

Modified wavenumber

노트 제목

2019-09-23

$$f = e^{ikx} \rightarrow f' = ik e^{ikx} = ikf$$

$$\text{CD2: } f_j' = \frac{f_{j+1} - f_{j-1}}{2h} = \frac{1}{2h} (e^{ikx_{j+1}} - e^{ikx_{j-1}}) = \frac{1}{2h} e^{ikx_j} (e^{ikh} - e^{-ikh})$$

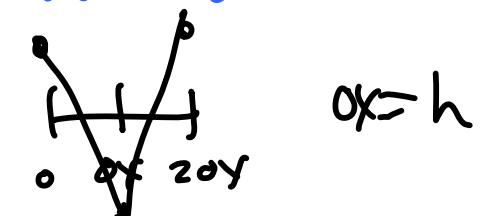
$$= i \frac{\sin kh}{h} e^{ikx_j} = i \frac{\sin kh}{h} f_j$$

k' : modified wavenumber

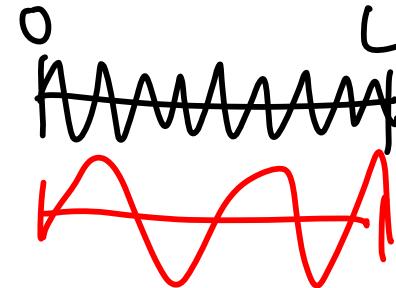
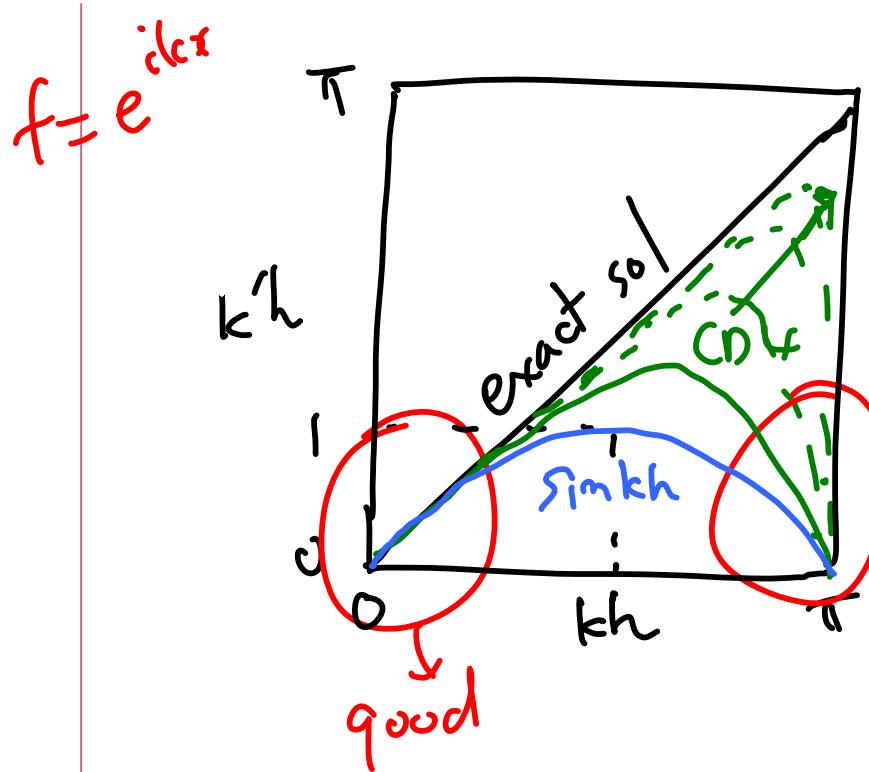
$$\rightarrow k'h = \sin kh \\ (\text{CD2})$$



$$h = \frac{L}{N}$$



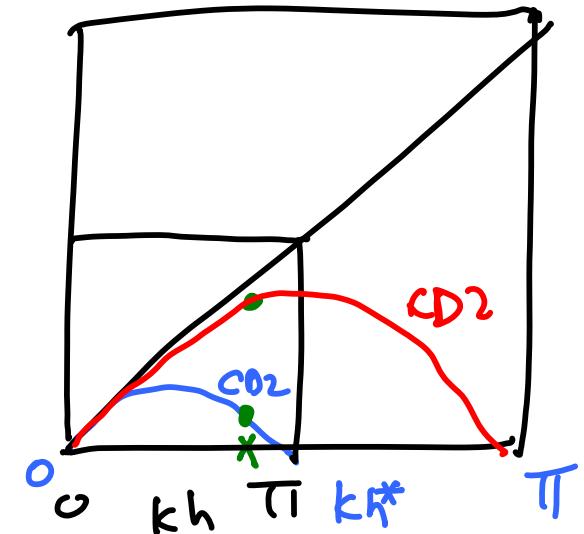
$$k \cdot (2ox) = 2\pi \rightarrow k = \frac{\pi}{h}$$



$$N \rightarrow 2N$$

$$h \rightarrow \frac{h}{2} = h^*$$

$CD2 : O(h^2)$



- 1st-order FD : $f_j' = \frac{f_{j+1} - f_j}{h} + O(h) = \frac{1}{h} (e^{ikx_{j+1}} - e^{ikx_j})$

$f' = ikf$ $F = e^{ikx}$

$$\rightarrow f_j' = i \left[\frac{\sin kh}{h} + i \frac{1 - \cos kh}{h} \right] f_j$$

" k' : modified wavenumber
complex number

k'_{CD2}

$k \rightarrow k'_{CD2}$

dispersion error

dissipation error

$k'_{FO} = a + ib$

$e^{ik'x} = e^{i(a+ib)x}$

$= e^{-bx} e^{iax}$

decaying ft.

2.4 Padé approximations

$$f_j' + a f_j + b f_{j-1} + c f_{j-2} + d f_{j+1} \dots = O(h^*)$$

Include derivatives too in the formula.

Ex) Find the most accurate formula of f_j' that involves

$$f_{j+1}, f_{j-1}, f_j, f_{j+1}', f_{j-1}'$$

$$\rightarrow f_j' + a_0 f_{j+1}' + a_1 f_{j-1}' + b_0 f_j + b_1 f_{j-1} + b_2 f_{j+1} = O(h^?)$$

Taylor
table

	f_j	f_j'	f_j''	f_j'''	$f_j^{(iv)}$	$f_j^{(v)}$	\dots
f_j'	0	1	0	0	0	0	- - -
$a_0 f_{j+1}'$	0	$a_0 \cdot 1$	$a_0 \cdot h$	$a_0 \cdot \frac{h^2}{2}$	$a_0 \cdot \frac{1}{6}h^3$	$a_0 \cdot \frac{1}{24}h^4$	- - -
$a_1 f_{j-1}'$	0	$a_1 \cdot 1$	$a_1 \cdot (-h)$	$a_1 \cdot \frac{1}{2}(-h)^2$	$a_1 \cdot \frac{1}{6}(-h)^3$	$a_1 \cdot \frac{1}{24}(-h)^4$	- - -
$b_0 f_j$	b_0	0	0	- - -	,	- 0	

$$+ \left[\begin{array}{c|cccccc} b_0 f_j & b_1 & b_1(-h) & b_1 \frac{1}{2}(-h)^2 & b_1 \frac{1}{6}(-h)^3 & \dots & \dots \\ b_2 f_{j+1} & b_2 & b_2 h & b_2 \frac{1}{2}h^2 & b_2 \frac{1}{6}h^3 & \dots & \dots \end{array} \right]$$

$$\rightarrow f_j' + a_0 f_{j+1}' + a_1 f_{j-1}' + b_0 f_j + b_1 f_{j-1} + b_2 f_{j+1}$$

$$= (b_0 + b_1 + b_2) f_j + (a_0 + a_1 - b_1 h + b_2 h) f_j' + (\underset{\text{||}}{a_0} \underset{\text{||}}{f_j''} + \underset{\text{||}}{a_1} \underset{\text{||}}{f_j'''} + \underset{\text{||}}{b_2} \underset{\text{||}}{f_j^{(4)}})$$

$$\rightarrow a_0 = \dots, a_1 = \dots, b_0 = \dots; b_1 = \dots; b_2 = \dots.$$

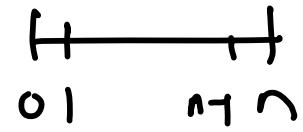
$$\rightarrow f_j' + \frac{1}{6} f_{j+1}' + \frac{1}{6} f_{j-1}' - \frac{3}{8h} f_{j+1} + \frac{3}{8h} f_{j-1} = \frac{h^4}{30} f_j^{(4)} + \dots$$

$$\rightarrow f_{j-1}' + 4f_j' + f_{j+1}' = \frac{3}{h} (f_{j+1} - f_{j-1}) + \frac{h^4}{30} f_j^{(4)} + \dots$$

3 pts. $\rightarrow 4^{\text{th}}$ -order accuracy

$O(h^4)$: 4^{th} -order accuracy

\Rightarrow 'compact' scheme. $j = 1, 2, \dots, n-1$



tri-diagonal matrix system \rightarrow easy to solve.

$n-1$ eqs. for $n+1$ unknowns (f'_0, \dots, f'_n)

$$j=0 : f'_0 = \frac{1}{2h}(-3f_0 + 4f_1 - f_2) + O(h^2)$$

$$@ j=1 : f'_2 + 4f'_1 = \frac{1}{2h}(7f_2 - 3f_0 - 4f_1)$$

$$\rightarrow \begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & & & 1 & 4 \end{pmatrix} \begin{pmatrix} f'_1 \\ f'_2 \\ \vdots \\ f'_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \vdots \\ \frac{3}{h}(f_{jn} - f_{j-1}) \end{pmatrix}}_{\rightarrow O(h^2 \sim h^4)}$$

$$\text{Or, } @ j=0 : f'_0 + a_0 f'_1 + b_0 f_0 + b_1 f_1 + b_2 f_2 = O(h^3)$$

$$\text{Using Taylor table, } f'_0 + 2f'_1 = \frac{1}{h} \left(-\frac{5}{2} f_0 + 2f_1 + \frac{1}{2} f_2 \right) + O(h^3)$$

$$\rightarrow \begin{pmatrix} 1 & 2 & * \\ 1 & 4 & 1 \\ \dots & \dots & \dots \\ 0 & & 2 \end{pmatrix} \begin{pmatrix} f'_0 \\ f'_1 \\ \vdots \\ f'_{j-1} \\ f'_{j+1} \\ f'_n \end{pmatrix} = \begin{pmatrix} \frac{3}{h} (f_{j+1} - f_{j-1}) \\ \vdots \\ \dots \end{pmatrix} \rightarrow O(h^3 \sim h^4)$$

$f'_0 + af'_1 + bf'_2 + \dots = \dots$

Padé approximations can be easily extended to higher derivatives.

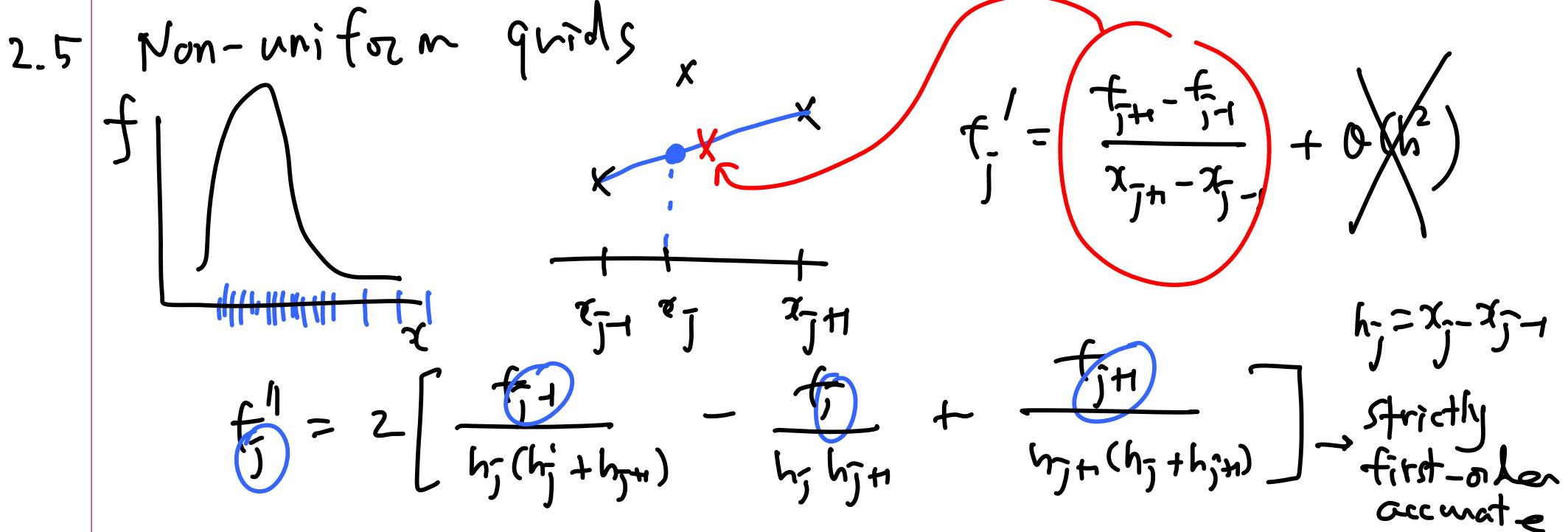
$$\text{Ex}) \frac{1}{12} f_j'' + \frac{10}{12} f_j'' + \frac{1}{12} f_{j+1}'' = \frac{1}{h^2} (f_{j+1} - 2f_j + f_{j-1}) + O(h^4)$$

4th-order accurate compact scheme.

S. Lele (JCP. 1992) — compact schemes

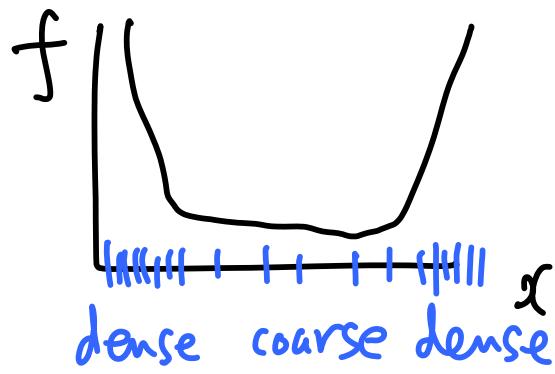
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citations



FD formula for non-uniform mesh generally has a lower order of accuracy than their counterpart with the same stencil for uniform mesh.

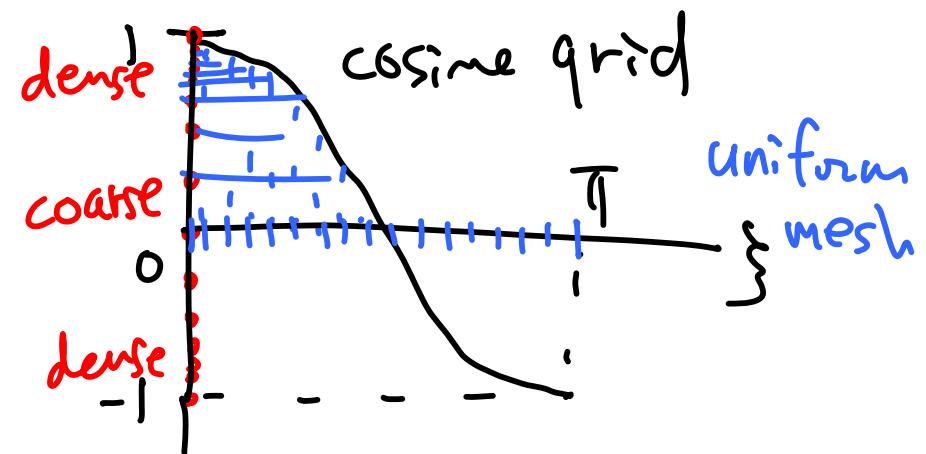
- Use a coordinate transformation



$$x \rightarrow \xi$$

ex) $\xi = \cos^{-1} x$

$$x = \cos \xi$$

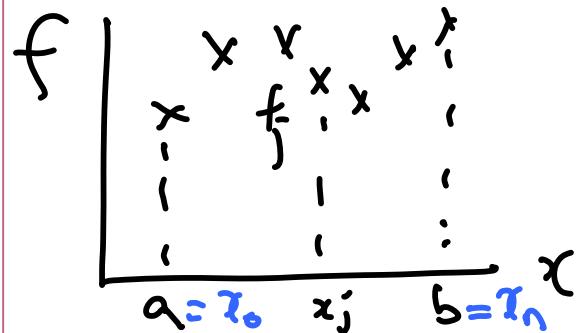


In general, $\xi = g(x)$

$$\frac{df}{dx} = \frac{df}{d\xi} \frac{d\xi}{dx} = g' \boxed{\frac{df}{d\xi}} \quad \text{FD on uniform mesh}$$

$$\begin{aligned} \frac{d^2f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) = \frac{d}{dx} \left(g' \frac{df}{d\xi} \right) = \left(g'' \frac{df}{d\xi} + g' \frac{d^2f}{d\xi^2} \frac{d\xi}{dx} \right) \\ &= g'' \boxed{\frac{df}{d\xi}} + g' \boxed{\frac{d^2f}{d\xi^2}} \end{aligned}$$

Ch. 3 Numerical integration (quadrature)



$$I = \int_a^b f(x) dx = \sum_{j=0}^n f_j w_j$$

↑ weight

3.1

Trapezoidal and Simpson's rules

$$\text{Lagrange polynomial } p(x) = \sum_{j=0}^n f_j L_j(x)$$

$$I = \int_a^b f(x) dx = \int_a^b p(x) dx = \int_a^b \sum_{j=0}^n f_j L_j(x) dx = \sum_{j=0}^n f_j \int_a^b L_j(x) dx$$

$$I = (b-a) \sum_{j=0}^n c_j f_j \quad \text{where } c_j = \frac{1}{b-a} \int_a^b L_j(x) dx$$

Newton - Cotes formula \tilde{x} Cotes number