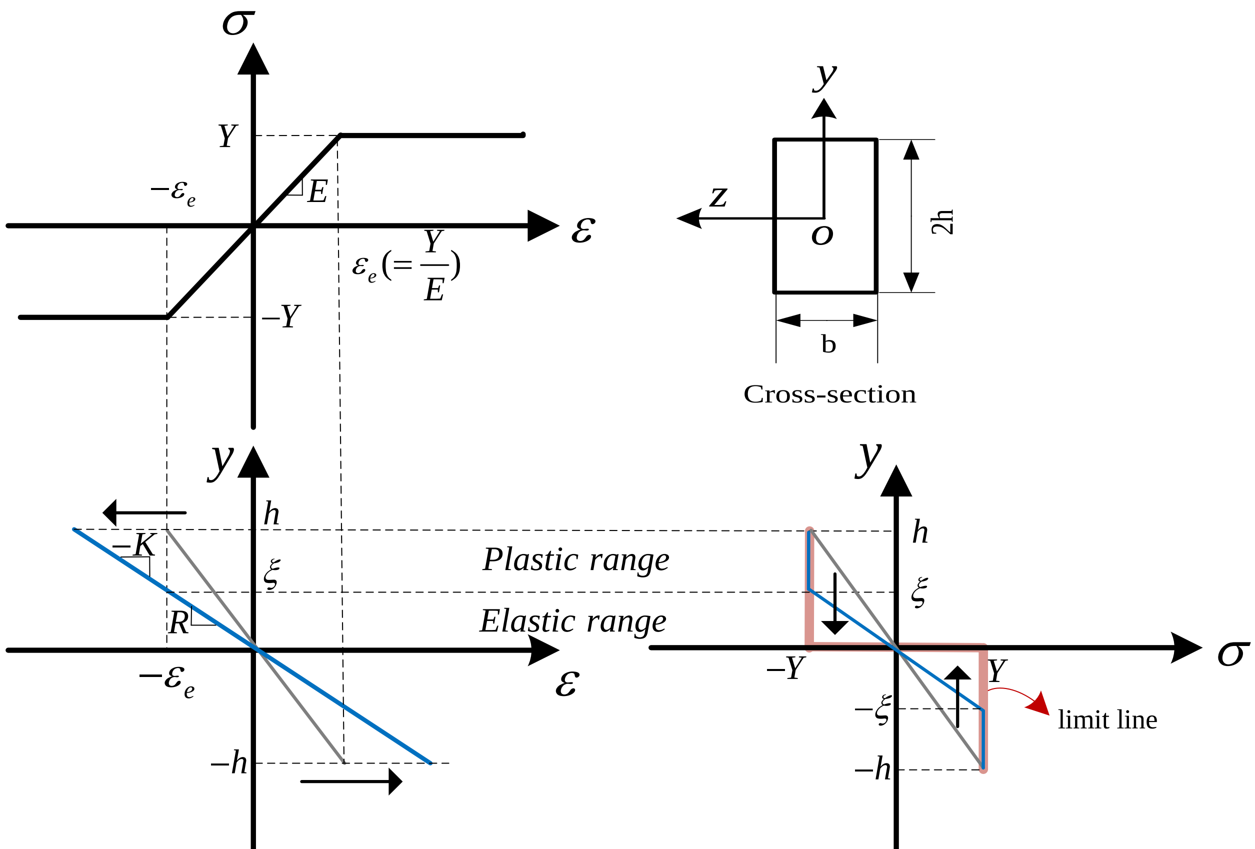
$$I = \int_{-h}^{h} y^2 dA = \int_{-h}^{h} by^2 dy = \frac{2bh^3}{3}$$

Pure bending theory

- **<u>Plastic range; elastic-plasticity</u>**

Kinematics $$\varepsilon = -\frac{y}{R} = -yK$$

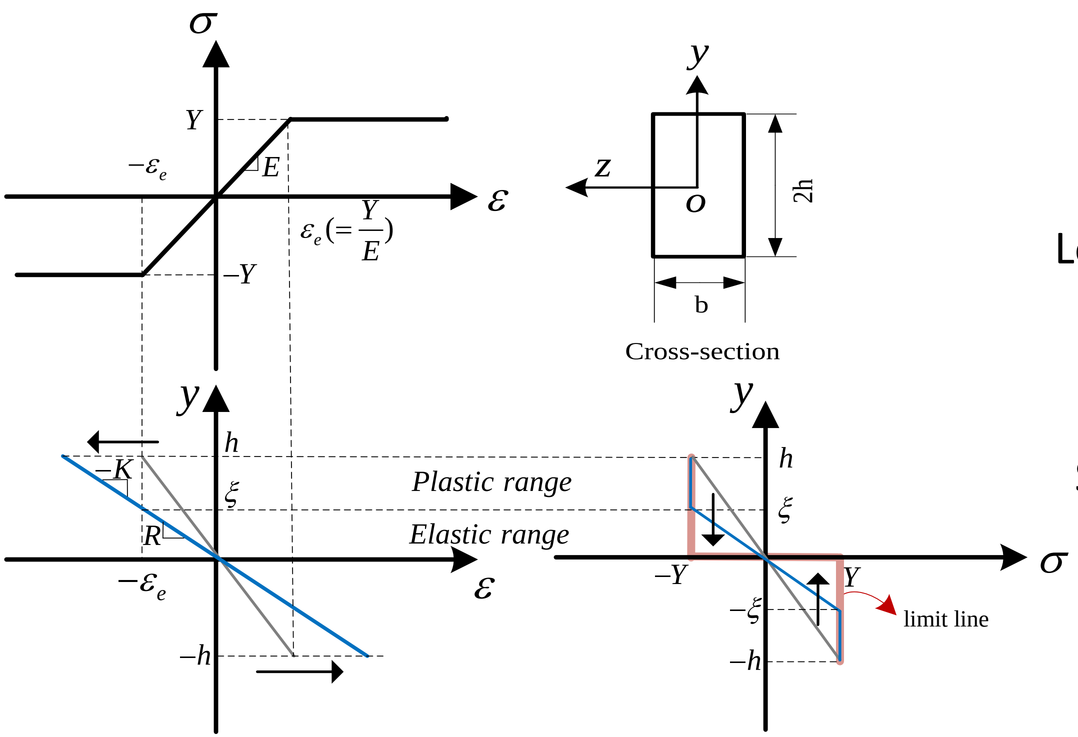
Constitutive law: elasto-perfect plasticity



ξ the location of the boundary between the elastic and e-p zones

Pure bending theory

- **<u>Plastic range; elastic-plasticity</u>**



Curvature when yield begins

$$K_e = \frac{Y}{Eh}$$

$$(\because \varepsilon_e = \frac{Y}{E} = -\frac{(-h)}{R_e} = hK_e)$$

Location of elastic/plastic boundary

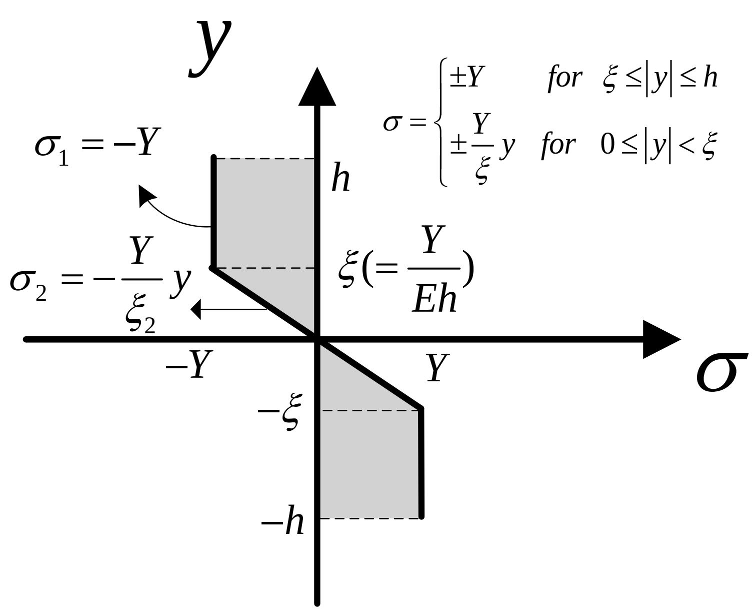
$$\frac{\xi}{h} = \frac{K_e}{K} \quad for\, K \geq K_e$$

Stress distribution through beam

$$\sigma = \begin{cases} \pm Y & for \quad \xi \leq |y| \leq h \\ \pm \dfrac{Y}{\xi} y & for \quad 0 \leq |y| < \xi \end{cases}$$

Pure bending theory

- **Plastic range; elastic-plasticity**



$$\sigma = \begin{cases} \pm Y & for \quad \xi \le |y| \le h \\ \pm \frac{Y}{\xi} y & for \quad 0 \le |y| < \xi \end{cases}$$

Equilibrium equation

$$F = 0, M \neq 0$$

<u>Moment acting on the beam</u>

$$M = -\int \sigma y dA = -2\left(\int_0^\xi \sigma_2 y dA + \int_\xi^h \sigma_1 y dA\right)$$

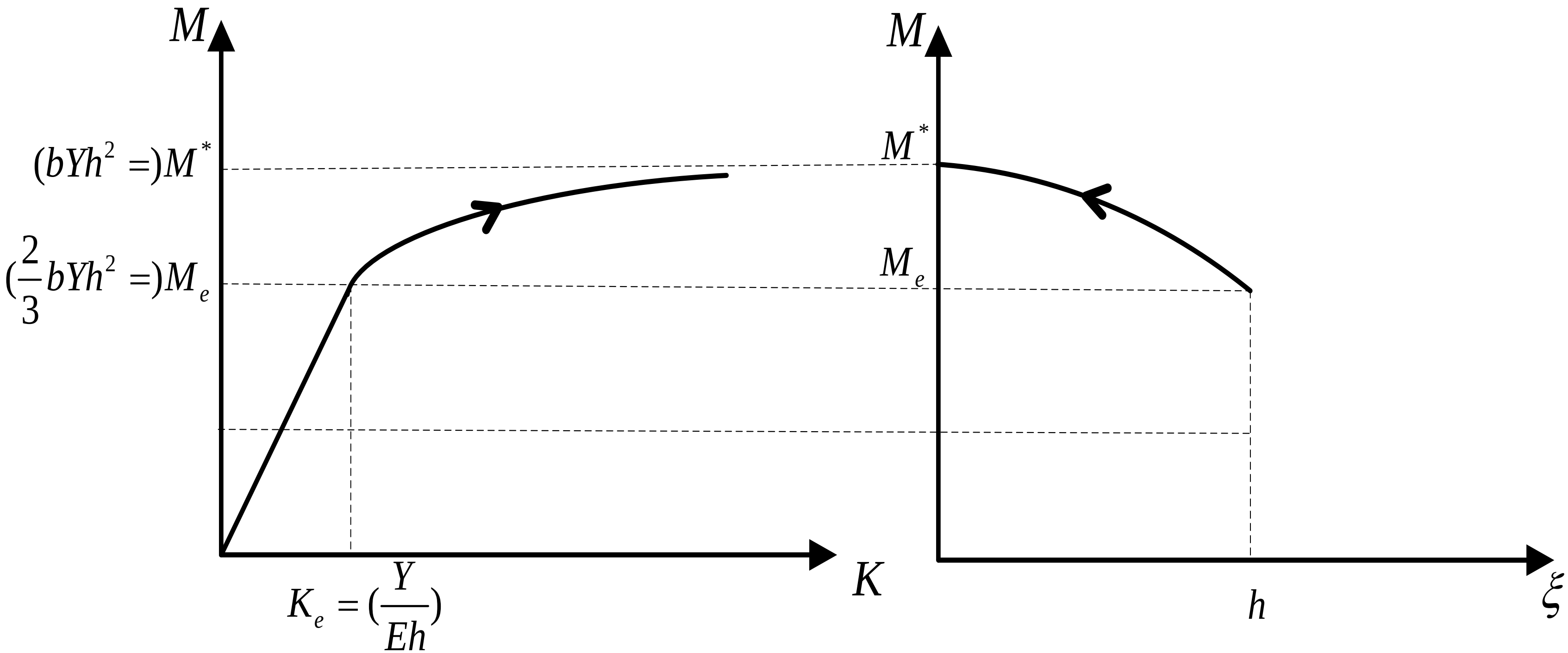
$$= 2\int_0^\xi \frac{Y}{\xi} y^2 (bdy) + 2\int_\xi^h Yy(bdy) = bY(h^2 - \frac{\xi^2}{3})(\xi \le h)$$

$$= bY(h^2 - \frac{h^2}{3}(\frac{K_e}{K})^2) = bYh^2(1 - \frac{1}{3}(\frac{K_e}{K})^2) \ (K \ge K_e)$$

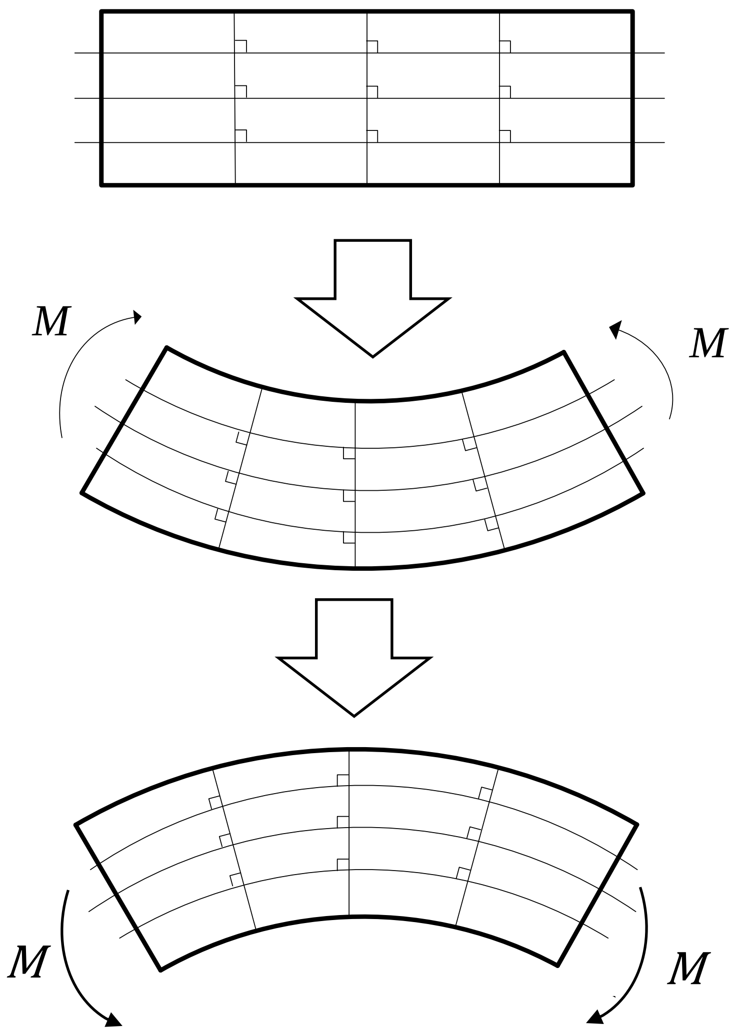
$$\boxed{\begin{aligned} M(K = K_e) &= M_e = \frac{2}{3} bYh^2 \\ M(K = \infty) &= M^* = bYh^2 \end{aligned}}$$

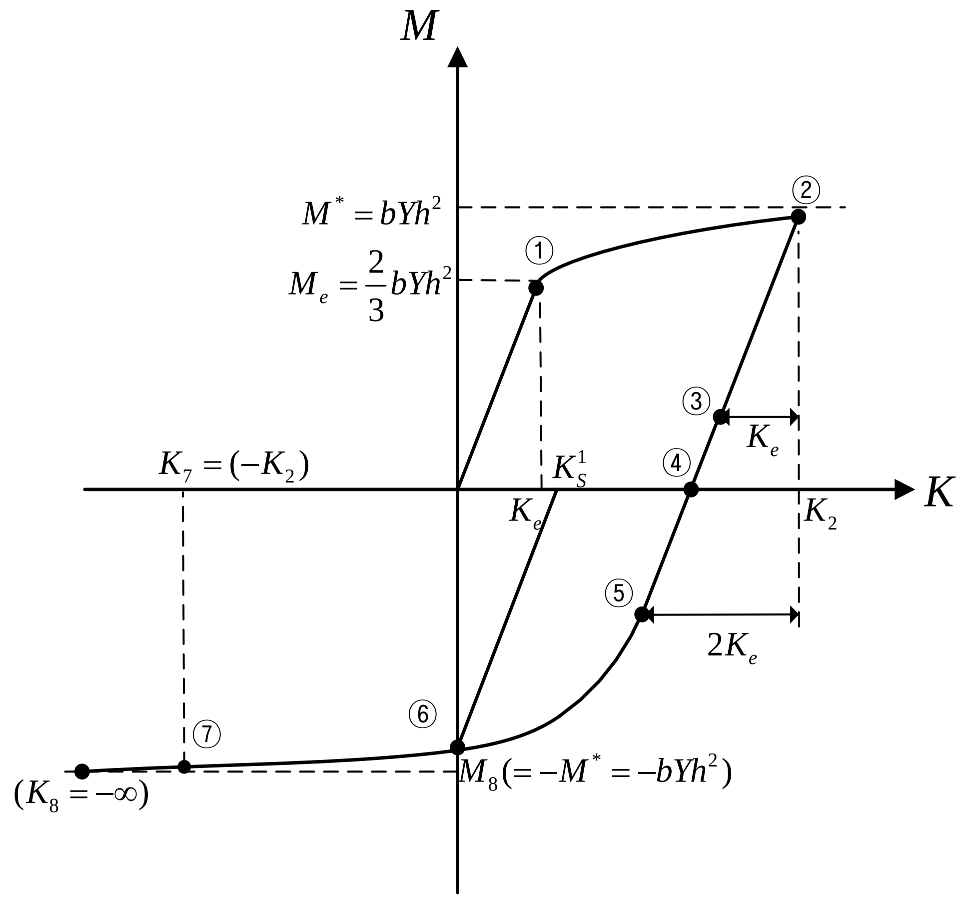
Pure bending theory

Relation between the bending moment and curvature



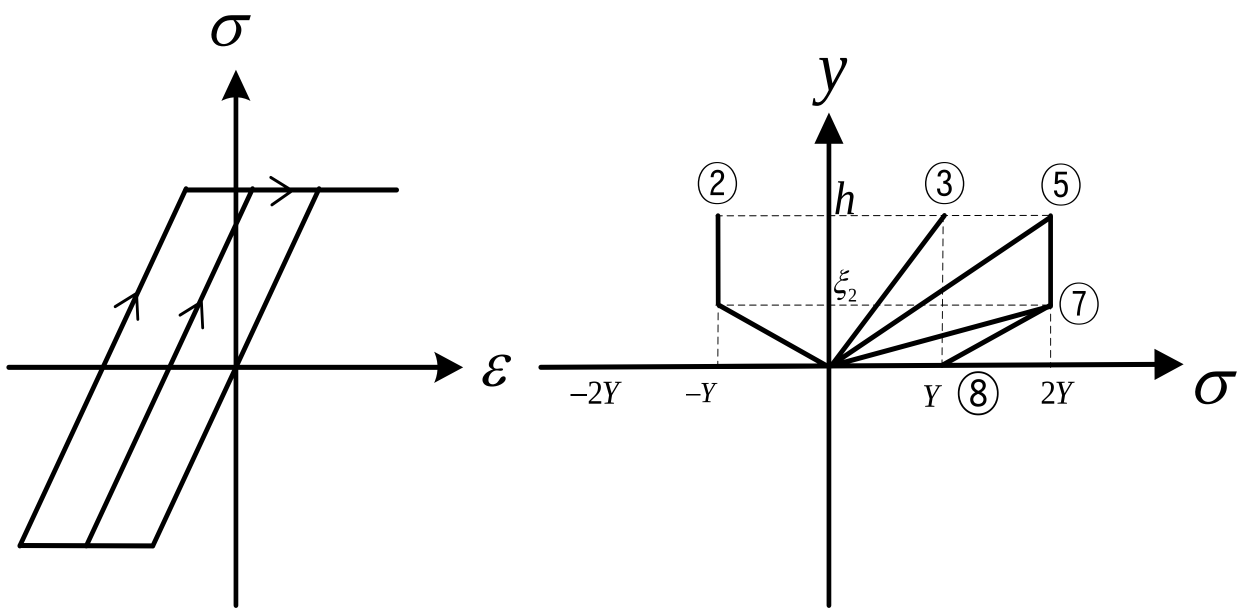
Pure bending theory
bending – unbending – reverse bending





Pure bending theory
bending – unbending – reverse bending

- Sequential superposition (for a non-linear system)

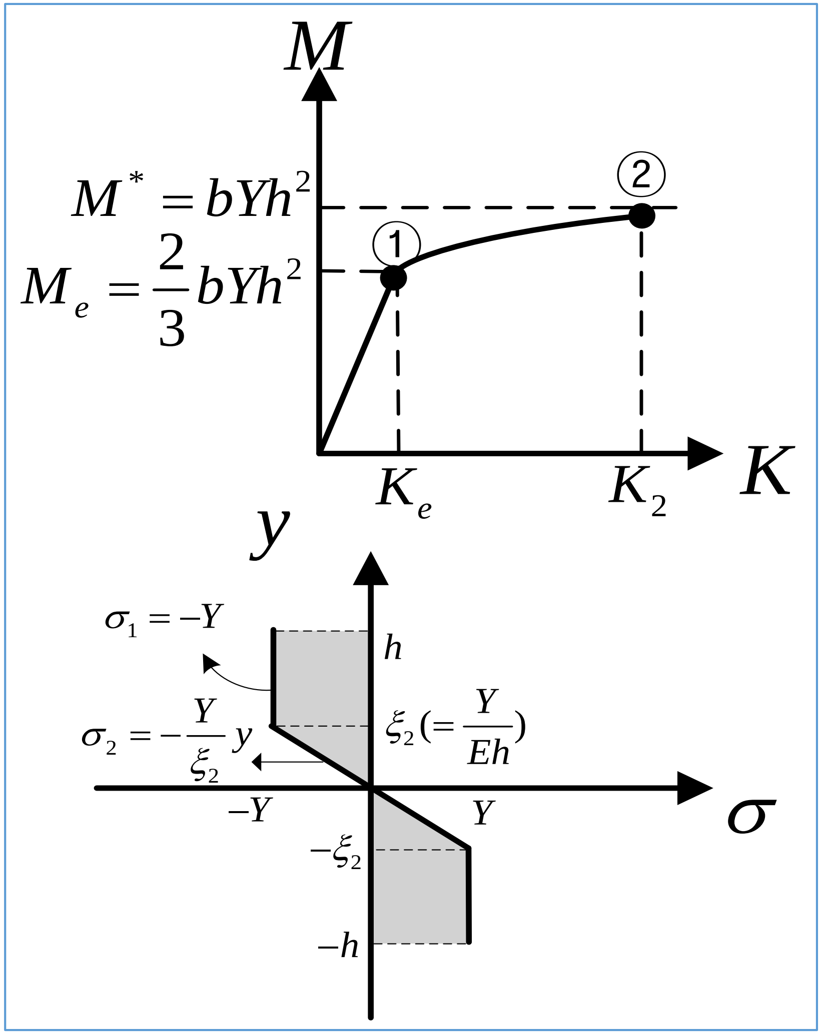


Remark

- M(K) relationship between 5 and 7 is the twice stretch of the M(K) relationship between 1 and 2.

Pure bending theory: initial bending

1. Point 1~2 $(K_e < K < K_2)$



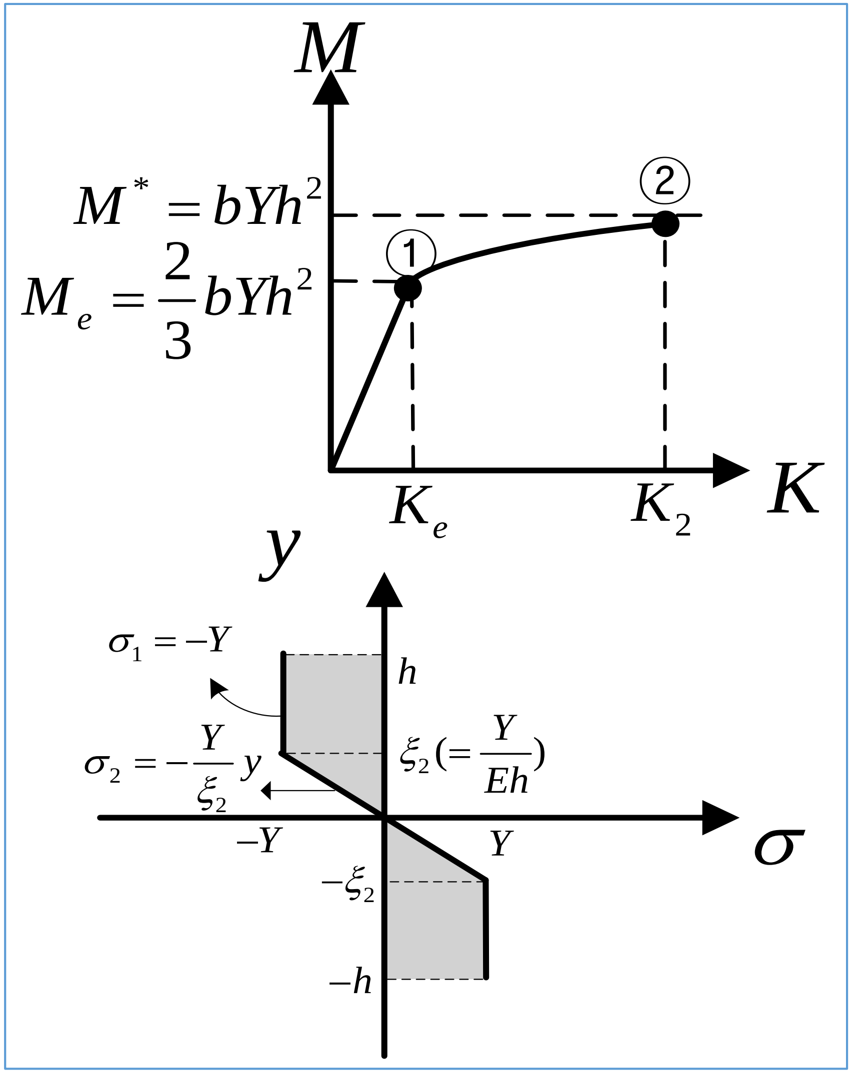
$$M_{1\sim2} = -2b\left(\int_0^{\xi} \sigma_2 y\,dy + \int_{\xi}^{h} \sigma_1 y\,dy\right)$$

$$= bYh^2\{1 - \frac{1}{3}(\frac{\xi}{h})^2\}$$

$$= bYh^2\{1 - \frac{1}{3}(\frac{K_e}{K_{1\sim2}})^2\}$$

Pure bending theory: initial bending

1. At point 2 $(K_e < K < K_2)$



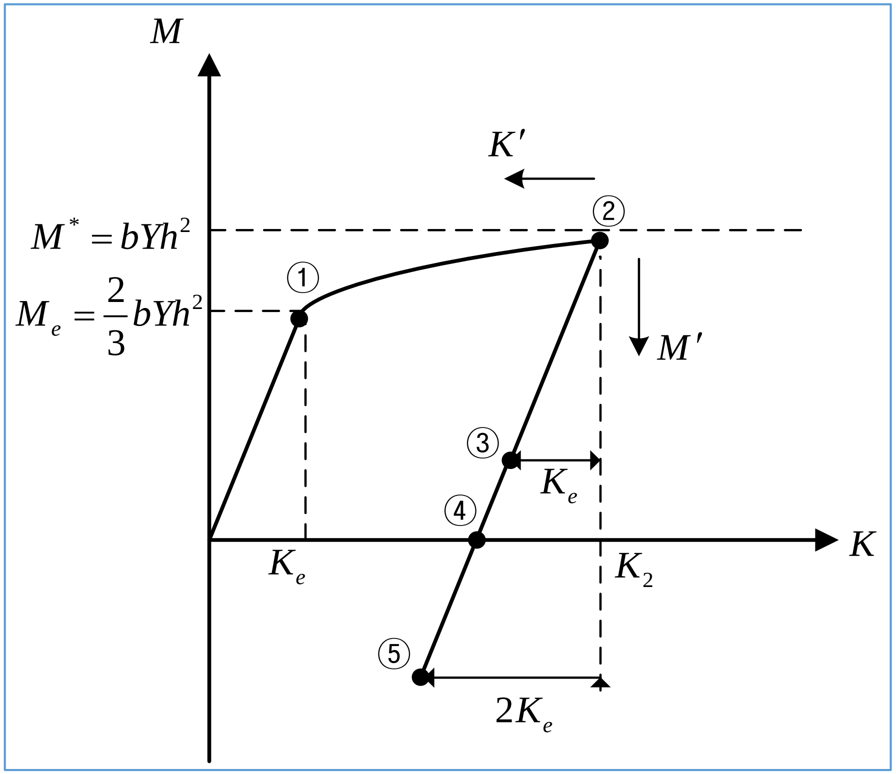
$$M_2 = bYh^2 (1 - \frac{1}{3} (\frac{K_e}{K_2})^2)$$

$$\varepsilon_e = K_e h = K_{1\sim2} \xi \rightarrow \frac{K_e}{K_{1\sim2}} = \frac{\xi}{h}$$

$$K_2 = \frac{h}{\xi_2} K_e$$

Pure bending theory: unbending

2. Point 2~5 $(0 < K' < 2K_e = K'_e)$



$$M'_{2\sim5} = \frac{2Ebh^3K'}{3}$$

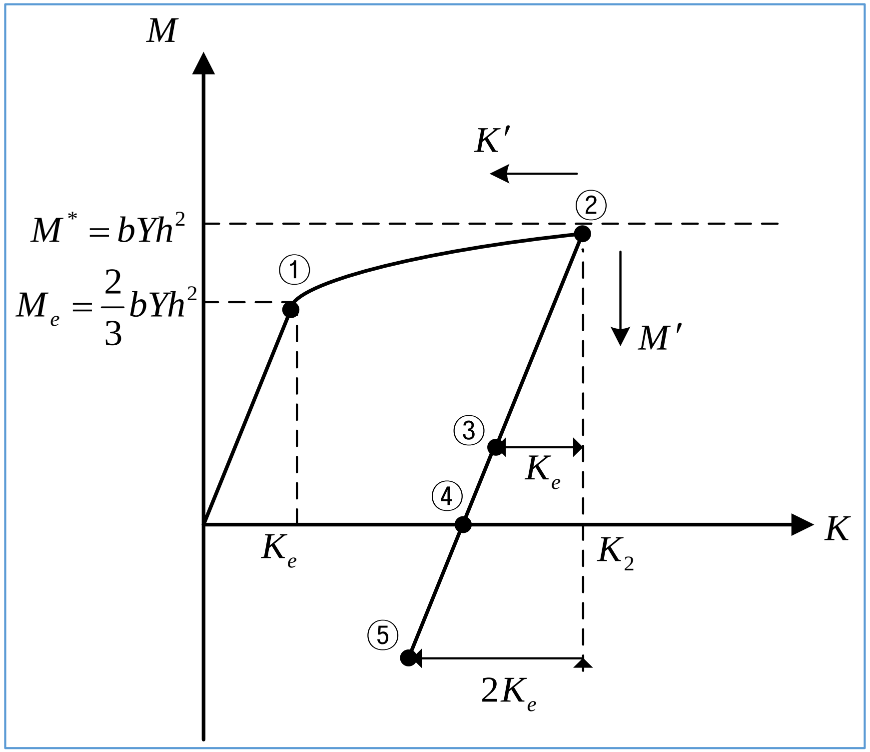
$$M_{2\sim5} = M_2 - M'_{2\sim5}$$

$$\sigma' = -EyK'$$

$$\sigma_{2\sim5}(y) = \sigma_2(y) - \sigma'_{2\sim5}(y)$$

Pure bending theory: unbending

2. Point 2~5 $(0 < K' < 2K_e = K_e')$



1) Point 3 $(K_3' = K_e)$

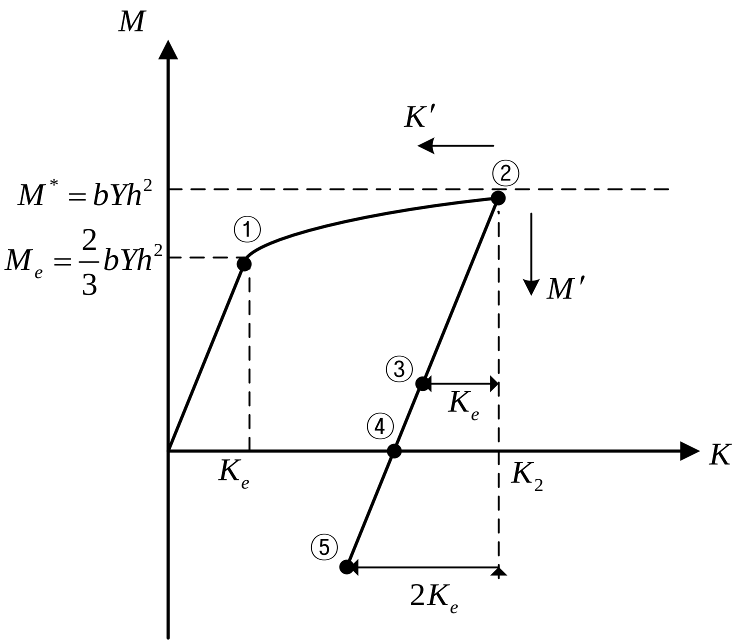
$$M_3' = M'(K' = K_e = \frac{Y}{Eh}) = \frac{2bYh^2}{3} = M_e$$

$$M_3 = M_2 - M_3' = \frac{bYh^2}{3}\{1 - (\frac{K_e}{K_2})^2\}$$

$$\sigma_3(y) = \sigma_2(y) - \sigma_3'(y)$$

Pure bending theory: unbending

2. Point 2~5 $(0 < K' < 2K_e = K_e')$



1) Point 4 $(M_4 = 0)$

$$M_4 = M_2 - M_4' = 0$$

$$bYh^2\left(1 - \frac{1}{3}\left(\frac{K_e}{K_2}\right)^2\right) - \frac{2Ebh^3K_4'}{3} = 0$$

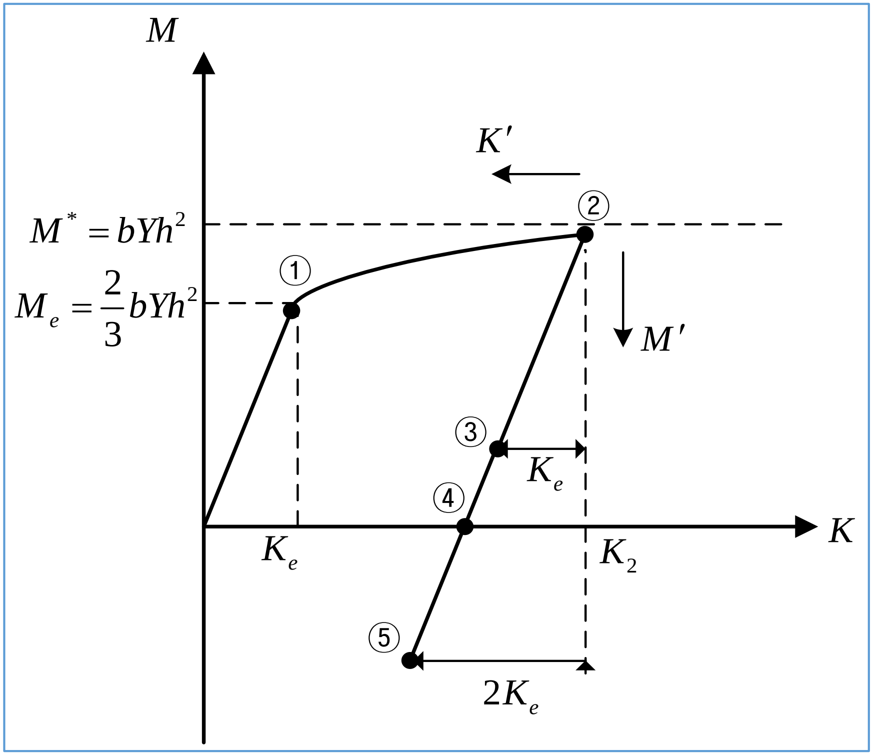
$$K_4{}' = \frac{3}{2}\frac{Y}{Eh}\left(1 - \frac{1}{3}\left(\frac{K_e}{K_2}\right)^2\right)$$

$$\sigma_4(y) = \sigma_2(y) - \sigma_4'(y)$$

Pure bending theory: unbending

<u>2. Point 2~5</u> $(0 < K' < 2K_e = K_e')$

1) Point 5



$$(K_5' = 2K_e = K_e' = \frac{2Y}{Eh}, \varepsilon_e' = 2\varepsilon_e = \frac{2Y}{E})$$

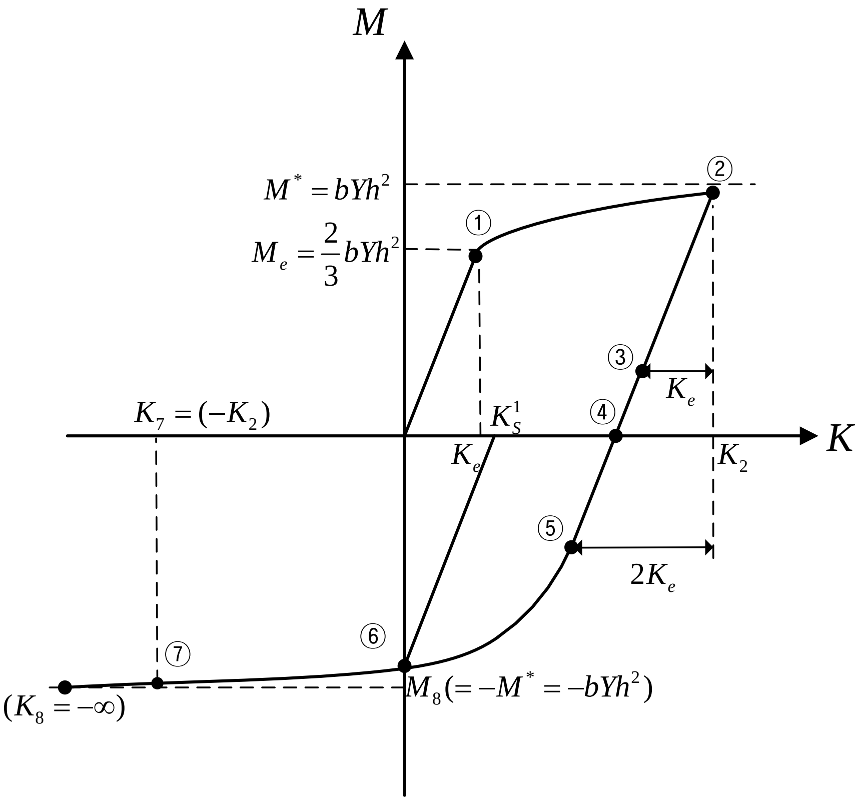
$$M_5' = \frac{4Ebh^3K_e}{3} = \frac{4Ybh^2}{3}$$

$$M_5 = M_2 - M_5' = -\frac{Ybh^2}{3}\{1 + (\frac{K_e}{K_2})^2\}$$

$$(\because M_5' = 2M_e)$$

Pure bending theory: re-bending

3. Point 5~7 $(2K_e = K_5' < K' < K_7', \xi_2 \le \eta \le h)$



$$M_{5\sim7}' = -2b\{\int_0^{\eta} -\sigma_2 y dy + \int_{\eta}^{h} -\sigma_1 y dy\}$$

$$= 2bYh^2\{1 - \frac{1}{3}(\frac{\eta}{h})^2\}$$

$$= 2bYh^2\{1 - \frac{1}{3}(\frac{K_e'}{K_{5\sim7}'})^2\} = 2M_{1\sim2}$$

$$M_{5\sim7} = M_2 - M_{5\sim7}'$$

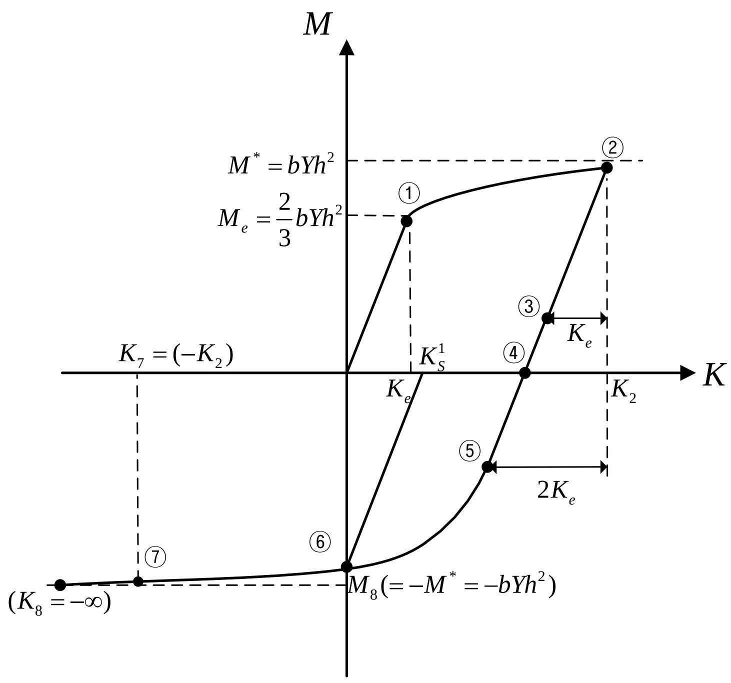
$$\varepsilon_e' = -(-\eta)K_{5\sim7}' = \frac{2Y}{E} = -(-h)K_e' = 2hK_e$$

$$\rightarrow K_{5\sim7}' = \frac{h}{\eta}K_e'$$

Pure bending theory: re-bending

3. Point 5~7

$$(2K_e = K_5' < K' < K_7', \xi_2 \le \eta \le h)$$



1) Point 6

$$K_6 = K_2 - K_6' = 0$$

$$\therefore K_6' = K_2 (> 2K_e)$$

$$M_6 = M_2 - M_6'$$

$$= bYh^2(1 - \frac{1}{3}(\frac{K_e}{K_2})^2) - 2bYh^2(1 - \frac{1}{3}(\frac{2K_e}{K_2})^2)$$

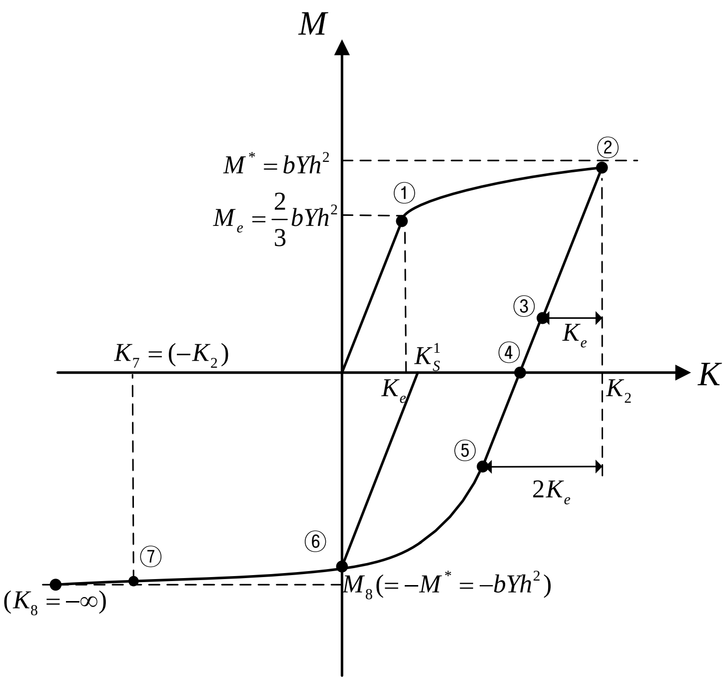
$$= -bYh^2(1 - \frac{7}{3}(\frac{K_e}{K_2})^2)$$

$$K_S^1 = \frac{|M_6|}{\frac{2}{3}Ebh^3} = \frac{3}{2}\frac{Y}{Eh}(1 - \frac{7}{3}(\frac{K_e}{K_2})^2)$$

Pure bending theory: re-bending

3. Point 5~7

$(2K_e = K_5' < K' < K_7', \xi_2 \le \eta \le h)$

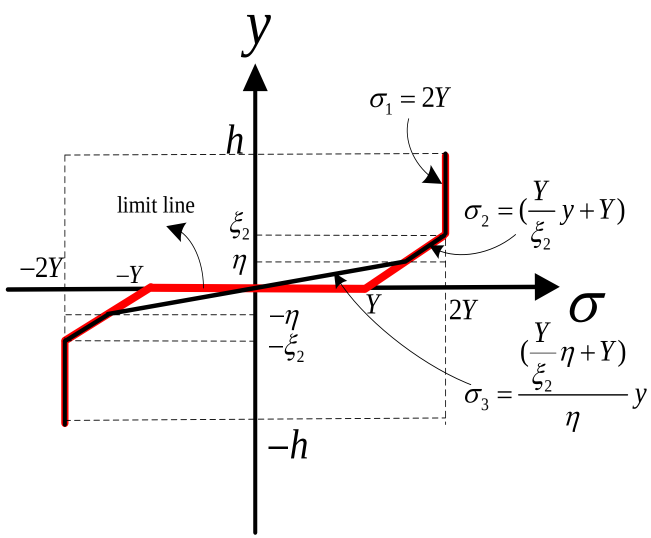


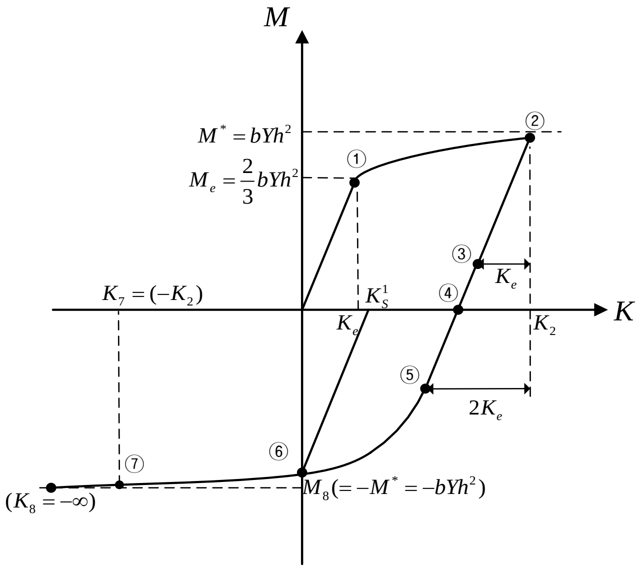
1) Point 7 $(K_7' = 2K_2, \eta = \xi_2)$

$$M_7 = M_2 - M_7' = M_2 - 2M_2 = -M_2$$

Pure bending theory: reverse bending

4. Point 7-8





$$M'_{7\sim 8} = -2b\left(\int_0^{\eta} -\sigma_3 y\,dy + \int_{\eta}^{\xi_2} -\sigma_2 y\,dy + \int_{\xi_2}^{h} -\sigma_1 y\,dy\right)$$

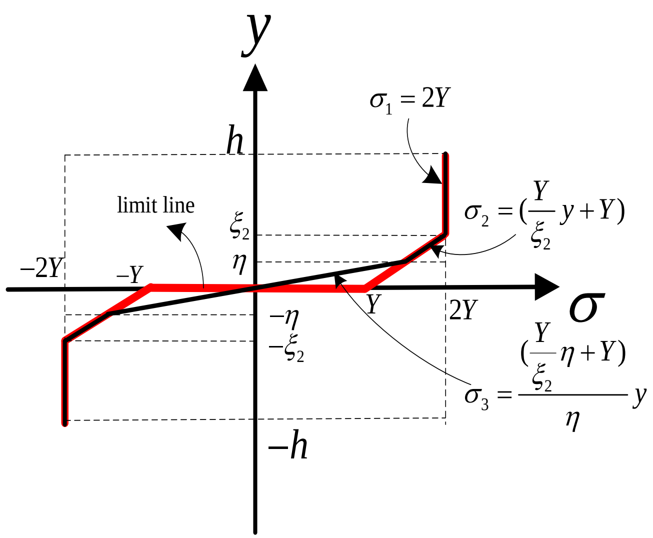
$$= 2bYh^2\left\{1 - \frac{1}{6}\left(\frac{\xi_2}{h}\right)^2 - \frac{1}{6}\left(\frac{\eta}{h}\right)^2\right\}$$

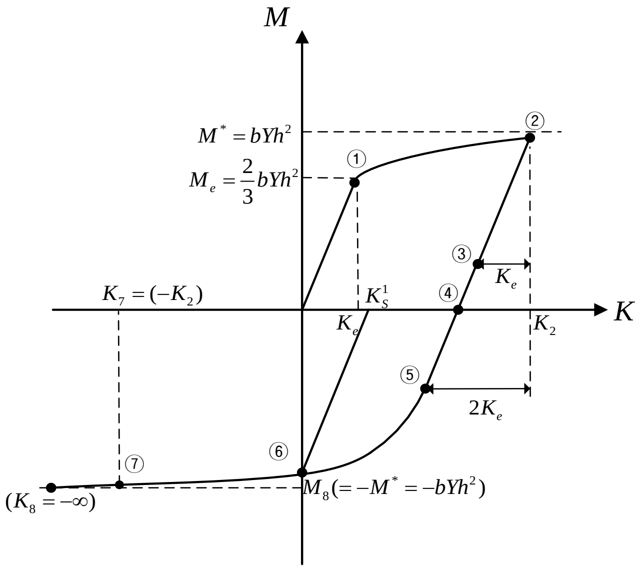
$$= 2bYh^2\left\{1 - \frac{1}{6}\left(\frac{K_e}{K_2}\right)^2 - \frac{1}{6}\left(\frac{K'_e}{2K'_{7\sim 8} - K'_7}\right)^2\right\}$$

$$\because \frac{Y}{\xi_2}\eta + \mathrm{Y} = \mathrm{E}\,\eta\,\mathrm{K}' \rightarrow \eta = \frac{Y}{E\left(K'_{7\sim 8} - \dfrac{Y}{E\xi_2}\right)}$$

Pure bending theory: reverse bending

4. Point 7-8





$$\eta = \frac{Y}{E(K'_{7\sim 8} - \frac{Y}{E\xi_2})} \qquad \textit{while} \quad h = \frac{2Y}{EK'_e}$$

$$\frac{\eta}{h} = \frac{K'_e}{2(K'_{7\sim 8} - \frac{Y}{E\xi_2})} = \frac{K'_e}{2(K'_{7\sim 8} - \frac{K'_7}{2})} = \frac{K'_e}{2K'_{7\sim 8} - K'_7}$$

$$(\because K'_7 = 2K_2 \leftarrow K_2\xi_2 = K_e h = \frac{Y}{E}))$$

$$M_{7\sim 8} = M_2 - M'_{7\sim 8}$$

Pure bending theory: reverse bending

(1) Point 7 $(K_7' = -2K_2, \eta = \xi_2)$

$$M_7' = 2bYh^2\{1 - \frac{1}{3}(\frac{K_e}{K_2})^2\} = 2M_2$$

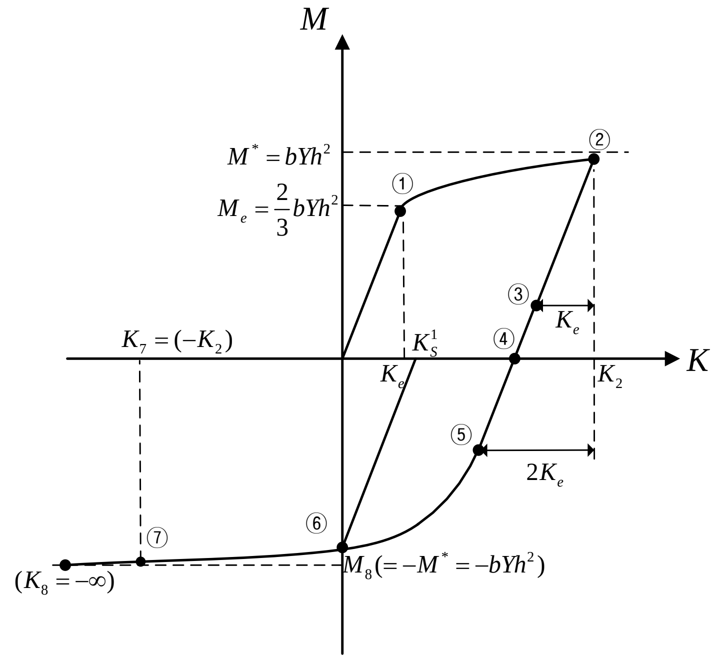
(Consistent)

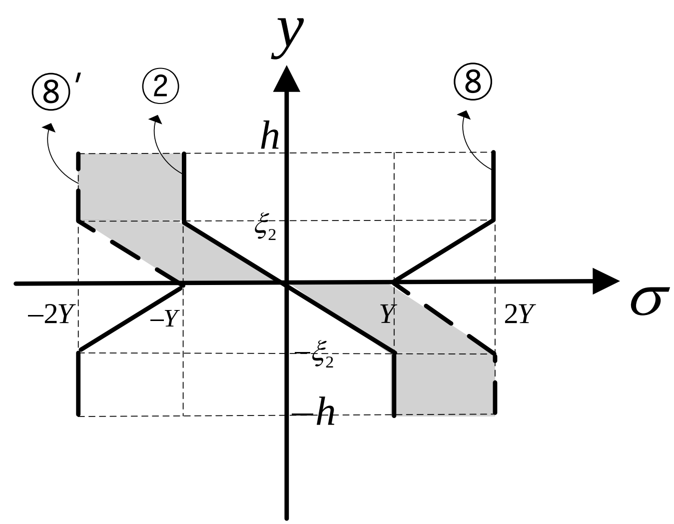
(2) Point 8 $(K_8' = \infty, \eta \to 0)$

$$M_8' = 2bYh^2(1 - \frac{1}{6}(\frac{K_e}{K_2})^2)$$

$$M_8 = M_2 - M_8' = -bYh^2$$

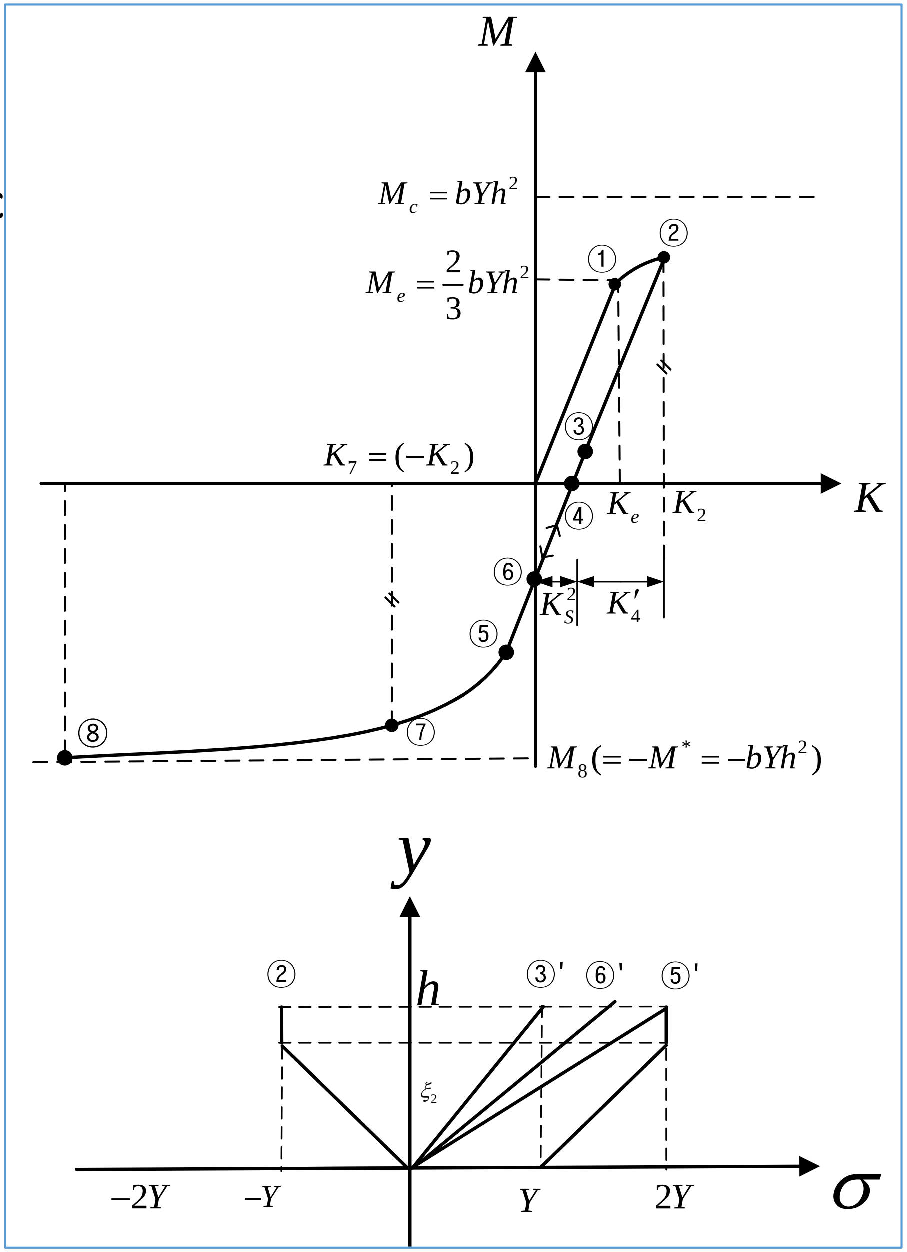
$$K_8 = K_2 - K_8{}' = -\infty$$





Pure bending theory $K_e \le K_2 \le 2K_e$

Full recovery of the initial bending curvature(point 6) occurs during elastic unloading from the initial bending



$$K_6' = K_2$$

$$M_6' = \frac{2Ebh^3K_6'}{3} = \frac{2Ebh^3K_2}{3}$$

$$M_6(\le 0) = M_2 - M_6' = bYh^2\{1 - \frac{1}{3}(\frac{K_e}{K_2})^2\} - \frac{2Ebh^3K_2}{3}$$

$$K_S^2 = \frac{|M_6|}{\frac{2}{3}Ebh^3} = K_2 - \frac{3}{2}\frac{Y}{Eh}\{1 - \frac{1}{3}(\frac{K_e}{K_2})^2\} = K_2 - K_4'$$

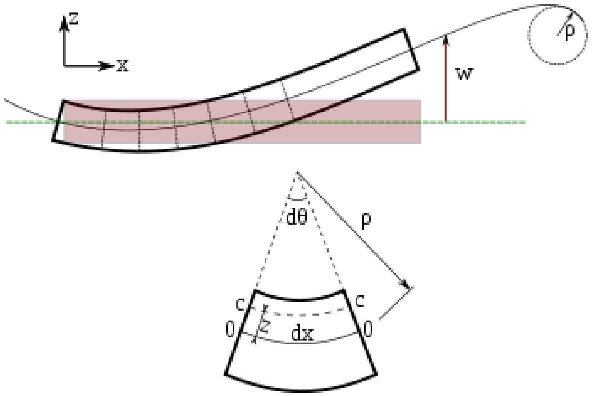
Beam theory

- Study on the deflection of a beam
- Long slender straight object under transverse loading
- Small deformation and a uniform symmetric cross-section shape

Euler- Bernoulli beam theory

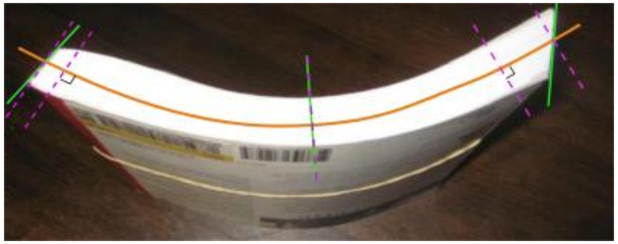
- Shear deformation is neglected (especially for a thin beam)

Traffic light
Long & slender shape

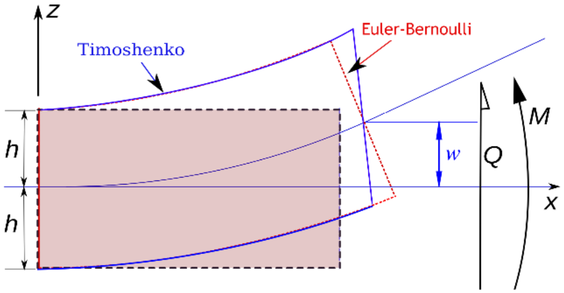


cf) Timoshenko beam theory

- Shear deformation is considered (especially for a thicker beam)



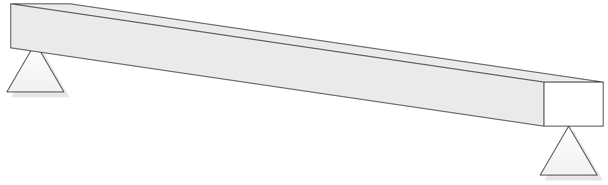
Bending of thick book
Mid-surface orientation and shape



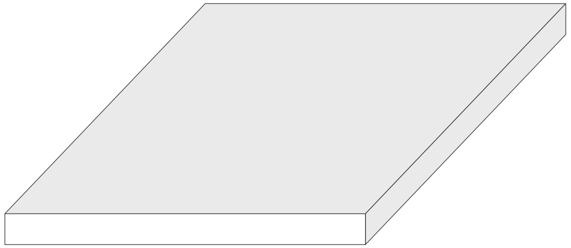
Images : https://en.wikipedia.org/wiki/Timoshenko_beam_theory
https://en.wikipedia.org/wiki/Euler%E2%80%93Bernoulli_beam_theory

Beam theory

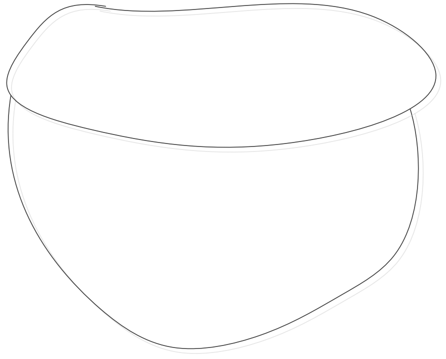
- Beam : 1-D



- Plate : 2-D

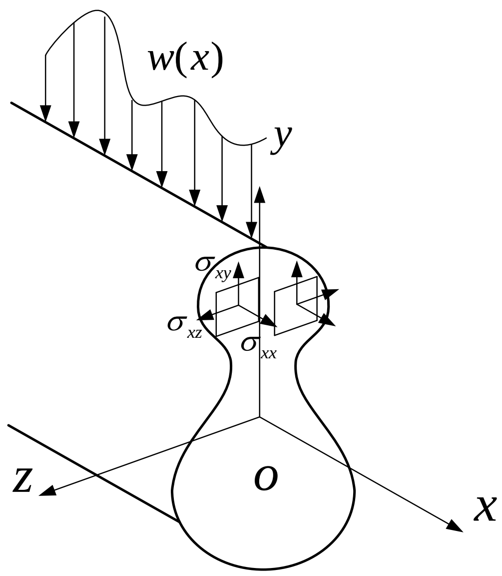


- Shell : 3-D

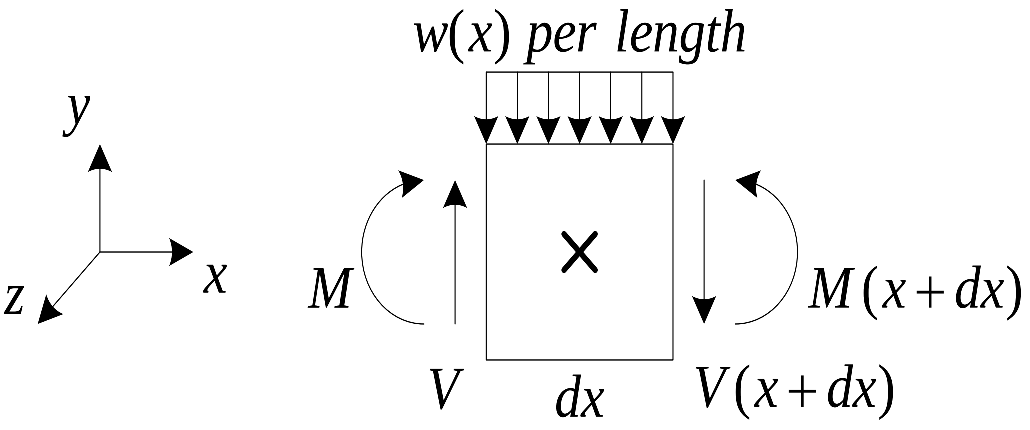


Equilibrium in beam – distributed load

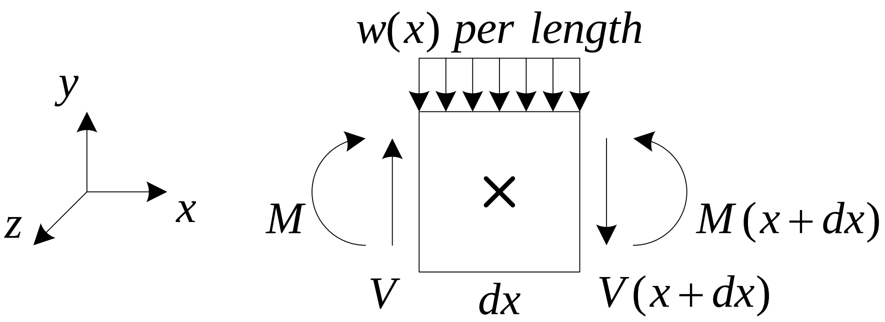
Symmetric stress distribution
on a beam with symmetric cross-section



Bending moment and shear force distribution on a beam



Equilibrium in beam – distributed load



- Force equilibrium

$$V(x+dx)+wdx = V(x)+dV+wdx = V(x)$$

$$\boxed{\therefore\ w(x) = -\frac{dV}{dx}}$$

- Moment equilibrium
  (reference at the center)

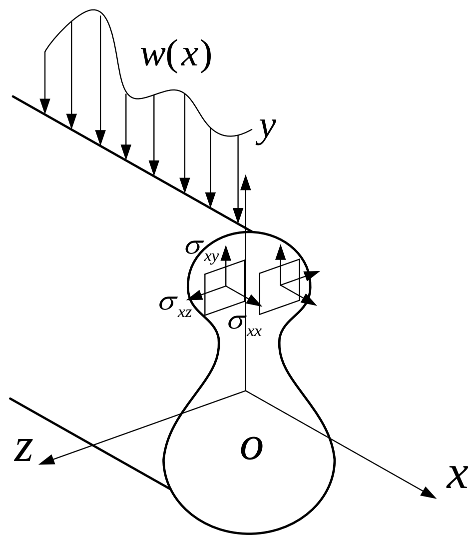
$$V(x)\frac{dx}{2}+V(x+dx)\frac{dx}{2}$$

$$= V(x)\frac{dx}{2}+(V(x)+dV)\frac{dx}{2} \quad = V(x)dx$$

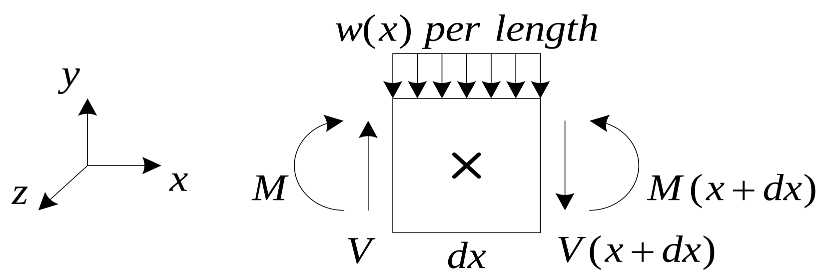
$$= M(x+dx)-M(x) \quad = dM$$

$$\boxed{\therefore\ V(x) = \frac{dM}{dx}}$$

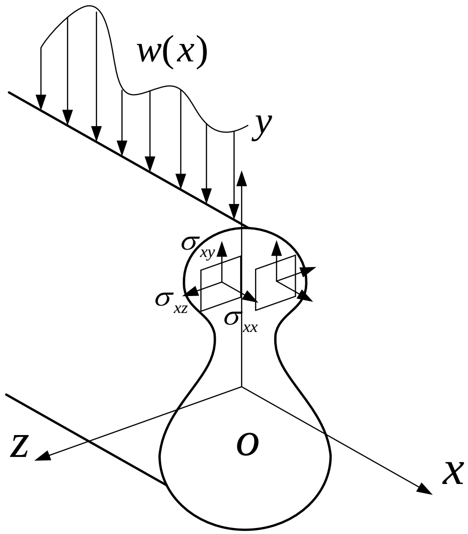
Kinematics : infinitesimal deformation theory

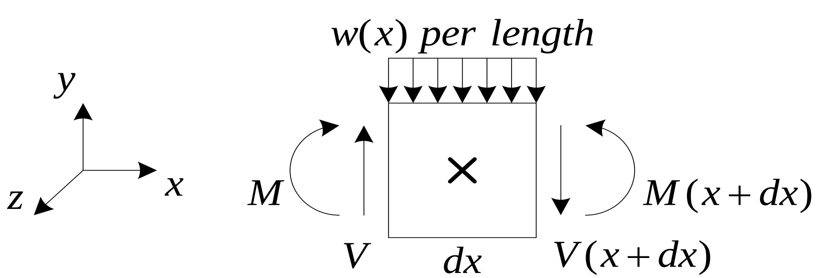


$$K\left(= \frac{d^2v \big/ dx^2}{\{1+(dv/dx)^2\}^{3/2}}\right) \approx \frac{d^2v}{dx^2}$$



Constitutive law & beam analysis





- Linear elasticity

$$M = EIK$$

$$w(x)(=-\frac{dV}{dx}=-\frac{d^2M}{dx^2})=-EI\frac{d^4v}{dx^4}$$

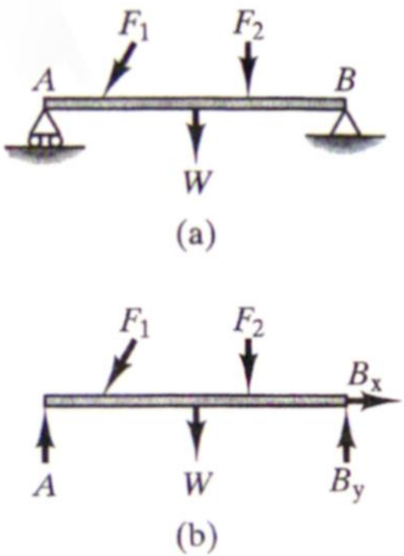
- Elasto-perfect plastic behavior

$$w(x)(=-\frac{dV}{dx}=-\frac{d^2M}{dx^2})=-\frac{d^2f}{dx^2}(K(x))$$

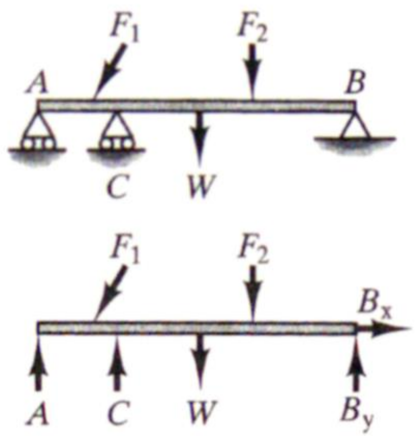
$$=-\frac{d^2f}{dx^2}(\frac{d^2v}{dx^2}(x))$$

Statically determinate vs indeterminate

- **Statically determinate**



- **Statically indeterminate**

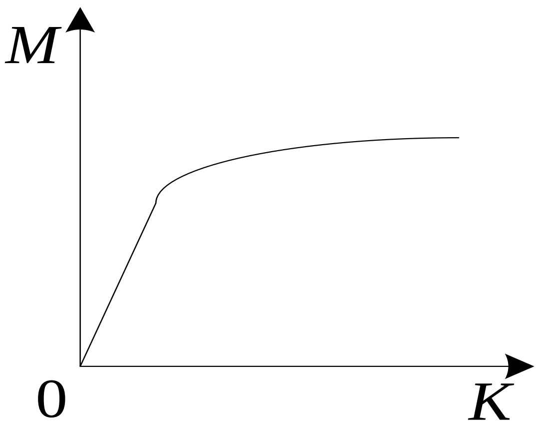


Figures : Shames, Introduction to solid mechanics

Solution procedure of the SD (statically determinate) problem

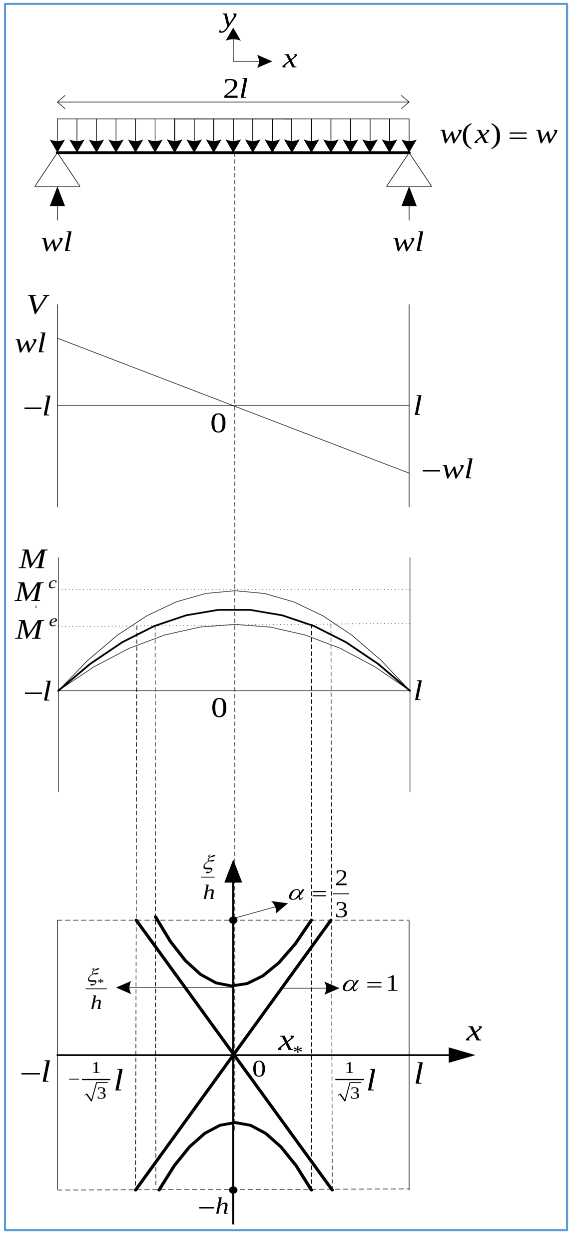
- Find $V(x), M(x)$ from the equilibrium condition

- Utilizing the moment-curvature relationship, find out the curvature distribution



- Integrate the curvature to find the deflection $K = \frac{d^2 v}{dx^2} \rightarrow v$

Example : Beam with elasto-plastic behavior



- Equilibrium equations

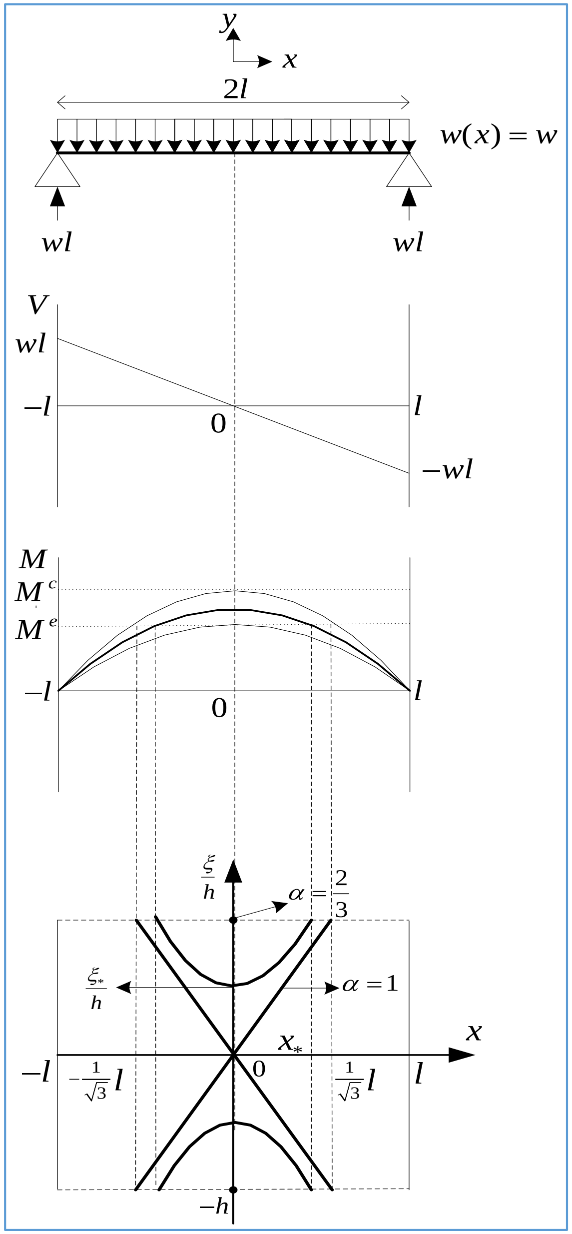
$$\therefore\ w(x) = -\frac{dV}{dx} \quad \therefore\ V(x) = \frac{dM}{dx}$$

$$V = -wx + C$$

$$at\ x = -l,\ V = wl \ \ \therefore\ C = 0$$

$$V = -wx$$

Example : Beam with elasto-plastic behavior



$$V = -wx + C$$

$$at\ x = -l,\ V = wl \quad \therefore\ C = 0$$

$$V = -wx$$

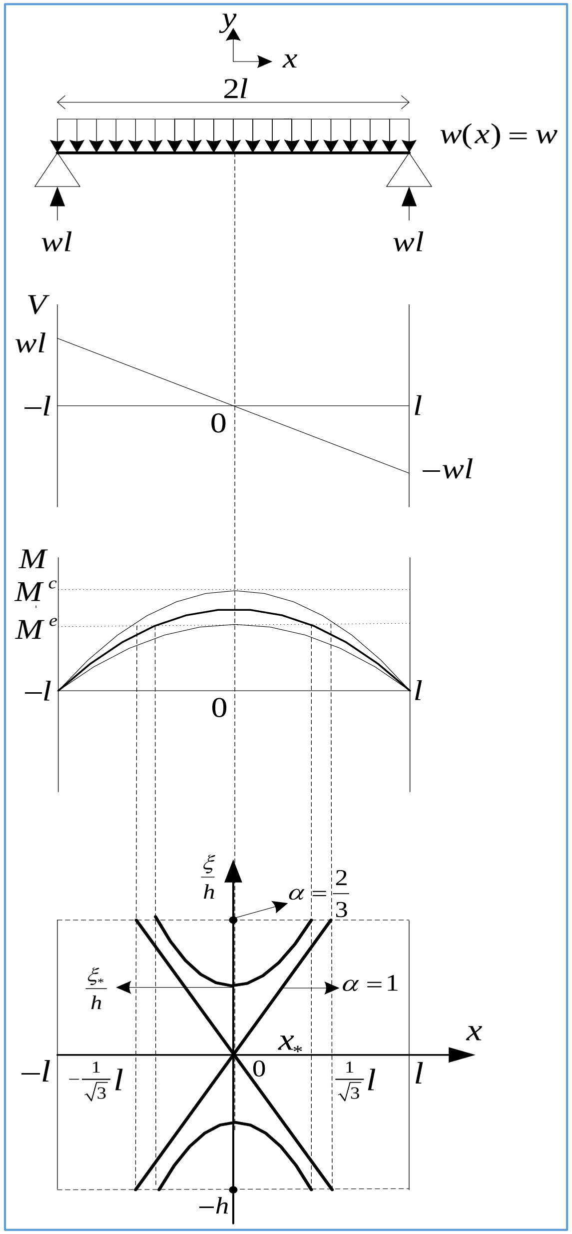
$$M = \int V dx + C_2$$

$$= -\frac{w_0}{2} x^2 + C_2$$

$$at\ x = \pm \ell,\ M = 0 \quad \therefore C_2 = \frac{w_0}{2} \ell^2$$

$$M = \frac{w_0}{2} (\ell^2 - x^2)$$

Example : Beam with elasto-plastic behavior



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$$V = -wx$$

$$M = \int V dx + C_2$$

$$= -\frac{w_0}{2} x^2 + C_2$$

$$at\ \ x = 0,$$

$$M_e = \frac{w_e}{2}\ell^2 = \frac{2}{3} bYh^2,\ therefore,\ w_e = \frac{4}{3}\frac{bYh^2}{\ell^2}$$

$$M_c = \frac{w_c}{2}\ell^2 = bYh^2,\ therefore,\ w_c = \frac{2bYh^2}{\ell^2} = \frac{3}{2} w_e$$

$$where$$

$$0 \le w_0 \le w_e\ \ :\ Elastic\ range$$

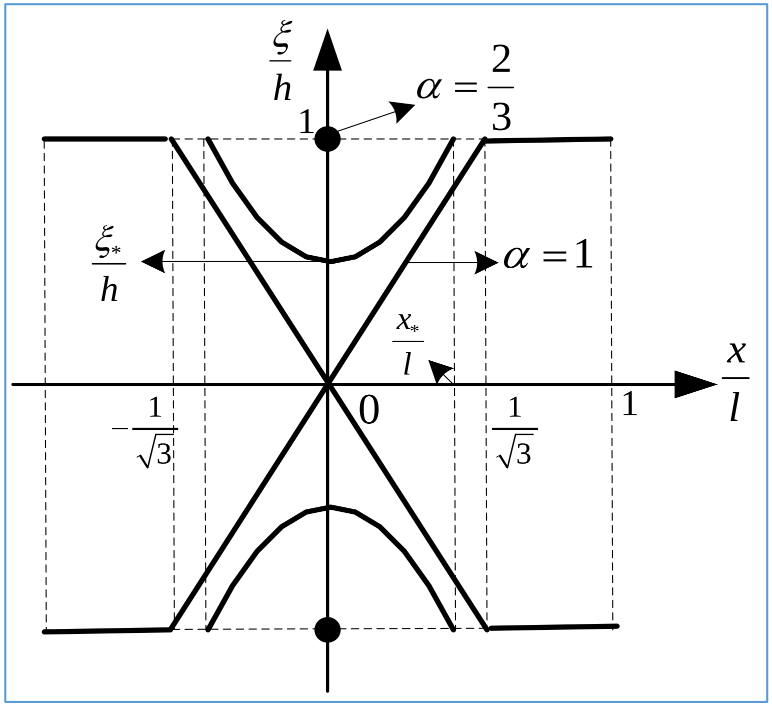
$$w_e \le w_0 \le w_c\ \ :\ Elasto\text{-}plastic\ range$$

Example : Beam with elasto-plastic behavior

- Find $\xi - x$ *relation*

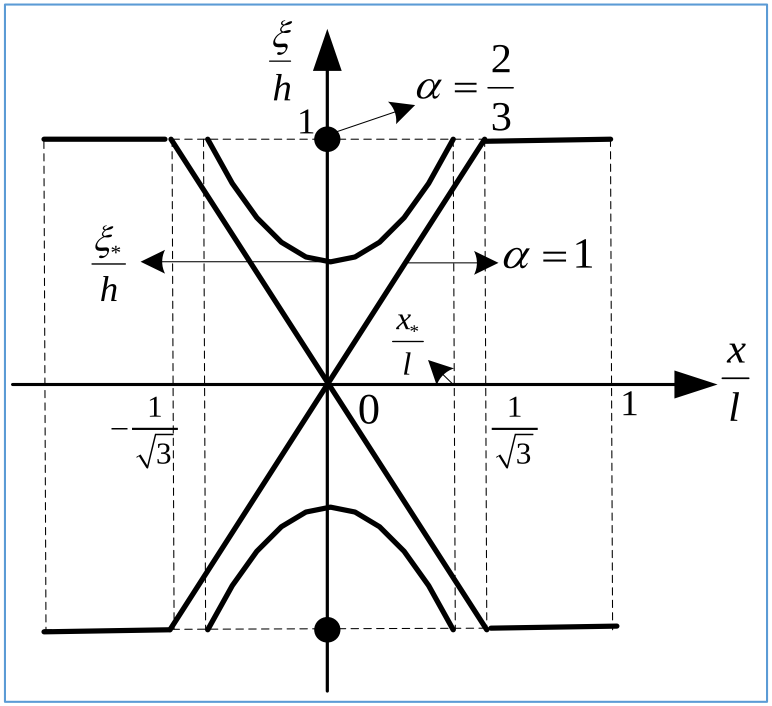
$$M = \frac{\alpha w_c}{2}(l^2 - x^2) \qquad \frac{2}{3} \le \alpha \le 1$$

$$M = bYh^2(1 - \frac{1}{3}(\frac{\xi}{h})^2) \quad \text{or} \quad bYh^2(1 - \frac{1}{3}(\frac{K_e}{K})^2)$$



Example : Beam with elasto-plastic behavior

- Find $\xi - x$ *relation*



$$M(x) = \frac{w}{2}(l^2 - x^2) = \frac{\alpha w_c}{2}(l^2 - x^2) = bYh^2(1 - \frac{1}{3}(\frac{\xi}{h})^2)$$

$$with \quad \frac{2}{3} \le \alpha \le 1 \quad and \quad w_c = \frac{2bYh^2}{l^2}$$

$$\alpha(1 - (\frac{x}{l})^2) = (1 - \frac{1}{3}(\frac{\xi}{h})^2)$$

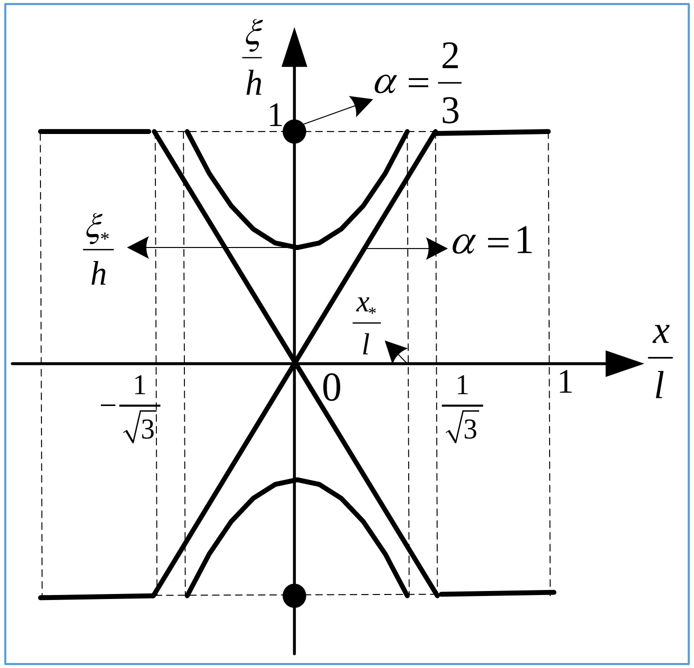
$$\therefore \quad \alpha - \alpha(\frac{x}{l})^2 = 1 - \frac{1}{3}(\frac{\xi}{h})^2$$

$$\frac{1}{3}(\frac{\xi}{h})^2 - \alpha(\frac{x}{l})^2 = 1 - \alpha$$

$$\therefore \quad \frac{1}{3(1-\alpha)}(\frac{\xi}{h})^2 - \frac{\alpha}{1-\alpha}(\frac{x}{l})^2 = 1 \qquad Hyperbola\ equation$$

Example : Beam with elasto-plastic behavior

- Find $\xi - x \ relation$



$$\therefore \ \frac{1}{3(1-\alpha)}\left(\frac{\xi}{h}\right)^2 - \frac{\alpha}{1-\alpha}\left(\frac{x}{l}\right)^2 = 1 \qquad \textit{Hyperbola equation}$$

$$i) \ \frac{x}{l} = 0, \quad \frac{\xi_*}{h} = \pm\sqrt{3(1-\alpha)}$$

$$ii) \ \frac{\xi}{h} = 1, \quad \left(\frac{x_*}{l}\right)^2 = \left(-1 + \frac{1}{3(1-\alpha)}\right)\left(\frac{1-\alpha}{\alpha}\right) = \frac{3\alpha - 2}{3\alpha}$$

$$\therefore \ \frac{x_*}{l} = \pm\sqrt{\frac{3\alpha-2}{3\alpha}}$$

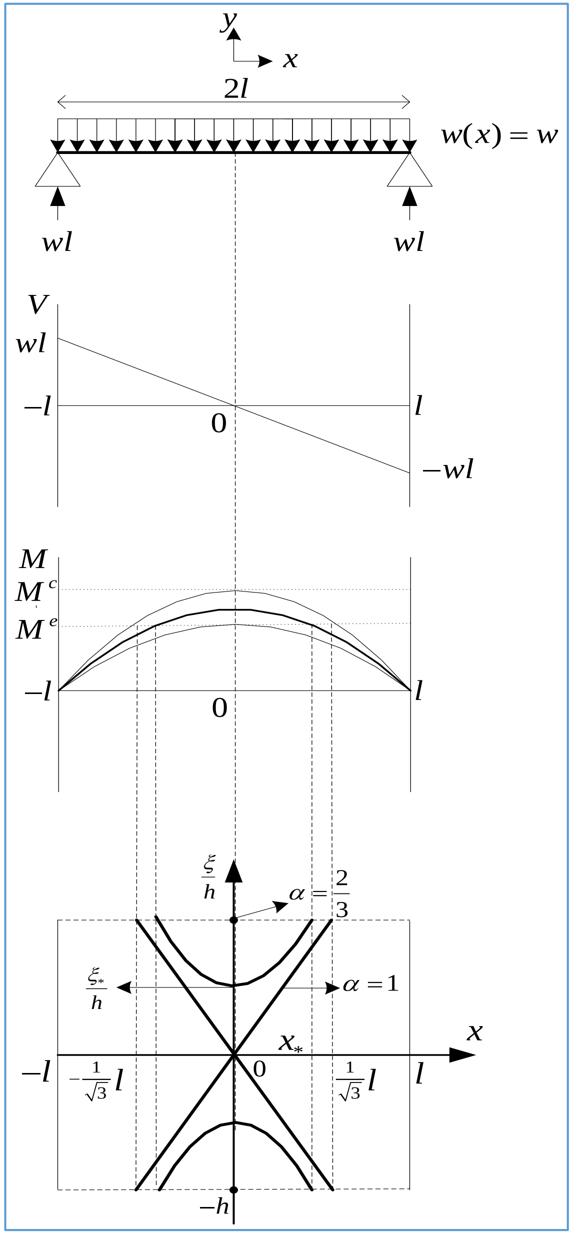
Further more

$$at \ \alpha = \frac{2}{3}, \ \left(\frac{\xi}{h}\right)^2 - 2\left(\frac{x}{l}\right)^2 = 1, \quad \frac{x_*}{l} = 0, \ \frac{\xi_*}{h} = \pm 1.0$$

$$at \ \alpha = 1, \ \left(\frac{\xi}{h}\right)^2 - 3\left(\frac{x}{l}\right)^2 = 0, \ \frac{\xi}{h} = \pm\sqrt{3}\,\frac{x}{l}, \quad \frac{x_*}{l} = \pm\sqrt{\frac{1}{3}} \approx \pm\frac{1}{1.732} \approx \pm 0.6$$

$$\begin{pmatrix} \textit{Plastic range is about } 60\% \\ \therefore \textit{the elastic property is important} \end{pmatrix} \frac{\xi_*}{h} = 0$$

Example : beam with elasto-plastic behavior



- Calculate M with K

Basic equations

$$w(x) = -\frac{dV}{dx}, \quad V(x) = \frac{dM}{dx}, \quad K \approx \frac{d^2v}{dx^2}$$

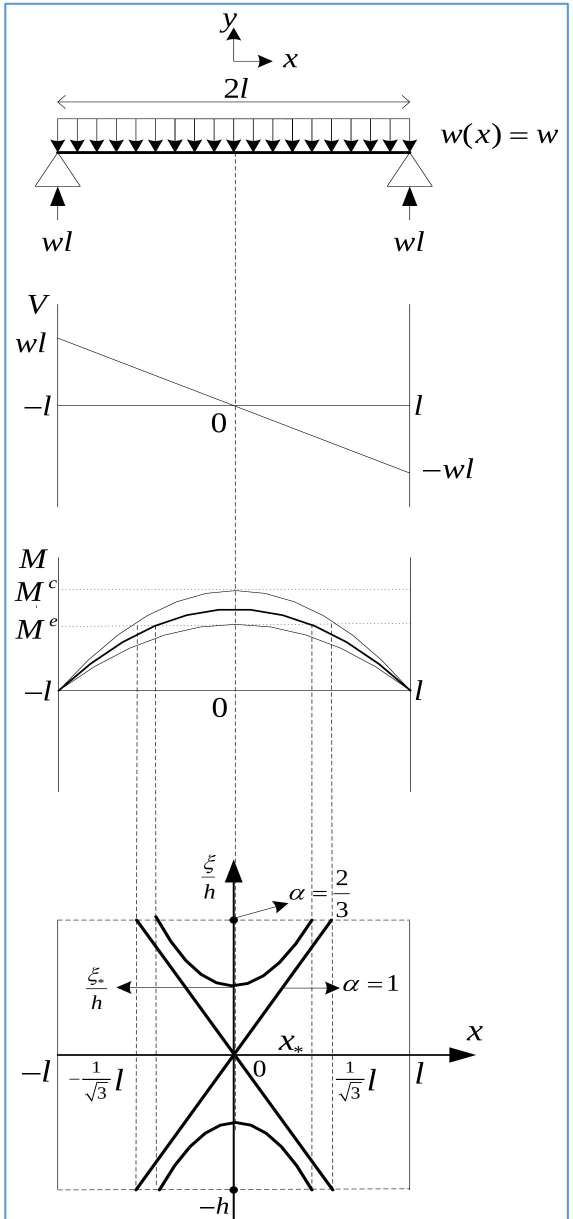
Elastic zone

$$\left|\frac{x}{\ell}\right| \geq \left|\frac{x_*}{\ell}\right| = \sqrt{\frac{3\alpha - 2}{3\alpha}}$$

$$M = \frac{\alpha w_c}{2}(\ell^2 - x^2) = EIK = EI\frac{d^2v}{dx^2}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} = \frac{\alpha w_c}{2EI}(\ell^2 - x^2) = \frac{3\alpha Y}{2Eh}\left(1 - \frac{x^2}{\ell^2}\right)$$

Example : beam with elasto-plastic behavior



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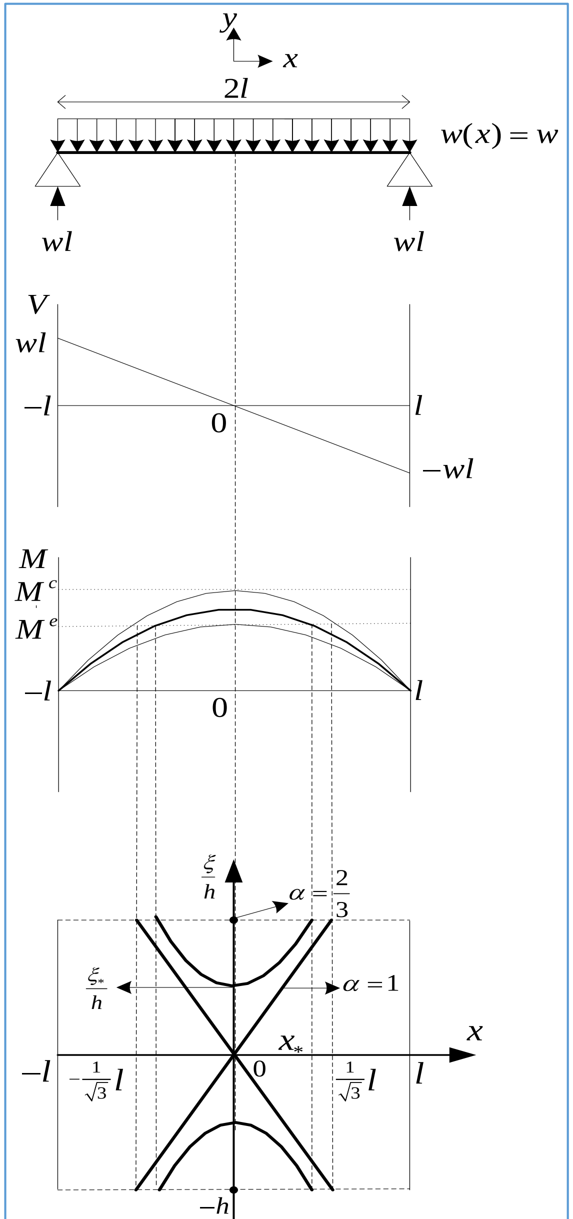
Plastic zone

$$\left|\frac{x}{\ell}\right| \le \left|\frac{x_*}{\ell}\right| = \sqrt{\frac{3\alpha - 2}{3\alpha}}$$

$$M = \frac{\alpha w_c}{2}(\ell^2 - x^2) = bYh^2\left(1 - \frac{1}{3}\left(\frac{K_e}{K}\right)^2\right)$$

$$\Rightarrow \quad \frac{d^2 v}{dx^2} = \frac{K_e}{\sqrt{3(1-\alpha(1-(\frac{x}{\ell})^2))}} = \frac{Y}{Eh\sqrt{3(1-\alpha(1-(\frac{x}{\ell})^2))}}$$

Example : beam with elasto-plastic behavior



- Boundary conditions

$$v(x=\pm\ell)=0,\quad \frac{dv}{dx}(x=0)=0$$

- Continuity of shear load

$$v\;and\;\frac{dv}{dx}\;\;are\;\;continuous$$

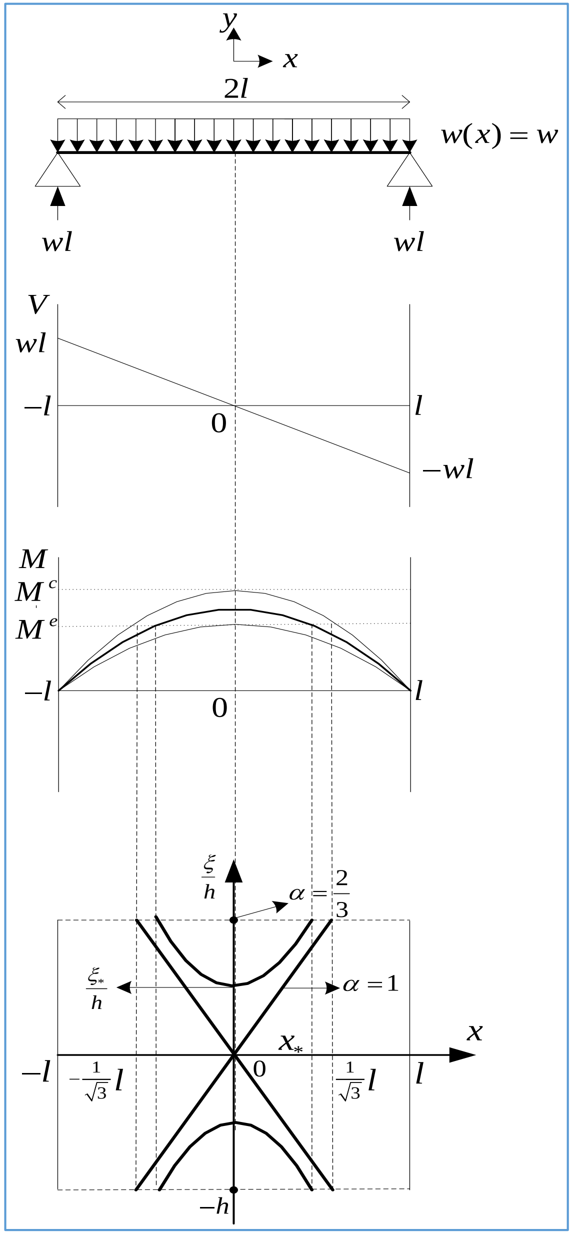
- Solutions

$$i)\;\frac{x}{\ell}\geq\frac{x_*}{\ell}=\sqrt{\frac{3\alpha-2}{3\alpha}}$$

$$v=\frac{Y}{Eh}\frac{\ell}{\sqrt{3\alpha}}\left\{\frac{3\alpha\sqrt{3\alpha}}{2\ell^3}\left(\frac{\ell^2}{2}x^2-\frac{1}{12}x^4-\frac{5}{12}\ell^4\right)\right.$$

$$\left.+\left(\ln\left(\frac{1+\sqrt{3\alpha-2}}{\sqrt{3(1-\alpha)}}\right)-\sqrt{3\alpha-2}\frac{(3\alpha+1)}{3}\right)(x-\ell)\right\}$$

Example : beam with elasto-plastic behavior



- Boundary conditions

$$v(x=\pm\ell)=0,\quad \frac{dv}{dx}(x=0)=0$$

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$$v\,and\quad \frac{dv}{dx}\quad are\quad continuous$$

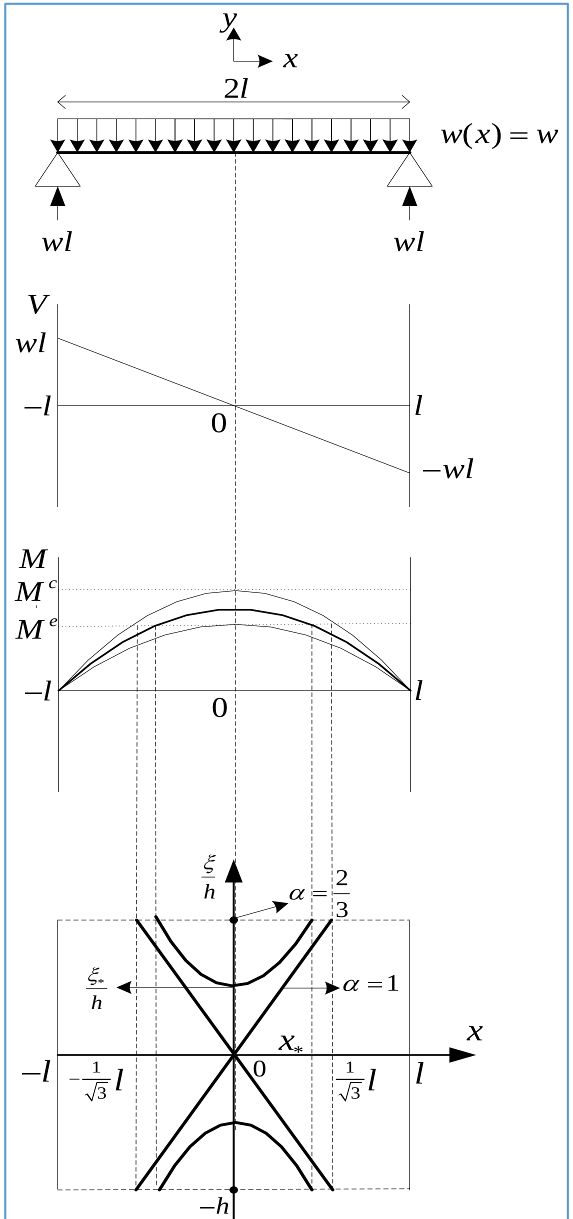
- Solutions

$$ii)\ 0\le\frac{x}{\ell}\le\frac{x_*}{\ell}=\sqrt{\frac{3\alpha-2}{3\alpha}}$$

$$v=\frac{Y}{Eh}\frac{\ell}{\sqrt{3\alpha}}\left\{x\ln\left(\frac{\sqrt{\alpha}x+\sqrt{\alpha(x^2-\ell^2)+\ell^2}}{\ell\sqrt{1-\alpha}}\right)-\frac{\sqrt{\alpha\left(x^2-\ell^2\right)+\ell^2}}{\sqrt{\alpha}}\right.$$

$$\left.-\ell\left(\ln\left(\frac{1+\sqrt{3\alpha-2}}{\sqrt{3\left(1-\alpha\right)}}\right)+\frac{\left(3\alpha+1\right)}{3}\sqrt{3\alpha-2}+\frac{1-2\alpha^2}{2\alpha}\sqrt{3\alpha}\right)\right\}$$

Example : beam with elasto-plastic behavior



- Boundary conditions

$$v(x=\pm\ell)=0,\quad \frac{dv}{dx}(x=0)=0$$

- Continuity of shear load

$$v \; and \; \frac{dv}{dx} \; are \; continuous$$

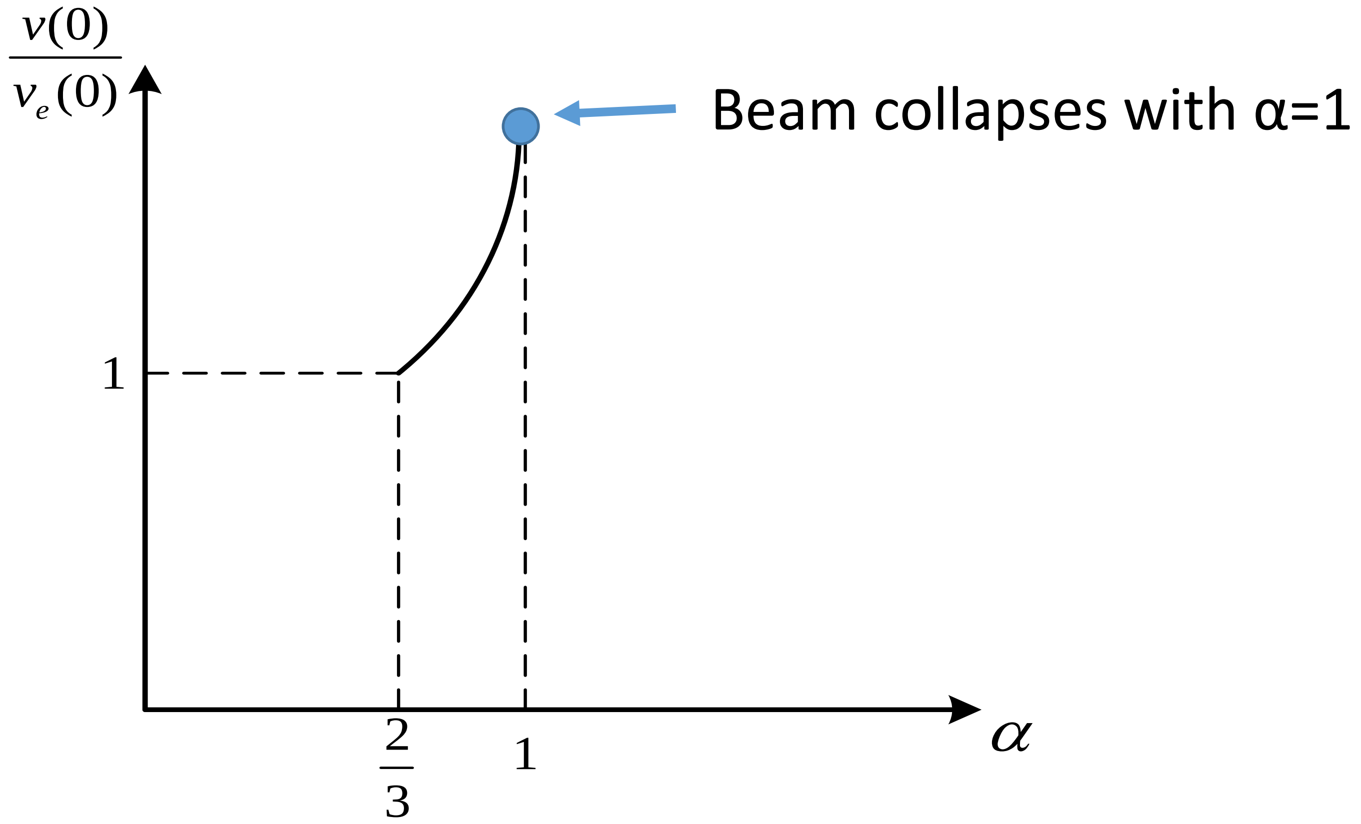
- Solutions

$$for \quad x=0 \qquad v=\frac{Y}{Eh}\frac{\ell^2}{\sqrt{3\alpha}}\left\{-\frac{\sqrt{1-\alpha}}{\sqrt{\alpha}}-\ln\left(\frac{1+\sqrt{3\alpha-2}}{\sqrt{3(1-\alpha)}}\right)+\frac{(3\alpha+1)}{3}\sqrt{3\alpha-2}+\frac{1-2\alpha^2}{2\alpha}\sqrt{3\alpha}\right\}$$

$$with \quad v_e\left(x=0,\alpha=2/3\right)=\frac{Y}{Eh}\frac{\ell^2}{\sqrt{3\alpha}}\left(-\frac{5\sqrt{2}}{12}\right)$$

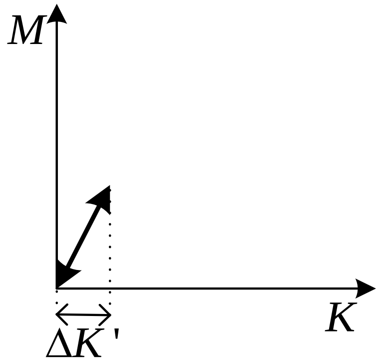
Example : beam with elasto-plastic behavior

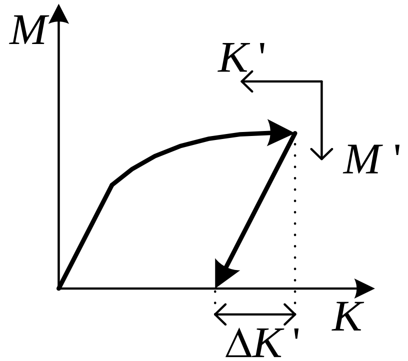
- Deflection at the center of the beam

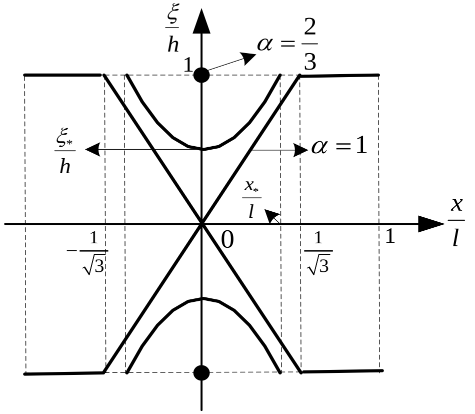


Example : unloading of beam(w(x)=0)

- Elastic recovery of curvatures during unloading







- The new curvature distribution after elastic unloading

$$\frac{d^2v}{dx^2} = K(x) - \Delta K'(x) = K(x) - \frac{M'}{EI} = K(x) - \frac{M(x)}{EI}$$

<u>Elastic zone</u> $\left|\frac{x}{\ell}\right| \geq \frac{x_*}{\ell} \quad K = \Delta K' \longrightarrow \frac{d^2v}{dx^2} = 0$

<u>Plastic zone</u> $0 \leq \left|\frac{x}{\ell}\right| \leq \frac{x_*}{\ell} \quad \Delta K' = \frac{M}{EI} = \frac{\alpha w_c}{2EI}(\ell^2 - x^2)$

$$\longrightarrow \frac{d^2v}{dx^2} = \frac{Y}{Eh\sqrt{3(1-\alpha(1-(\frac{x}{\ell})^2))}} - \frac{\alpha w_c}{2EI}(\ell^2 - x^2) = \frac{Y}{Eh}\left(\frac{1}{\sqrt{3(1-\alpha(1-(\frac{x}{\ell})^2))}} - \frac{3}{2}\alpha(1-\frac{x^2}{\ell^2})\right)$$

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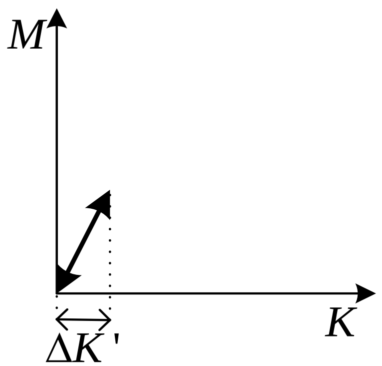

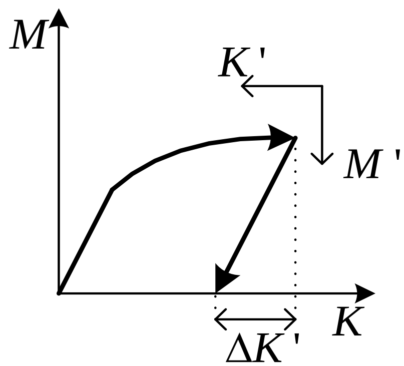

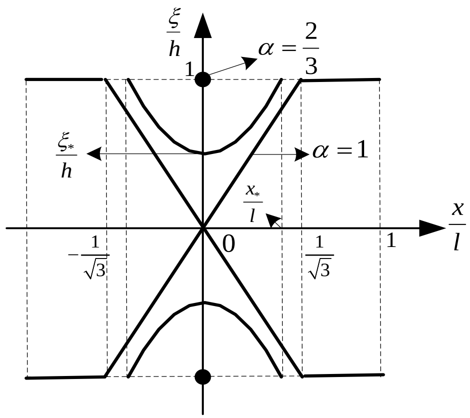

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<u>Plastic zone</u> $\quad 0 \le \left|\frac{x}{\ell}\right| \le \frac{x_*}{\ell} \qquad \Delta K' = \frac{M}{EI} = \frac{\alpha w_c}{2EI}(\ell^2 - x^2)$

$$\longrightarrow \quad \frac{d^2v}{dx^2} = \frac{Y}{Eh\sqrt{3(1-\alpha(1-(\frac{x}{\ell})^2))}} - \frac{\alpha w_c}{2EI}(\ell^2 - x^2) = \frac{Y}{Eh}\left(\frac{1}{\sqrt{3(1-\alpha(1-(\frac{x}{\ell})^2))}} - \frac{3}{2}\alpha(1-\frac{x^2}{\ell^2})\right)$$

Example : unloading of beam(w(x)=0)

- Schematic view of residual stress in plastic zone after unloading