Engineering Mathematics 2

Lecture 7

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Previously,

- complex numbers
- polar form
- complex fns and analyticity
- Cauchy-Riemann eqns

13.5 Exponential function

- $e^z = e^{x+iy} = e^x e^{iy}$

$= e^x(\cos y + i\sin y)$

- e^z is analytic: $e^z = u + iv$ 'entire'

$u = e^x\cos y$, $v = e^x\sin y$

$u_x = e^x\cos y = v_y$

$u_y = -e^x\sin y = -v_x$

$$(e^z)' = \frac{\partial}{\partial x}\left[e^x(\cos y + i\sin y)\right]$$

$$= e^x(\cos y + i\sin y)$$

$$= e^z$$

$$(e^z)' = \frac{\partial}{i\partial y}$$

Trigonometric function

Euler's formula : $e^{\pm ix} = \cos x \pm i \sin x$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Similarly,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

· $\cos z$ & $\sin z$ are entire.

· $$(\cos z)' = \left(\frac{e^{iz}+e^{-iz}}{2}\right)'$$

$$= \frac{i}{i}\,\frac{ie^{iz}-ie^{-iz}}{2} = -\,\frac{e^{iz}-e^{-iz}}{2i}$$

$$= -\sin z$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{-y+ix} - e^{y-ix}}{2i}$$

$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$

$$= \sin x \frac{e^{y}+e^{-y}}{2} + \cos x \frac{-(e^{y}-e^{-y})}{2i}$$

$$\cos^2 z + \sin^2 z$$

$$= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$$

$$= \frac{\cancel{e^{2iz}}+2+\cancel{e^{-2iz}}}{4} + \frac{\cancel{e^{2iz}}+2+\cancel{e^{-2iz}}}{4}$$

$$= 1$$

Hyperbolic functions

- $\cosh z = \dfrac{e^{z}+e^{-z}}{2}$
- $\sinh z = \dfrac{e^{z}-e^{-z}}{2}$
- $\cos iz = \cosh z \quad , \quad \cosh iz = \cos z$
- $\sin iz = i \sinh z \quad , \quad \sinh iz = \sin z$

Logarithm

$w = \ln z \quad , \quad e^{w} = z$

$w = u + iv \quad , \quad z = re^{i\theta}$

$e^{u+iv} = re^{i\theta}$

$e^{u}e^{iv} = re^{i\theta}$

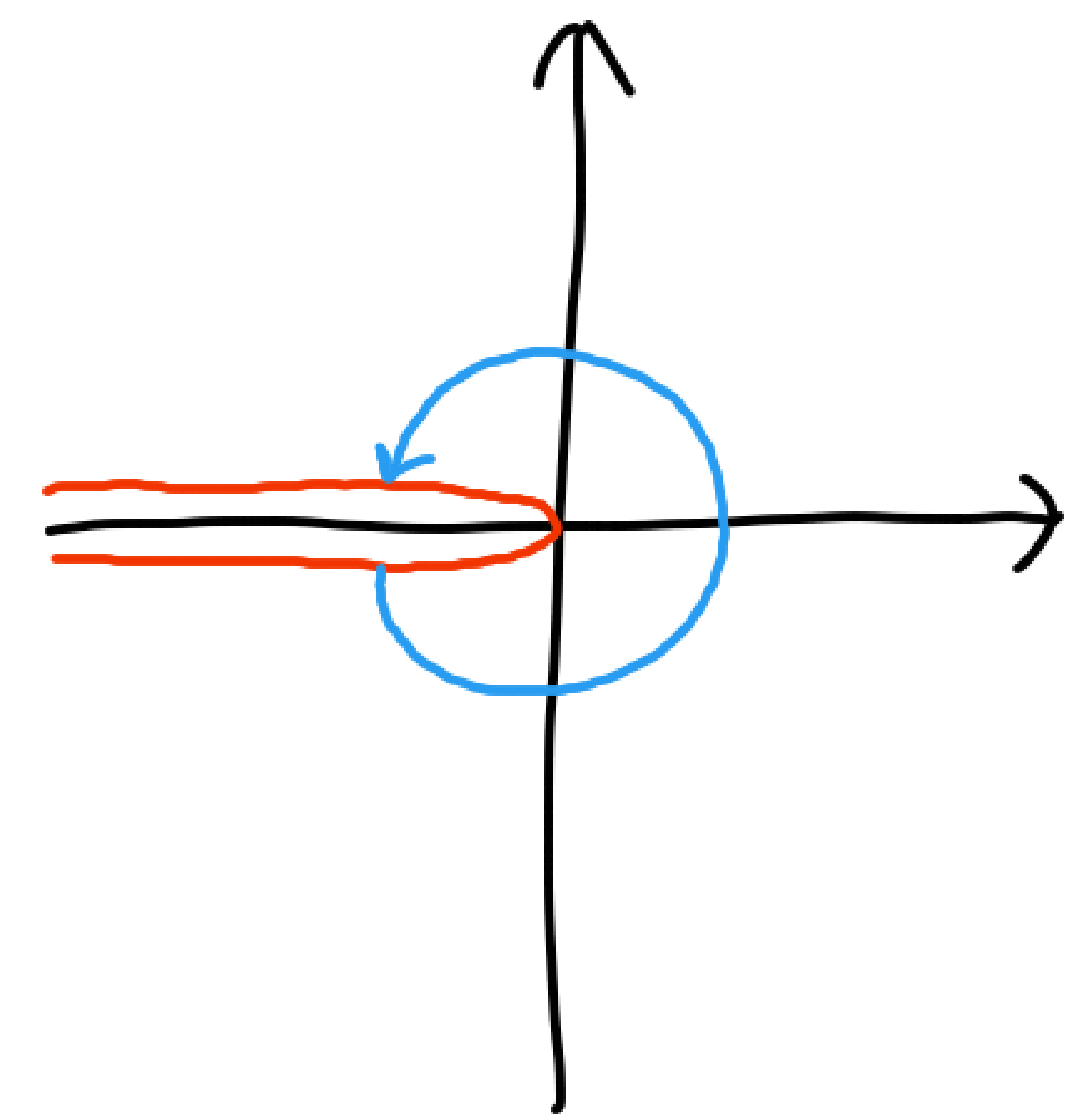
$u = \ln r \quad , \quad v = \theta + 2n\pi \quad (n = \cdots, -1, 0, 1, \cdots)$

$$\ln z = \ln|z| + i\,\underline{\arg z}$$

The principal value

$$\mathrm{Ln}\, z = \ln|z| + i\,\mathrm{Arg}\, z$$

$$-\pi < \mathrm{Arg}\, z \le \pi$$



$$\ln(-1) = \ln 1 + i\arg(-1)$$

$$= i(\pi + 2n\pi), \quad (n = \cdots, -1, 0, 1, \cdots)$$

- Logarithm is <u>analytic</u> except at zero and on the <u>branch cut</u>
- $\ln z = \operatorname{Ln} z + i(2n\pi)$
- For each n, $\ln z$ is called a branch and the negative real axis the branch cut.
- $(\ln z)' = \dfrac{1}{z}$

General power

- $z^c = e^{c \ln z}$
- $a^z = e^{z \ln a}$

$$i^{i} = e^{i \ln i} = e^{i\left[i\left(\frac{\pi}{2}+2n\pi\right)\right]}$$

$$= e^{-\left(\frac{\pi}{2}+2n\pi\right)}, \quad n = \cdots -1, 0, 1, \cdots$$

Principal value, $e^{i \mathrm{Ln}\, i} = e^{-\frac{\pi}{2}}$