4/12/2021, current / charge / drift

$$\frac{dn}{dt} + n(\vec{\nabla}\cdot\vec{u}) = S_0$$

$$mn\frac{d\vec{u}}{dt} = en(\vec{E} + \vec{u}\times\vec{B}) - \vec{\nabla}\cdot\overleftrightarrow{P} + \vec{R}_1 + \vec{S}_1 - m\vec{u}S_0$$

$$\frac{d}{dt}\left(\frac{3}{2}P\right) + \frac{5}{2}P(\vec{\nabla}\cdot\vec{u}) + \overleftrightarrow{\Pi}:\vec{\nabla}\vec{u} + \vec{\nabla}\cdot\vec{q} = R_2 - \vec{u}\cdot\vec{R}_1 + S_2 - \vec{u}\cdot\vec{S}_1 + \frac{1}{2}mu^2 S_0$$

☆ Two-fluids relations $s = (\alpha, \beta)$

(1) momentum conservation

$$\vec{R}_{1,\alpha} = \int d^3v_\alpha\, m\vec{v}_\alpha C_{\alpha\beta} \equiv \vec{F}_{\alpha\beta}$$

$$\sum_s \vec{R}_1 = \vec{F}_{\alpha\beta} + \vec{F}_{\beta\alpha} = \int d^3v\, m_\alpha\vec{v}_\alpha C_{\alpha\beta} + m_\beta\vec{v}_\beta C_{\beta\alpha} = 0$$

(2) energy conservation

$$R_{2,\alpha} = \int d^3v_\alpha \frac{1}{2}mv_\alpha^2 C_{\alpha\beta}$$

$$= \int d^3v_\alpha \frac{1}{2}m(\vec{v}_\alpha - \vec{u}_\alpha)^2 C_{\alpha\beta} + \int d^3v_\alpha\, m(\vec{u}_\alpha\cdot\vec{v}_\alpha)C_{\alpha\beta}$$

$$\equiv Q_{\alpha\beta} + \vec{u}_\alpha\cdot\vec{F}_{\alpha\beta}$$

$$\sum_s R_2 = Q_{\alpha\beta} + Q_{\beta\alpha} + \vec{u}_\alpha\cdot\vec{F}_{\alpha\beta} + \vec{u}_\beta\cdot\vec{F}_{\beta\alpha} = 0$$

(3) Current & Resistivity

- Krook collision approximation

$$\vec{F}_{\alpha\beta} = -m_\alpha n_\alpha \nu_{\alpha\beta}(\vec{u}_\alpha - \vec{u}_\beta)$$

↑ collision frequency

- For electron $\alpha = e$, D/T ions $\beta = i$

$$\vec{F}_{ei} = -m_e n_e \nu_{ei} (\vec{u}_e - \vec{u}_i)$$

- momentum conservation

$$\vec{F}_{ei} + \vec{F}_{ie} = 0 \rightarrow m_e n_e \nu_{ei} = m_i n_i \nu_{ie}$$

if $\underline{n_e = n_i}$ $\quad \nu_{ei} \gg \nu_{ie}$

- proportional to currents

$$\vec{j} = -en(\vec{u}_e - \vec{u}_i) \qquad \vec{F}_{ei} = \frac{m_e \nu_{ei}}{e} \vec{j}$$

- parallel momentum eq. for electron

$$\cancel{m_e \frac{dv_{\parallel e}}{dt}} = -e n_e E_\parallel + \hat{b} \cdot (\cancel{\vec{\nabla} p}) + \vec{F}_{ei,\parallel} + \cancel{S..}$$

small inertia $\qquad$ p - homogeneous

- $$E_\parallel = \frac{1}{e n_e} \vec{F}_{ei,\parallel} = \left(\frac{m_e \nu_{ei}}{e^2 n_e} \right) \vec{j}_\parallel \qquad \left(\frac{m_e \nu_{ei}}{e^2 n_e}\right) \equiv \eta$$

- plasma resistivity η.

$$\eta \sim \frac{10^{-3}}{T_e^{3/2}(eV)} \Omega m$$

10keV

$$\sim 10^{-9} \Omega m$$

$$\eta_{copper} \sim 10^{-8} \Omega m$$

- In general,

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E} \simeq \begin{pmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \cdot \vec{E}$$

$$\sigma_\parallel \simeq 2\sigma_\perp \quad \text{i.e } \eta_\perp \simeq 2\eta_\parallel$$

- In general, $\vec{j}$ can't be determined alone by $\vec{E}$

↗

need $\vec{\nabla} p$

(4) Ohmic heating

energy conservation

$$Q_{ei} + Q_{ie} = -\vec{u}_e \cdot \vec{F}_{ei} - \vec{u}_i \cdot \vec{F}_{ie}$$

$$= -\vec{F}_{ei} (\vec{u}_e - \vec{u}_i)$$

$$= -\frac{m_e \nu_{ei}}{e} \vec{j} \cdot \left(-\frac{\vec{j}}{en}\right) = \eta j^2$$

(5) charge density

Typically, we don't use $\epsilon_0 \vec{\nabla} \cdot \vec{E} = e(n_i - n_e)$ to determine $\vec{E} = -\vec{\nabla}\phi$.

Rather, parallel momentum balance gives a useful info for ϕ

steady state. $e n_e E_{\parallel} + \nabla_{\parallel} p_e \approx 0$

const. T_e. $e n_e \nabla_{\parallel} \phi \approx \nabla_{\parallel} p_e = T_e \nabla_{\parallel} n_e$

$$n_e \propto \exp(e\phi/T_e)$$

$$\phi \propto (T_e/e) \ln(n_e)$$

very good approximation.

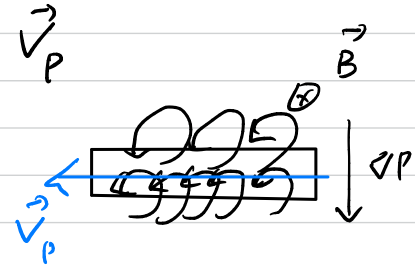
(b) Drift of fluid

In steady state, w/o source/exchange

$$\vec{B} \times (en\vec{E} + en\vec{u}\times\vec{B} - \vec{\nabla}p) \simeq 0$$

$$\vec{u}_\perp = \frac{\vec{E}\times\vec{B}}{B^2} + \frac{\vec{B}\times\vec{\nabla}p}{neB^2} \equiv \vec{V}_E + \vec{V}_p$$

$\vec{E}\times\vec{B}$ drift ; ↑ diamagnetic drift



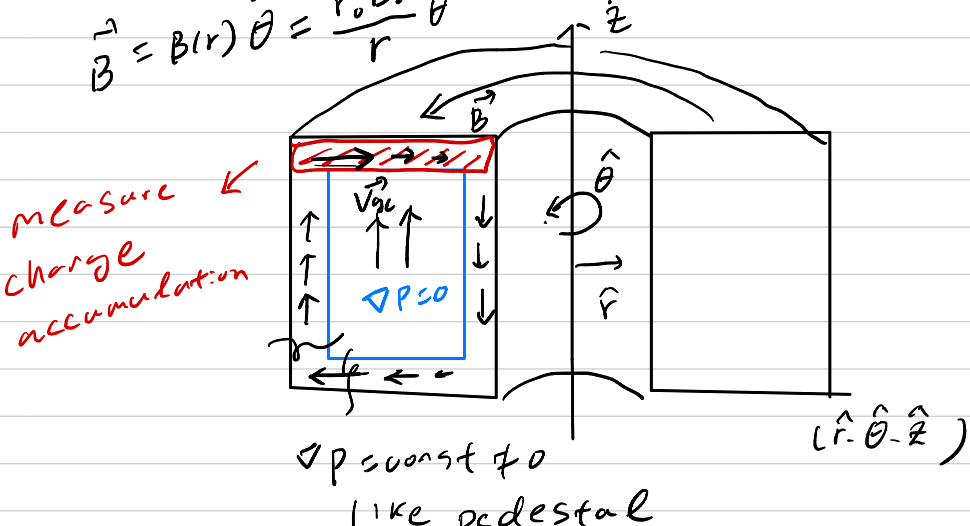
$$\boxed{\begin{array}{l}\text{Q) where are } \vec{\nabla}B\text{, Curvature drifts?} \\ \text{consistent with particle motion?} \\ \vec{V}_{gc} = \vec{V}_E + \vec{V}_{\nabla B} + \vec{V}_{curv}\end{array}}$$

* Assump
- $\vec{V}\times\vec{B} = 0$ (low-β assumption) magnetic field by external coils
- Maxwellian

(A) charge accumulation

A simplified example.

$$\vec{B} = B(r)\hat{\theta} = \frac{r_0 B_0}{r}\hat{\theta}$$



consider Ion drifts

– particle picture

$$\vec{v}_{gc} = \frac{2T}{e}\frac{\vec{B}\times\vec{\nabla}B}{B^3} = \frac{2T}{eBr}\hat{z}$$

$$\frac{d\sigma_{s,gc}}{dt} = en\vec{V}_{gc} = \frac{2P}{Br} = \frac{2P}{r_0 B_0}$$

– Fluid picture $\vec{V}_p = \frac{\vec{B}\times\vec{\nabla}P}{enB^2}$

$$\vec{j}_p = j_p \hat{r} = env_p \hat{r} = en\frac{B\hat{\theta}\times(-|\nabla P|)\hat{z}}{enB^2} = \frac{-|\nabla P|}{B}\hat{r}$$

$$= -\frac{r|\nabla P|}{r_0 B_0}\hat{r}$$

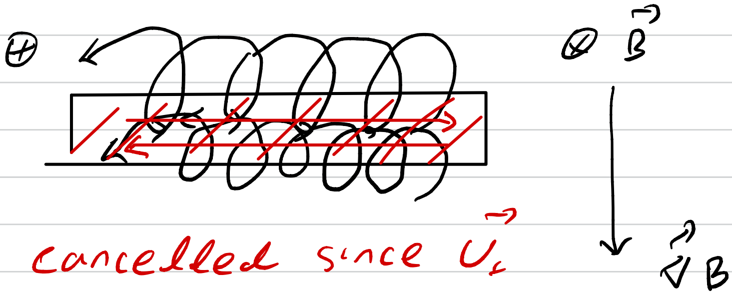
$$\frac{dP}{dt} = -\vec{\nabla}\cdot\vec{j}_p = \frac{1}{r}\frac{d}{dr}\left(r\cdot\frac{r|\nabla P|}{r_0 B_0}\right) = \frac{2|\nabla P|}{r_0 B_0}$$

$$\frac{d\sigma_{s,p}}{dt} = \int dz \frac{2\left|\frac{dP}{dz}\right|}{r_0 B_0} = \frac{2P}{r_0 B_0} = \frac{d\sigma_{s,gc}}{dt}$$

A bit more generally,

$\vec{\nabla}\cdot\vec{u}_\perp = \vec{\nabla}\cdot\vec{V}_{gc}$ (Try it for $\vec{\nabla}\times\vec{B}=0$)

(B) drift (or current) itself?



Goldston's book Chapter 7.4.

cancelled since $\vec{U}_\perp$ remains same

(Laboratory frame) (moving frame)

A bit more generally, $\vec{j}_\perp = \vec{j}_{gc} + \vec{\nabla}\times\vec{M}$ $\vec{M} = -\hat{b}\langle\mu\rangle = -\hat{b}\frac{P}{B}$

$\vec{u}_\perp = \vec{V}_{gc} + \frac{1}{en}\vec{\nabla}\times\vec{M}$

(1) $\vec{\nabla}\cdot\vec{u}_\perp = \vec{\nabla}\cdot\vec{V}_{gc}$

(2) $\vec{\nabla}\times\vec{M} = -\vec{\nabla}\times\left(\vec{B}\frac{P}{B^2}\right) = \frac{\vec{B}\times\vec{\nabla}P}{B^2} - 2P\frac{\vec{B}\times\vec{\nabla}B}{B^3}$

$$\vec{u}_\perp = \frac{\vec{E}\times\vec{B}}{B^2} + \frac{2P}{en}\cancel{\frac{\vec{B}\times\vec{\nabla}B}{B^3}} + \frac{1}{en}\frac{\vec{B}\times\vec{\nabla}P}{B^2} - \frac{2P}{en}\cancel{\frac{\vec{B}\times\vec{\nabla}B}{B^3}}$$

4/14/21 Single fluid MagnetoHydroDynamics (MHD)

Ion & electron

- plasma density $\quad n \simeq n_e \simeq n_i$
- Mass density $\quad \rho_m = \sum_s m_s n_s \simeq m_i n_i$
- Charge density $\quad \sigma = \sum_s e_s n_s \simeq 0$
- Mass velocity $\quad \vec{u} = \sum_s m_s n_s \vec{u}_s / \rho_m \simeq \vec{u}_i + \frac{m_e}{m_i} \vec{u}_e$
- current density $\quad \vec{j} = \sum_s e_s n_s \vec{u}_s \simeq en(\vec{u}_i - \vec{u}_e)$

$$\left\{ \begin{array}{l} \vec{u}_i \simeq \vec{u} + \frac{m_e}{m_i} \vec{j} \\ \vec{u}_e \simeq \vec{u} - \frac{\vec{j}}{en} \end{array} \right.$$

- Total pressure tensor $\quad \overleftrightarrow{P} = \sum_s m_s n_s \langle (\vec{v}_s - \vec{u})(\vec{v}_s - \vec{u}) \rangle$
- Total heat flux $\quad \vec{q} = \sum_s \frac{1}{2} m_s n_s \langle (\vec{v}_s - \vec{u})^2 (\vec{v}_s - \vec{u}) \rangle$

(Ignore source terms)

(1) continuity

$\sum_s m_s \times$ (continuity for each)

$$\longrightarrow \quad \frac{\partial \rho_m}{\partial t} + \vec{\nabla} \cdot (\rho_m \vec{u}) = 0$$

$\sum_s e_s \times$ (continuity for each)

$$\longrightarrow \quad \frac{\partial \sigma}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

(2) Equation of motion

$\sum_s$ (momentum conservation for each)

$$\longrightarrow \quad \rho_m \frac{d\vec{u}}{dt} = \sigma \vec{E} + \vec{j} \times \vec{B} - \vec{\nabla} \cdot \overleftrightarrow{P} + \cancel{\sum_s \vec{R}_{is}}$$

(3) Equation of energy

$\sum_s$ (energy conservation for each)

$$\rightarrow \quad \frac{d}{dt}\left(\frac{3}{2}p\right) + \frac{5}{2}p(\vec{\nabla}\cdot\vec{u}) + \overleftrightarrow{\pi} : \vec{\nabla}\vec{u} + \vec{\nabla}\cdot\vec{q}$$

$$= (\vec{j} - \sigma\vec{u})\cdot(\vec{E} + \vec{u}\times\vec{B}) + \cancel{\sum_s R_{2s}}$$

(4) Generalized Ohm's law

$\sum_s \frac{e_s}{m_s} \times$ (momentum conservation for each)

or electron equation of motion (two species)

$$\cancel{m_e \frac{d\vec{v_e}}{dt}} = -en_e\vec{E} - en_e\underline{\vec{u_e}}\times\vec{B} - \vec{\nabla}\cdot\overleftrightarrow{P_e} + en_e\eta\vec{j}$$

electron inertia

$$\vec{u} - \frac{\vec{j}}{en_e}$$

$$\vec{E} + \vec{u}\times\vec{B} = \eta\vec{j} + \frac{\vec{j}\times\vec{B} - \vec{\nabla}\cdot\overleftrightarrow{P_e}}{en_e} \leftarrow \text{Hall term}$$

\+ Maxwell equations

$$\begin{cases} \vec{\nabla}\times\vec{B} = \mu\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\cdot\vec{E} = \sigma/\epsilon_0 \end{cases}$$

Approximation for (low frequency phenomena
long wavelength
high collisionality

a : scale of interest $\qquad \sigma \sim \epsilon_0 E/a$

$$\omega = \frac{\partial}{\partial t} \sim \frac{V_{thi}}{a}, \quad \vec{u} \sim \frac{\vec{E}\times\vec{B}}{B^2} \sim \frac{E}{B}, \quad \vec{j} \sim \frac{\vec{\nabla} p}{B} \sim \frac{\rho_m V_{thi}^2}{Ba}$$

(A) small gyroradius $\frac{r_L}{a} \ll 1$

(B) high collisionality $\frac{\omega}{\nu_{ii}} \sim \frac{V_{thi}}{\nu_{ii} a} \ll 1$

(C) low resistivity ?

(0) Maxwell

Goldston Friedberg $u \sim V_{thi}$

$$\frac{\partial \vec{E}/\partial t}{c^2 \vec{\nabla}\times\vec{B}} \sim \frac{E V_{thi}}{B c^2} \sim \frac{u V_{thi}}{c^2} \ll 1$$

$$\rightarrow \vec{\nabla}\times\vec{B} = \mu_0 \vec{j} + \cancel{\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}$$

(1) continuity

$$\frac{\partial \sigma/\partial t}{\vec{\nabla}\cdot\vec{j}} \sim \frac{\epsilon_0 E V_{thi}/\cancel{a^2}}{\rho_m V_{thi}^2/B\cancel{a^2}} \sim \frac{\epsilon_0 B^2}{\rho_m}\frac{u}{V_{thi}} \sim \frac{V_A^2}{c^2}\frac{u}{V_{thi}} \ll 1$$

$$* \quad \frac{\epsilon_0 B^2}{\rho_m} \sim \frac{1}{c^2}\frac{B^2}{\mu_0 m n} \sim \frac{V_A^2}{c^2} \ll 1$$

$$\rightarrow \cancel{\frac{\partial \sigma}{\partial t}} + \vec{\nabla}\cdot\vec{j} = 0$$

Many instabilities

Incompressible,

$$\star \quad \vec{\nabla}\cdot\vec{u} = 0 \leftarrow$$

$$\left(\frac{d\rho_m}{dt} + \rho_m \cancel{(\vec{\nabla}\cdot\vec{u})} \right) = 0 \leftarrow \frac{\partial \rho_m}{\partial t} + \vec{\nabla}\cdot(\rho_m \vec{u}) = 0$$

(2) momentum

$$\frac{\sigma \vec{E}}{\vec{j}\times\vec{B}} \sim \frac{\epsilon_0 E^2/a}{\rho_m V_{thi}^2/a} \sim \frac{\epsilon_0 B^2}{\rho_m}\frac{u^2}{V_{thi}^2} \sim \frac{V_A^2}{c^2}\left(\frac{u^2}{V_{thi}^2}\right) \ll 1$$

$$\frac{\rho_m\, \partial\vec{u}/\partial t}{\vec{j}\times\vec{B}} \sim \frac{\rho_m u V_{thi}/a}{\rho_m V_{thi}^2/a} \sim \frac{u}{V_{thi}}$$

$$\frac{\rho_m\, \vec{u}\cdot\vec{\nabla}\vec{u}}{\vec{j}\times\vec{B}} \sim \frac{u^2}{V_{thi}^2}$$

$\mathcal{O}(1)$ MHD-ordering

$\mathcal{O}\left(\frac{r_L}{a}\right)^2$ drift-MHD ordering

"Hazeltine"

$$\frac{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}{\vec{\nabla}P} \sim \frac{\mu\nabla^2 u}{\vec{\nabla}P} \sim \frac{\mu u}{Pa} \sim \frac{u}{\nu_{ii}a} \sim \left(\frac{u}{V_{thi}}\right)\left(\frac{V_{thi}}{\nu_{ii}a}\right) \ll 1$$

$\mu \sim$ momentum viscosity $\sim P/\nu_{ii}$

$$\rho_m\frac{d\vec{u}}{dt} = \cancel{\sigma\vec{E}} + \vec{j}\times\vec{B} - \vec{\nabla}P - \cancel{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}$$

(3) Energy

$$\frac{\vec{\nabla}\cdot\vec{q}}{(3/2)\,\partial P/\partial t} \sim \frac{K_{\parallel}\nabla_{\parallel}^2 T}{PV_{thi}/a} \sim \frac{\cancel{P}T/\nu_{ei}a^2 m_e}{\cancel{P}V_{thi}/a} \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\left(\frac{V_{thi}}{\nu_{ii}a}\right) \ll 1$$

$m_i V_{thi}^2$

$K_{\parallel} \sim P/m_e\nu_{ei}$, $\nu_{ei} \sim \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}}\nu_{ii}$

$$\frac{d}{dt}\left(\frac{3}{2}P\right) + \frac{5}{2}P(\vec{\nabla}\cdot\vec{u}) + \cancel{\overleftrightarrow{\Pi}:\vec{\nabla}\vec{u} + \vec{\nabla}\cdot\vec{q}}$$
$$= (\vec{j} - \cancel{\sigma\vec{u}})(\vec{E} + \vec{u}\times\vec{B})$$
$$= \eta j^2$$

(4) Ohm's law

$$\frac{\eta \vec{j}}{\vec{u}\times\vec{B}} \sim \frac{m_e \nu_{ei}\, m_i^2 n V_{thi}^2\, a}{m_i n e e^2 B^2 a^2 u} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \left(\frac{r_L}{a}\right)^2 \left(\frac{V_{thi}}{\nu_{ii} a}\right)^{-1} \left(\frac{V_{thi}}{u}\right)$$

since $\vec{j}\times\vec{B} - \vec{\nabla} P_e \sim \vec{\nabla} P_i$

$$\frac{\vec{\nabla} P_i / en}{\vec{u}\times\vec{B}} \sim \frac{m_i V_{thi}^2/e}{uBa} \sim \left(\frac{r_L}{a}\right)\left(\frac{V_{thi}}{u}\right)$$

$\mathcal{O}\left(\frac{r_L}{a}\right)$ MHD-ordering

$\mathcal{O}(1)$ drift MHD

$u \sim \mathcal{O}\left(\frac{r_L}{a}\right) \cdot V_{thi}$

$$\vec{E} + \vec{u}\times\vec{B} = \eta \vec{j} + \cancel{\frac{\vec{j}\times\vec{B} - \vec{\nabla} P_e}{e n_e}}$$

→ resistive MHD

For parallel motion, $j_\parallel \sim \frac{B}{\mu_0 a}$

$$\frac{\eta \vec{j}}{\vec{u}\times\vec{B}} \sim \frac{\eta}{\mu_0 u a} \sim \frac{1}{R_M}$$

(Magnetic Reynolds / Landquist) number

$\vec{E} + \vec{u}\times\vec{B} = 0$ → ideal MHD

* MHD equations

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{d\rho_m}{dt} + \rho_m (\vec{\nabla} \cdot \vec{u}) = 0$$

$$\rho_m \frac{d\vec{u}}{dt} = \vec{j} \times \vec{B} - \vec{\nabla} p$$

$$\frac{dp}{dt} + \frac{5}{3} p (\vec{\nabla} \cdot \vec{u}) = 0$$

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j}$$

14 equations

for $(\vec{B}, \vec{E}, \vec{u}, \vec{j}, \rho_m, p)$

by eliminating $\vec{E}$, $\vec{j}$

$$\frac{d\rho_m}{dt} + \rho_m (\vec{\nabla} \cdot \vec{u}) = 0$$

$$\frac{dp}{dt} + \frac{5}{3} p (\vec{\nabla} \cdot \vec{u}) = 0$$

$$\rho_m \frac{d\vec{u}}{dt} + \vec{\nabla}\left(p + B^2/2\mu_0\right) = \vec{B} \cdot \vec{\nabla} \vec{B} / \mu_0$$

$$\frac{d\vec{B}}{dt} - \vec{B} \cdot \vec{\nabla} \vec{u} + \vec{B}(\vec{\nabla} \cdot \vec{u}) = \frac{\eta}{\mu_0} \nabla^2 \vec{B}$$

8 equations

for $(\vec{u}, \vec{B}, \rho_m, p)$

simple geometry
$$\begin{cases} \vec{u} = \dfrac{\vec{B} \times \vec{\nabla} \phi}{B^2} + u_\parallel \hat{b} \\ \vec{B} = b_z \hat{b} + \hat{b} \times \vec{\nabla} \psi \end{cases}$$

$(\psi, \phi, b_z, u_\parallel)$ ρ_m, p

Four field equations.