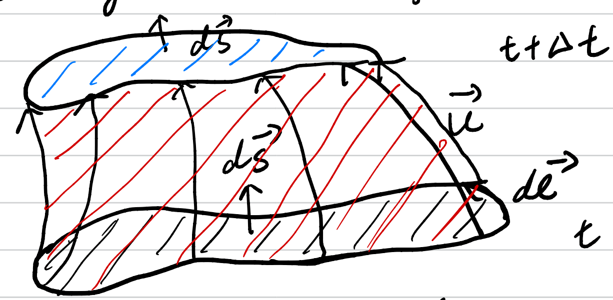
4/26 MHD equilibrium / pinches

(Goldston Ch. 9, Freidberg Ch. 2)

1. Frozen-Flux theorem in ideal MHD

$\vec{E} + \vec{u} \times \vec{B} = 0$: ideal Ohm's law

→ plasma moves together with field lines



$$\frac{\partial \psi}{\partial t} = \oint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\oint (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\oint \vec{E} \cdot d\vec{\ell}$$

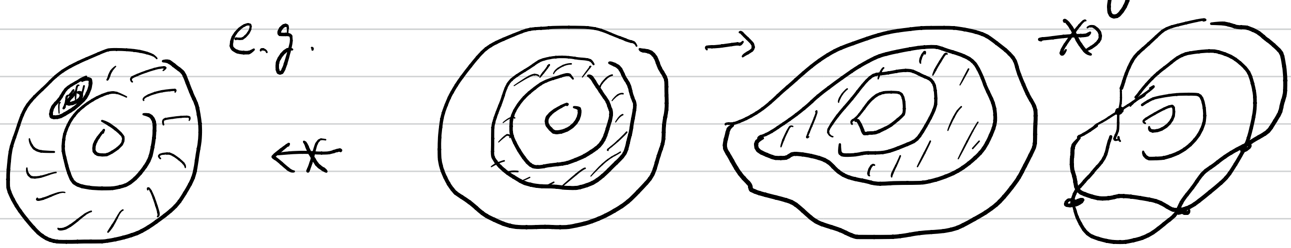
$$= \oint (\vec{u} \times \vec{B}) \cdot d\vec{\ell} = -\oint \vec{B} \cdot (\underline{\vec{u} \times d\vec{\ell}})$$

$$\underline{\frac{d\psi}{dt}} = \frac{\partial \psi}{\partial t} + \oint \vec{B} \cdot (\vec{u} \times d\vec{\ell}) = 0$$

→ Flux through every plasma element conserved

→ no change in magnetic topology

e.g.



→ Further reading

Newcomb '62 $\nabla \times (\vec{E} + \vec{u} \times \vec{B}) = 0$

Boozer '05 Review paper

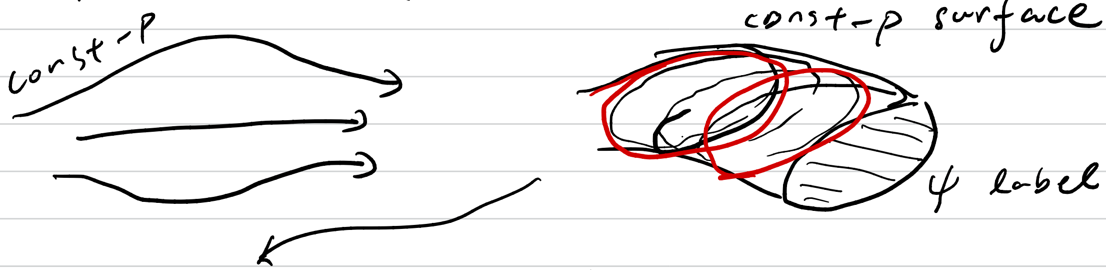
Biskamp's book Ch. 2.

2. Ideal MHD equilibrium ($\vec{u} = 0$)

$$\left[\begin{array}{l} \vec{j} \times \vec{B} = \vec{\nabla} p \\ \vec{j} = \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \\ \vec{\nabla} \cdot \vec{B} = 0 \end{array}\right. \qquad \left(\begin{array}{l} \vec{E} = 0 \\ \frac{\partial \rho}{\partial t} = 0 \\ \frac{\partial p}{\partial t} = 0 \end{array}\right.$$

(1) $\vec{B} \cdot \vec{\nabla} p = 0$

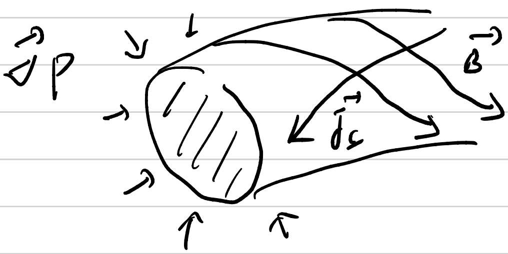
- pressure const. along the field lines



- p flux surface fcn. $p(\psi)$

(2) $\vec{j} \cdot \vec{\nabla} p = 0$ same implications

its $\vec{j}_\perp$ confines plasma by $\vec{j} \times \vec{B}$



using $\vec{\nabla} \cdot \vec{j} = 0$,

$$\vec{\nabla} \cdot (j_{\parallel} \hat{b} + \vec{j}_\perp) = 0$$

$$\vec{B} \cdot \vec{\nabla} \left(\frac{j_{\parallel}}{B} \right) = - \vec{\nabla} \cdot \vec{j}_\perp$$

return pfirsch-schlüter current.

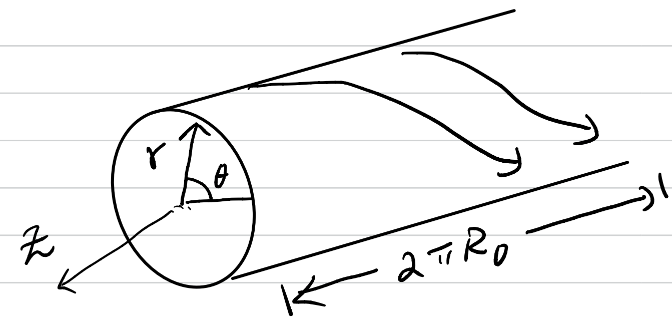
(3) with $\vec{\nabla} \cdot \vec{B} = 0$

Ideal equilibrium has 3 unknowns

e.g.) $(B_\theta(r), B_z(r), p(r))$ $(\psi, q(\psi), p(\psi))$

3. Screw pinch (circular, cylindrical approx.)

(r, θ, z)



symmetry in (θ, z) only r-dependency

(1) $\vec{\nabla} \cdot \vec{B} = 0 \qquad \frac{\partial B_r}{\partial r} = 0 \qquad B_r = 0$

(2) $\vec{j} = \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) = \frac{1}{\mu_0}\left(\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{z} - \frac{\partial B_z}{\partial r}\hat{\theta}\right)$

(3) $\vec{j} \times \vec{B} = \vec{\nabla} p \qquad j_\theta B_z - j_z B_\theta = \frac{dp}{dr}$

$$-\frac{1}{\mu_0}\left(\frac{1}{2}\frac{dB_z^2}{dr} + \frac{1}{2}\frac{dB_\theta^2}{dr} + \frac{B_\theta^2}{r}\right) = \frac{dp}{dr}$$

$$\boxed{\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}}$$

thermal pressure

magnetic pressure

curvature term

$(p(r), B_\theta(r), B_z(r))$

For θ-pinch: $B_\theta = 0$, $B_z(r)$ (or $j_\theta(r)$), $p(r)$

For z-pinch: $B_z = 0$, $B_\theta(r)$ (or $j_z(r)$), $p(r)$

4. concept of "beta" β

with strong B_z

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) \approx 0$$

magnetic fusion confinement efficiency

$$\beta \equiv \frac{\langle p \rangle}{\langle B^2 \rangle / 2\mu_0}$$

$\langle A \rangle$: volume average

e.g.) screw pinch

$$\langle A \rangle = \frac{\int_0^a A\, r\,dr\,d\theta\,dz}{\int r\,dr\,d\theta\,dz}$$

a : plasma radius

$$= \frac{2\int_0^a A\, r\,dr}{a^2}$$

typically, use

$$\boxed{\beta = \frac{\langle p \rangle}{B^2 / 2\mu_0}}$$

, where $B^2 = B_{z0}^2 + B_{\theta a}^2$

B_{z0}: at magnetic axis

$B_{\theta a}$: $r=a$ at boundary

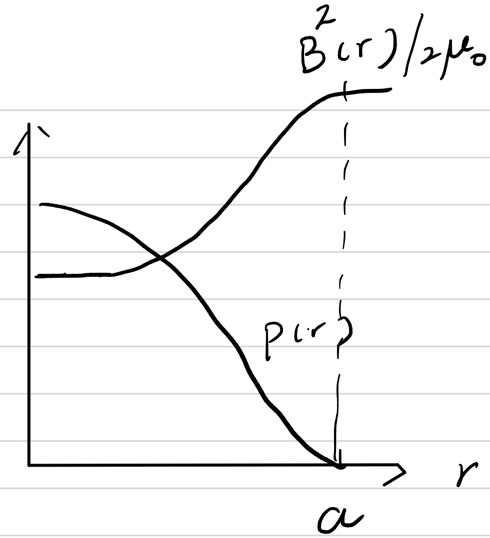
$$2\pi \int_0^a r\, \mu_0 j_z\, dr = 2\pi \int_0^a \cancel{r} \frac{1}{\cancel{r}} \frac{d}{dr}(rB_\theta)\, dr$$

$$\mu_0 I = 2\pi a B_{\theta a}$$

toroidal $\beta_t = \dfrac{\langle p \rangle}{B_{z0}^2 / 2\mu_0}$

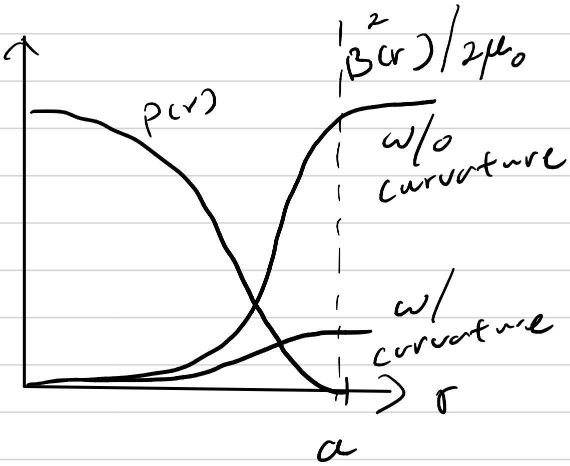
poloidal $\beta_p = \dfrac{\langle p \rangle}{B_{\theta a}^2 / 2\mu_0} = \dfrac{8\pi^2 a^2 \langle p \rangle}{\mu_0 I^2}$

* $\dfrac{1}{\beta} = \dfrac{1}{\beta_t} + \dfrac{1}{\beta_p}$



tokamak $\beta < 1$

Spherical torus $\beta \simeq 1$



Astrophysical plasma

$\beta > 1$

5. Safety factor "q"

$$q(r) = \frac{r B_z}{R_0 B_\theta} = \frac{B_z/2\pi R_0}{B_\theta/2\pi r} = \frac{m}{n}$$

# of toroidal transit (m)

# of poloidal transit. (n)

- measure of pitch
- measure of ideal stability
- more generally $q(\psi) = \frac{d\psi_T}{d\psi_P}$

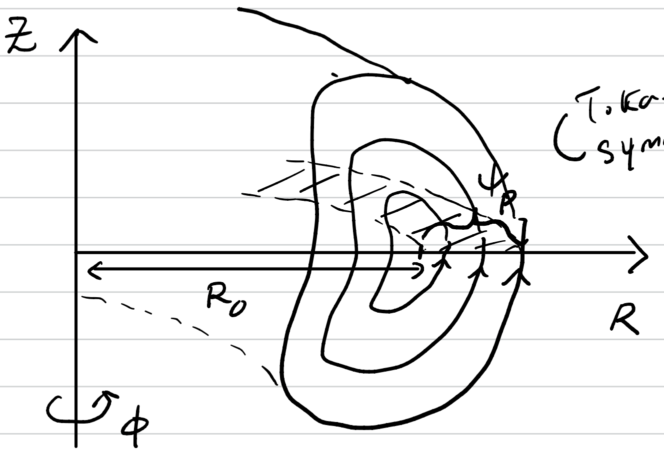
  toroidal flux (ψ_T)

  poloidal flux (ψ_P)

- "Kink" safety factor

$$q_* = \frac{a B_{z0}}{R_0 B_{\theta a}} = \frac{2\pi a^2 B_{z0}}{\mu_0 R_0 I}$$

4/28 Tokamak equilibrium / Grad - Shafranov



(R, ϕ, Z) coordinates

(Tokamak symmetry) $\frac{\partial A}{\partial \phi} = 0 \longrightarrow$ scalar quantity $(j_p, B, \cdots)$

$\vec{j} \times \vec{B} = \vec{\nabla} p$

Goal : shape of magnetic field given two scalar fcns.

(1) $\vec{\nabla} \cdot \vec{B} = 0$

$$\frac{1}{R}\frac{\partial}{\partial R}(R B_R) + \frac{\partial B_Z}{\partial Z} = 0$$

$$\rightarrow \quad B_R = -\frac{1}{R}\frac{\partial \psi}{\partial Z} \qquad B_Z = \frac{1}{R}\frac{\partial \psi}{\partial R} \qquad \psi(R, Z)$$

$$\vec{\nabla}\phi = \frac{1}{R}\hat{\phi}$$

$$\vec{B} = B_R \hat{R} + B_Z \hat{Z} + B_\phi \hat{\phi} = \vec{B}_p + B_\phi \hat{\phi}$$

$$= \frac{1}{R}\vec{\nabla}\psi \times \hat{\phi} + B_\phi \hat{\phi} = \vec{\nabla}\psi \times \vec{\nabla}\phi + R B_\phi \vec{\nabla}\phi$$

$$\psi_p = \int_0^{2\pi} d\phi \int_{R_0}^{R} R' B_Z(R', 0)\, dR' = 2\pi \psi$$

poloidal flux fcn

(2) $\mu_0 \vec{j} = \vec{\nabla} \times \vec{B}$

$$= \vec{\nabla} \times (\vec{\nabla}\psi \times \vec{\nabla}\phi + R B_\phi \vec{\nabla}\phi)$$

$$= \vec{\nabla}\psi\,\cancel{(\nabla^2 \phi)} - \vec{\nabla}\phi\, \nabla^2 \psi + \vec{\nabla}\phi \cdot \vec{\nabla}(\vec{\nabla}\psi) - \vec{\nabla}\psi \cdot \vec{\nabla}(\vec{\nabla}\phi)$$

$$+ \vec{\nabla}(R B_\phi) \times \vec{\nabla}\phi$$

$$= -\frac{1}{R}\nabla^2\psi\,\hat{\phi} + \frac{1}{R^2}\frac{\partial \psi}{\partial R}\hat{\phi} - \frac{\partial \psi}{\partial R}\frac{\partial}{\partial R}\left(\frac{1}{R}\hat{\phi}\right) + \frac{1}{R}\vec{\nabla}(R B_\phi) \times \hat{\phi}$$

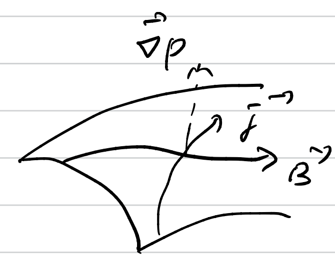
$$= -\frac{1}{R}\nabla^2\psi\,\hat{\phi} + \frac{2}{R^2}\frac{\partial\psi}{\partial R}\hat{\phi} + \frac{1}{R}\vec{\nabla}(RB_\phi)\times\hat{\phi}$$

$$= -\frac{1}{R}\left\{\frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial z^2}\right\}\hat{\phi} + \frac{1}{R}\vec{\nabla}(RB_\phi)\times\hat{\phi}$$

$$\equiv -\frac{1}{R}\Delta^*\psi\,\hat{\phi} + \frac{1}{R}\vec{\nabla}(RB_\phi)\times\hat{\phi}$$

$$\Delta^*\psi = \{\quad\} = R^2\vec{\nabla}\cdot\left(\frac{\vec{\nabla}\psi}{R^2}\right) = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial z^2}$$

$$= \mu_0\left(j_\phi\hat{\phi} + \vec{j}_p\right)$$



(3) $\vec{j}\times\vec{B} = \vec{\nabla}p$

(i) parallel component (w.r.t $\vec{B}$)

$$\vec{B}\cdot\vec{\nabla}p = 0 = (\vec{\nabla}\psi\times\hat{\phi})\cdot\vec{\nabla}p$$

$$\hat{\phi}\cdot(\vec{\nabla}p\times\vec{\nabla}\psi) = 0 \xrightarrow{\text{generally}} p = p(\psi)$$

(ii) parallel component (w.r.t $\vec{j}$)

$$\vec{j}\cdot\vec{\nabla}p = 0 = (\vec{\nabla}(RB_\phi)\times\hat{\phi})\cdot\vec{\nabla}p$$

$$p'\hat{\phi}\cdot(\vec{\nabla}(RB_\phi)\times\vec{\nabla}\psi) = 0 \qquad ' \equiv \frac{d}{d\psi}$$

$$\xrightarrow{\text{generally}} RB_\phi \equiv F(\psi)$$

* $\mu_0 I_p = \int_0^{2\pi}d\phi\int_{R_0}^{R} R' j_z(R',0)\,dR' = 2\pi F$

poloidal current

$$\vec{B} = \frac{1}{R}\vec{\nabla}\psi \times \hat{\phi} + \frac{1}{R}F(\psi)\hat{\phi}$$

$$\mu_0 \vec{j} = -\frac{1}{R}\Delta^*\psi\,\hat{\phi} + \frac{1}{R}F'\vec{\nabla}\psi \times \hat{\phi}$$

c(iii) Radial force balance

$$\vec{\nabla}\psi \cdot (\vec{j} \times \vec{B}) = \vec{\nabla}\psi \cdot \vec{\nabla}p = p'|\vec{\nabla}\psi|^2$$

$$\frac{\vec{\nabla}\psi}{\mu_0} \cdot \left(-\frac{1}{R^2}\Delta^*\psi - \frac{FF'}{R^2} \right) \underbrace{\hat{\phi} \times (\vec{\nabla}\psi \times \hat{\phi})}_{\vec{\nabla}\psi} = p'|\vec{\nabla}\psi|^2$$

$$-\Delta^*\psi - FF' = \mu_0 R^2 p'$$

$$\boxed{\Delta^*\psi = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial z^2} = -\mu_0 R^2 p' - FF'}$$

Grad-Shafranov (GS) equation.

$\longrightarrow$ Gives $\psi(R,z)$ for $\begin{cases} p(\psi), F(\psi) \text{ specified} \\ \text{boundary condition.} \end{cases}$

: direct equilibrium

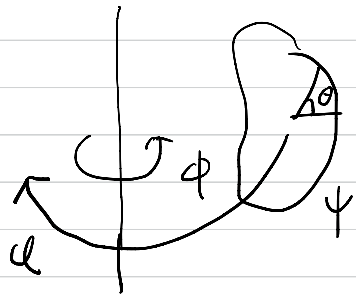
$\longrightarrow$ $R(\psi, \theta)$, $z(\psi, \theta)$

: Inverse equilibrium

(4) Flux surface - average for physical quantities

$$d\vec{r} = R\,dR\,d\phi\,dz = J\,d\psi\,d\theta\,d\varphi$$

$$(R, \phi, z) \qquad (\psi, \theta, \varphi)$$



$$\varphi = -\phi$$

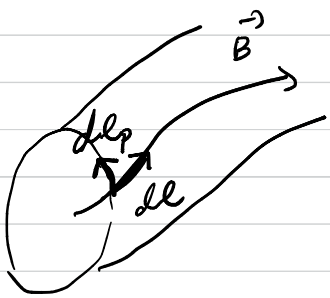
Jacobian $J^{-1} = (\vec{\nabla}\psi \times \vec{\nabla}\theta)\cdot\vec{\nabla}\varphi$

$$= (\vec{\nabla}\psi \times \vec{\nabla}\phi)\cdot\vec{\nabla}\theta$$

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\phi + F\vec{\nabla}\phi$$

$$\vec{B}\cdot\vec{\nabla}\theta = \frac{1}{J} \quad \text{V}$$

$$d\vec{r} = \frac{1}{\vec{B}\cdot\vec{\nabla}\theta}\,d\psi\,d\theta\,d\varphi = d\psi\,d\varphi\,\frac{dl}{B} = d\psi\,d\varphi\,\frac{dl_p}{B_p}$$

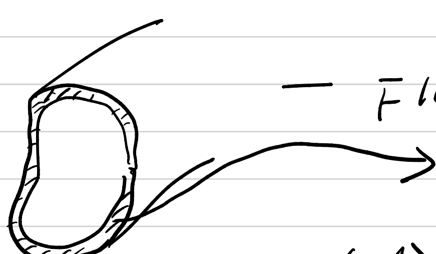


∵ Field line equation

$$\vec{B}\times d\vec{R} = 0 \rightarrow \frac{d\psi}{\vec{B}\cdot\vec{\nabla}\psi} = \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta} = \frac{d\varphi}{\vec{B}\cdot\vec{\nabla}\varphi} = \frac{dl}{B} = \frac{dl_p}{B_p}$$

— Volume average of A

$$\langle A\rangle_v = \frac{\int d\vec{r}\,A}{\int d\vec{r}} = \frac{\int_0^\psi d\psi \oint \frac{dl_p}{B_p} A}{\int_0^\psi d\psi \oint \frac{dl_p}{B_p}}$$



— Flux surface average of A

→ $d\psi$ Annulus volume average

$$\langle A\rangle = \frac{\oint \frac{dl_p}{B_p} A}{\oint \frac{dl_p}{B_p}}$$

– q profile

$$q(\psi) = \frac{\langle \text{toroidal pitch} \rangle}{\langle \text{poloidal pitch} \rangle}$$

$$= \frac{\oint \vec{B}\cdot\vec{\nabla}\phi \frac{dl_p}{B_p}}{\oint \vec{B}\cdot\vec{\nabla}\theta \frac{dl_p}{B_p}} = \frac{\int \frac{F\,dl_p}{R^2 B_p}}{\oint d\theta}$$

$$= \frac{F(\psi)}{2\pi} \oint \frac{dl_p}{R^2 B_p}$$

(5) Alternative representations of G-S.

using $I_p(\psi) = \frac{2\pi F(\psi)}{\mu_0}$

(i) $\Delta^* \psi = -\mu_0 R^2 p' - \frac{\mu_0^2}{8\pi^2} (I_p^2)'$

using $F(\psi) = q(\psi) \left(\frac{1}{2\pi} \oint \frac{dl_p}{R^2 B_p} \right)^{-1}$

(ii) $\Delta^* \psi = -\mu_0 R^2 p' - \frac{1}{2} \left[q^2(\psi) \left(\frac{1}{2\pi} \oint \frac{dl_p}{R^2 B_p} \right)^{-2} \right]'$

Integro-differential equation

Iterative. q-solver

Using, $\bar{j}_\phi = -\frac{1}{\mu_0 R} \Delta^* \psi = +\frac{1}{\mu_0 R} (\mu_0 R^2 p' + FF')$

$\langle \bar{j}_\phi \rangle$

(iii) $\Delta^* \psi = -\mu_0 R^2 p' + \left(\mu_0 p' \oint \frac{dl_p}{B_p} - \mu_0 \langle \bar{j}_\phi \rangle \oint \frac{dl_p}{R B_p} \right) \left(\oint \frac{dl_p}{B_p} \right)^{-1}$

Iterative, J-solver

(iv) $\langle \bar{j}_\parallel B \rangle$ Stacey' book Y