

Spline interpolation

노트 제목

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$$\frac{\alpha_{i-1}}{6} g''_{i-1} + \frac{\alpha_{i-1} + \alpha_i}{3} g''_i + \frac{\alpha_i}{6} g''_{i+1} = \frac{y_{i+1} - y_i}{\alpha_i} + \frac{y_{i-1} - y_i}{\alpha_{i-1}} \quad i=1, 2, \dots, n-1$$

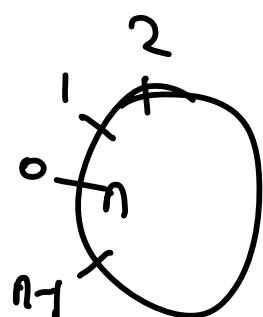
$g''_0 \dots g''_n$ $n+1$ unknowns
↑
 $A-1$ egs.

We need two more conditions. $\rightarrow g''_0$ & g''_n

$$\begin{cases} g''_0 = \lambda g''_1 \\ 0 \leq \lambda \leq 1 \end{cases}$$

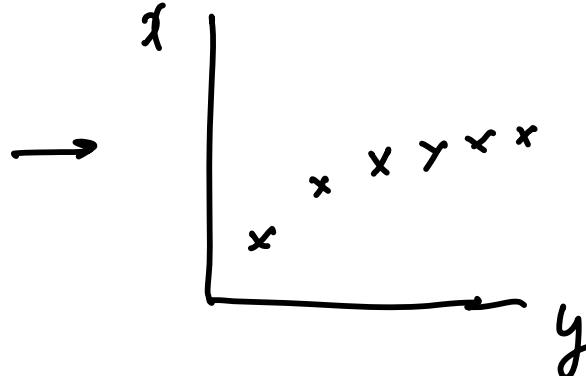
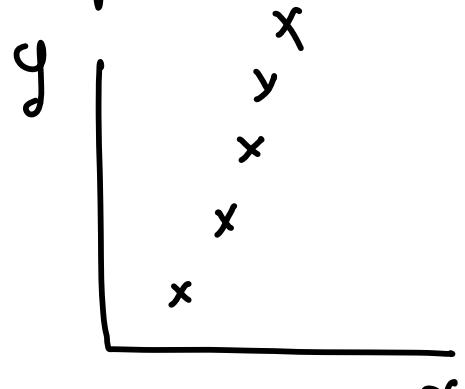
$$g''_n = \lambda g''_{n-1}$$

$$\begin{cases} g''_0 = g''_n & \text{periodic b.c.} \\ g''_n = g''_0 & \text{b.c.} \end{cases}$$



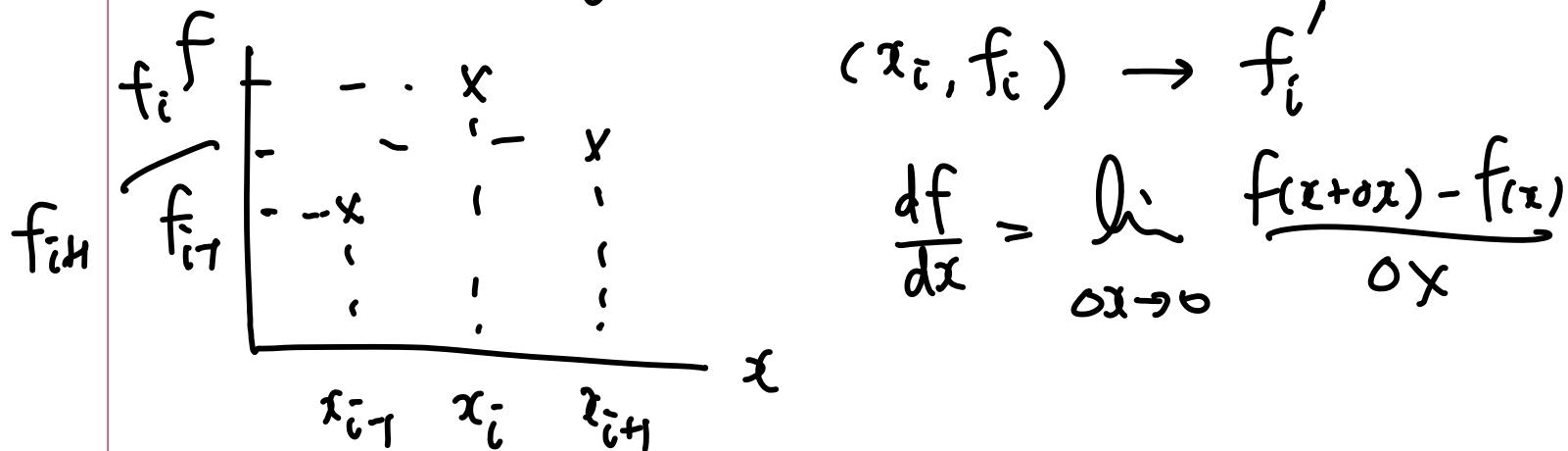
$$\left(\begin{array}{cccccc|c} * & * & 0 & \dots & 0 & * & g''_0 \\ & \backslash & & & & / & g''_1 \\ 0 & . & 0 & * & * & x & \vdots \\ & & & \backslash & & / & g''_m \end{array} \right)$$

- Spline has difficulties with large slopes.



Ch.2 Numerical differentiation - finite difference

2.1 Construction of difference formulae using Taylor series



Finite difference using Taylor series expansion

$$f(x_j + ox) = f(x_j) + ox f'(x_j) + \frac{1}{2} ox^2 f''(x_j) + \frac{1}{6} ox^3 f'''(x_j) + \dots$$

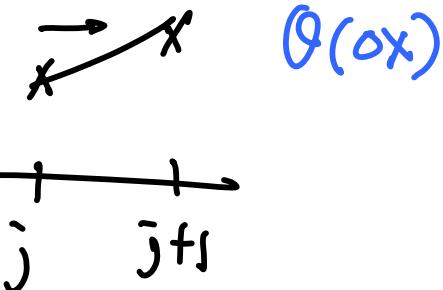
$$\rightarrow \frac{f(x_j + ox) - f(x_j)}{ox} = \underbrace{f'(x_j)}_{\text{circled}} + \frac{1}{2} ox f''(x_j) + \frac{1}{6} ox^2 f'''(x_j) + \dots$$

$$f'(x_j) = \frac{f(x_{j+1}) - f(x_j)}{\Delta x} - \frac{1}{2} \Delta x f''(x_j) - \frac{1}{6} \Delta x^2 f'''(x_j) - \dots$$

1st-order accurate

$$f'_j = \frac{f_{j+1} - f_j}{\Delta x} + O(\Delta x)$$

leading-error truncation error



forward 1st-order FD method

Similarly,

$$f'_j = \frac{f_j - f_{j-1}}{\Delta x} + O(\Delta x)$$

backward 1st-order FD method

$$f_{j+1} = f_j + \dots$$

$\underline{+ f_{j-1} = f_j + \dots}$

→ $f'_j = \frac{f_{j+1} - f_{j-1}}{2\alpha x} - \frac{1}{6}\alpha x^2 f''_j + \dots$

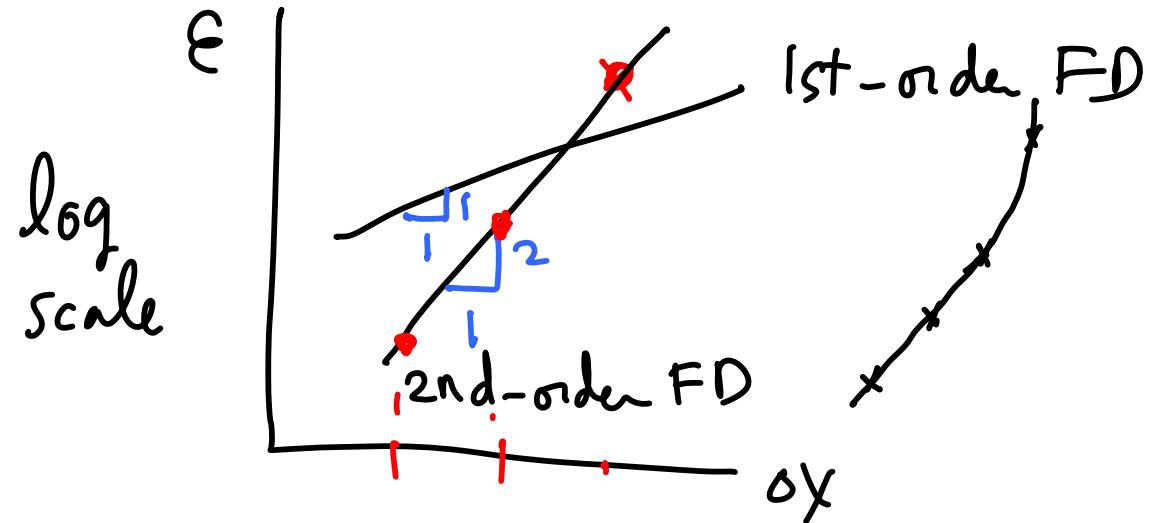
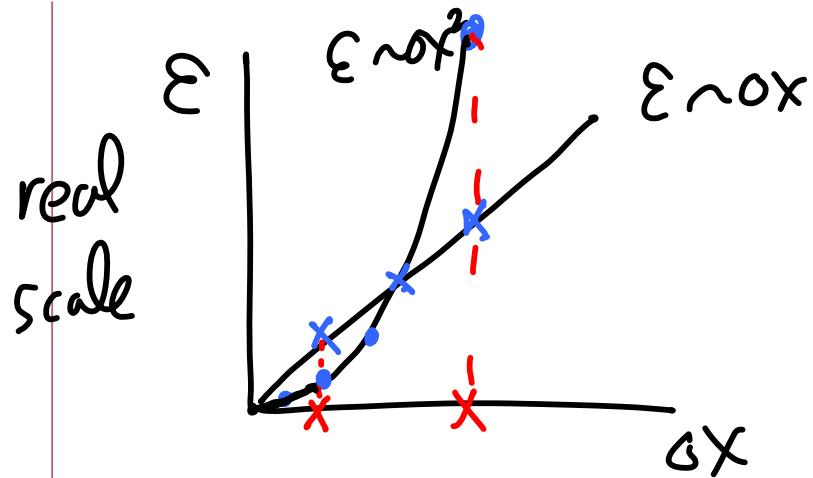
$\Theta(\alpha x^e)$

2nd-order central

FD method (CFD2)

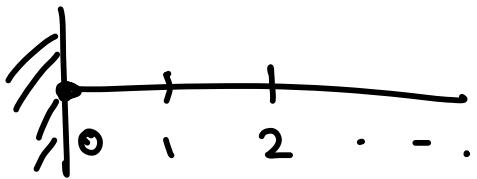
$$\left\{ \begin{array}{l} f'_j = \frac{f_{j+1} - f_j}{\alpha x} - \frac{1}{2}\alpha x f''_j + \dots \rightarrow \varepsilon \sim \alpha x \rightarrow \ln \varepsilon \sim \ln \alpha x \\ f'_j = \frac{f_j - f_{j-1}}{\alpha x} + \frac{1}{2}\alpha x f''_j + \dots \\ f'_j = \frac{f_{j+1} - f_{j-1}}{2\alpha x} - \frac{1}{6}\alpha x^2 f'''_j + \dots \rightarrow \varepsilon \sim \alpha x^2 \rightarrow \ln \varepsilon \sim 2 \ln \alpha x \end{array} \right.$$

error = ε



$$f'_j = \underbrace{\frac{f_{j+2} - 8f_{j+1} + 8f_{j+0} - f_{j-2}}{12 \delta x}}_{\text{12 } \delta x} + O(\delta x^4) \quad \text{CD4}$$

↑
more grid pts. → higher accuracy
problem → near/on boundary
how to obtain this formula?



lower-order accurate formula

e.g. @ $j=1$, CD 2

f'_j using f_0, f_1, f_2, f_3

2.2 A general technique for construction of FD schemes

- f'_j using function values at $j, j+1, j+2$

Q : what is the most accurate formula?



$$f'_j + a_0 f_j + a_1 f_{j+1} + a_2 f_{j+2} = O(h^2) \quad (h=0x)$$

Taylor table

| | f_j | f_j' | f_j'' | f_j''' | \dots |
|-----------------|---------------|------------------|-------------------------------|-------------------------------|---------|
| f_j' | 0 | 1 | 0 | 0 | - - |
| $a_0 f_j$ | a_0 | 0 | 0 | 0 | - - . |
| $a_1 f_{j+1}$ | $a_1 \cdot 1$ | $a_1 \cdot h$ | $a_1 \cdot \frac{1}{2}h^2$ | $a_1 \cdot \frac{1}{6}h^3$ | - - . |
| $+ a_2 f_{j+2}$ | $a_2 \cdot 1$ | $a_2 \cdot (2h)$ | $a_2 \cdot \frac{1}{2}(2h)^2$ | $a_2 \cdot \frac{1}{6}(2h)^3$ | - - . |

$$\rightarrow f_j' + a_0 f_j + a_1 f_{j+1} + a_2 f_{j+2} = (a_0 + a_1 + a_2) f_j + (1 + a_1 h + 2a_2 h) f_j' + \left(\frac{1}{2} a_1 h^2 + 2a_2 h^2\right) f_j'' + \left(\frac{1}{6} a_1 h^3 + \frac{4}{3} a_2 h^3\right) f_j''' + \dots$$

Set as many lower-order coeffs. to zero as possible

$$a_0 + a_1 + a_2 = 0$$

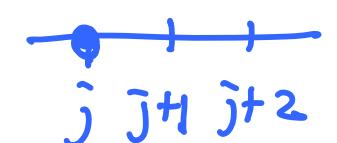
$$1 + a_1 h + 2a_2 h = 0 \rightarrow a_0 = \frac{3}{2h}, a_1 = -\frac{2}{h}, a_2 = \frac{1}{2h}$$

$$\frac{1}{2} a_1 h^2 + 2a_2 h^2 = 0$$

Substitute these into formula.

$$\rightarrow f_j' + \frac{3}{2h} f_j - \frac{2}{h} f_{j+1} + \frac{1}{2h} f_{j+2} = \frac{1}{3} h^2 f_j''' + \dots$$

$$\rightarrow f_j' = \frac{-3f_j + 4f_{j+1} - f_{j+2}}{2h} + \frac{1}{3} h^2 f_j''' + \dots$$


j j+1 j+2

2nd-order FD

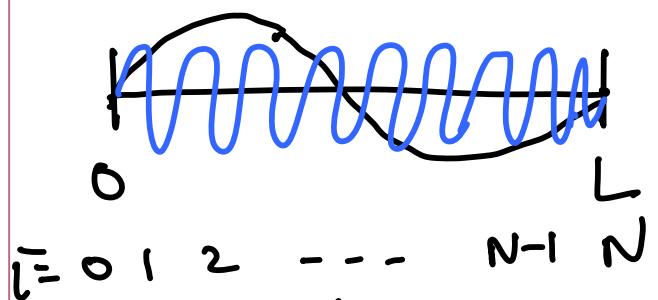
one-side
difference

2.3

An alternative measure for the accuracy of FD

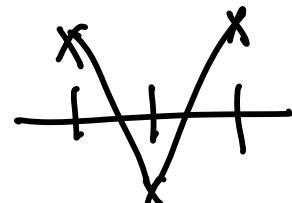
* modified wavenumber approach for measuring the ~~order~~
~~of~~ accuracy.

• $f(x) = e^{ikx} = \cos kx + i \sin kx$: pure harmonic ft. of period L.
 k : wavenumber



uniform mesh $\Delta x = h = L/N$: grid spacing

$$\Delta k \cdot L = 2\pi \rightarrow \Delta k = \frac{2\pi}{L} : \text{smallest wavenumber}$$



$$K \cdot (2\Delta x) = 2\pi \rightarrow K = \frac{\pi}{\Delta x} = \frac{\pi}{h} = \frac{\pi N}{L} : \text{largest wavenumber}$$

$$f(x) = e^{ikx} \rightarrow f' = ike^{ikx} = ikf \text{ exact sol.}$$

Q: how accurately CD2 computes the derivatives of f for different values of k ? o C

$$x_j = h \cdot j = \frac{L}{N} j . \quad j=0, 1, 2, \dots, N$$

On this grid, e^{ikx} ranges from a constant for $k=0$ to a highly oscillatory ft. of period equal to mesh widths ($2\alpha x$) for $k = \pi N/L (= \pi/h)$.

$$f'_j = \frac{f_{j+1} - f_{j-1}}{2h} \text{ (CD2)}$$

$$= \frac{1}{2h} (e^{ikx_j''} - e^{ikx_{j'}}) = \frac{1}{2h} e^{\bar{ik}x_j} (e^{ikh} - e^{-ikh})$$

$$= i \cdot \frac{\sin kh}{h} \cdot e^{\bar{ik}x_j} = i \frac{\sin kh}{h} f_j \quad f = ikf_j$$

\bar{k}' : modified wavenumber