Engineering Math 2

Lecture 8

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Previously, we discussed

- Some complex functions

- Can you calculate $\ln(-1)$, i^i ?

Line integral

$$\int_C f(z)\,dz ,$$

if C is closed,

$$\oint_C f(z)\,dz$$

Method I

For analytic function in a simply-connected domain, find $F(z)$ s.t. $F'(z) = f(z)$, then

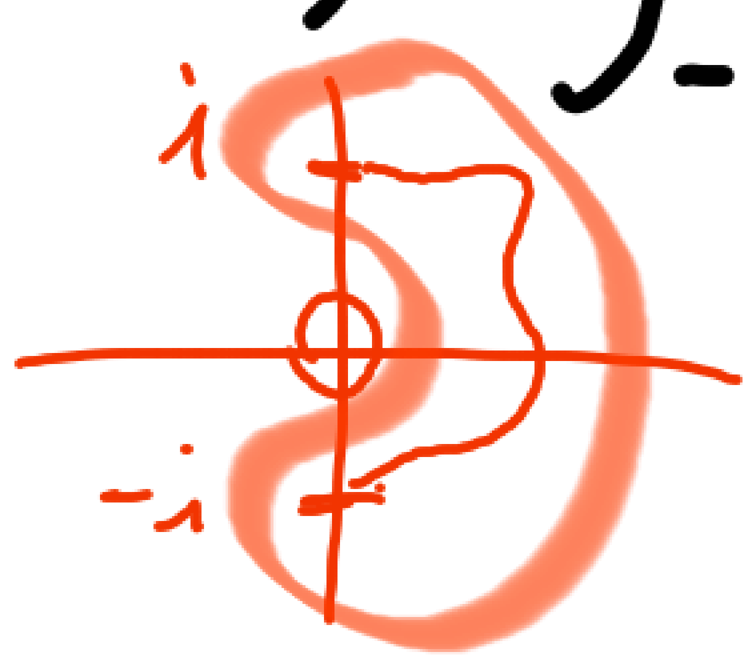
$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

Example

1) $\displaystyle\int_0^{1+i} z\,dz = \frac{z^2}{2}\Bigg|_0^{1+i} = \frac{(1+i)^2}{2} = -i$

2) $\displaystyle\int_{-i}^{i} \frac{dz}{z} = Ln(i) - Ln(-i)$

$\displaystyle = i\frac{\pi}{2} - \left(-i\frac{\pi}{2}\right) = i\pi$

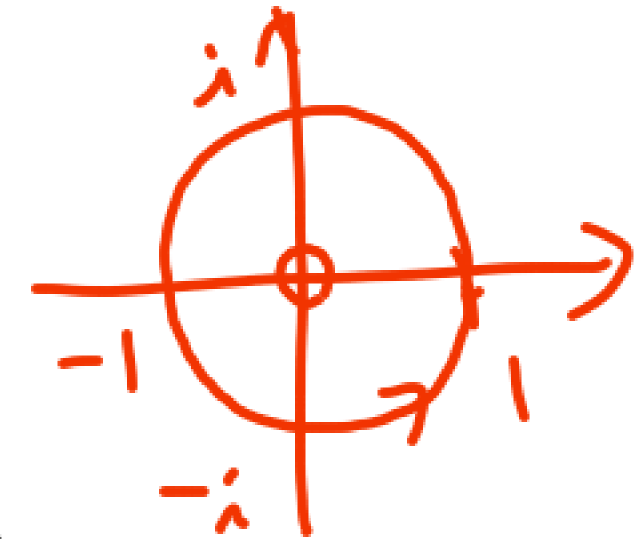


Method II

For $f(z)$ continuous and C piecewise smooth,

$$\int_C f(z)\,dz = \int_a^b f(z)\frac{dz}{dt}\,dt$$

Example



$dz = ie^{it}dt$

1) $\oint_C \frac{dz}{z}$ on the unit circle $\quad z = e^{it}, \; 0 \le t < 2\pi$

$$= \int_0^{2\pi} \frac{i\cancel{e^{it}}\,dt}{\cancel{e^{it}}} = 2\pi i$$

2) $\oint_C (z-z_0)^m dz$, C: $z = z_0 + \rho e^{it}$, $0 \le t < 2\pi$

$$= \int_0^{2\pi} (\rho e^{it})^m i\rho e^{it}\,dt = \int_0^{2\pi} i\rho^{m+1} e^{i(m+1)t}\,dt$$

$$= \begin{cases} 2\pi i, & m = -1 \\ 0, & m \neq -1, \text{ integer} \end{cases}$$

ML inequality

$$\left| \int_C f(z)\,dz \right| \le ML$$

where $|f(z)| \le M$, L = length of C

Cauchy's integral theorem

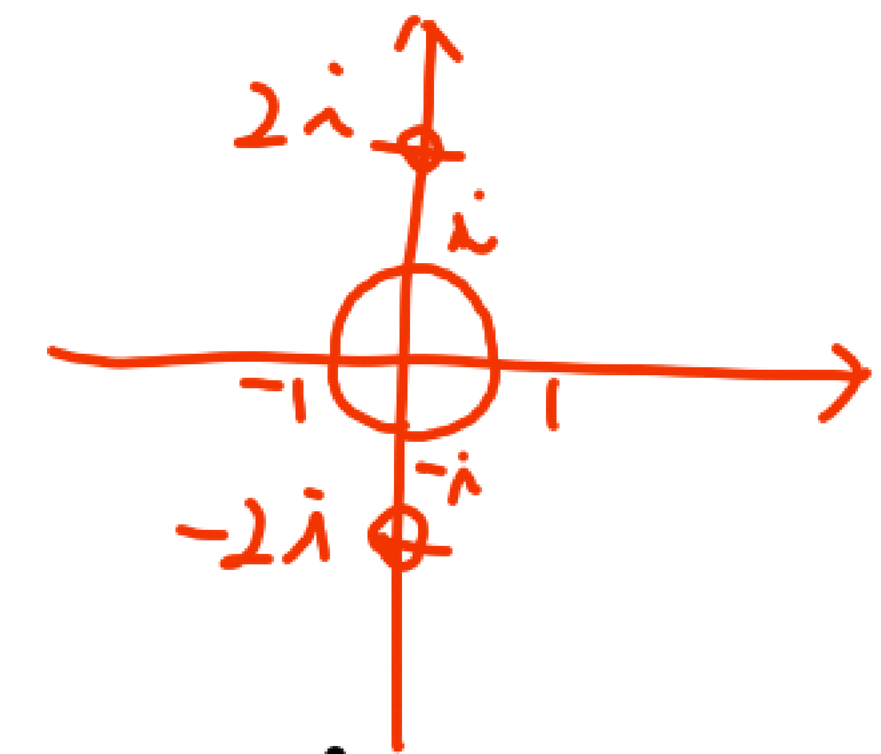
$f(z)$ analytic in a simply-connected domain D,

$$\oint_C f(z)\,dz = 0$$

for every C in D.

Example

1) $\oint_c e^z dz = 0$ for any c



2) $\oint_c \frac{dz}{z^2+4} = 0$ on a unit circle

3) $\oint_c \bar{z}\, dz = \int_0^{2\pi} \cancel{e^{-it}}\, i\, \cancel{e^{it}}\, dt = 2\pi i$

Given $f'(z)$ continuous,

$$\oint_C f(z)\,dz = \oint (u+iv)(dx+idy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

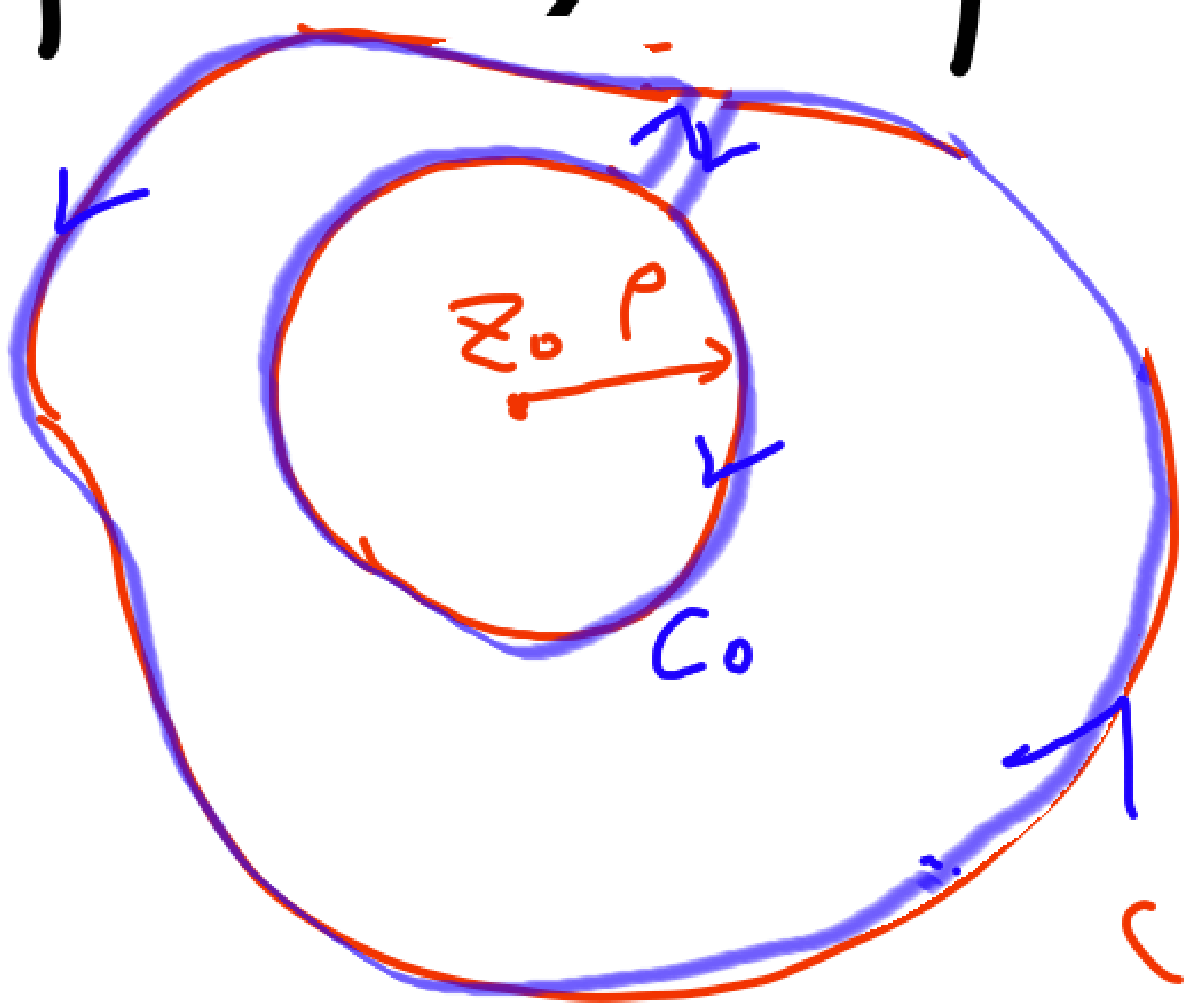
$$= \iint\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dA + i\iint\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dA$$

$$= 0$$

Example

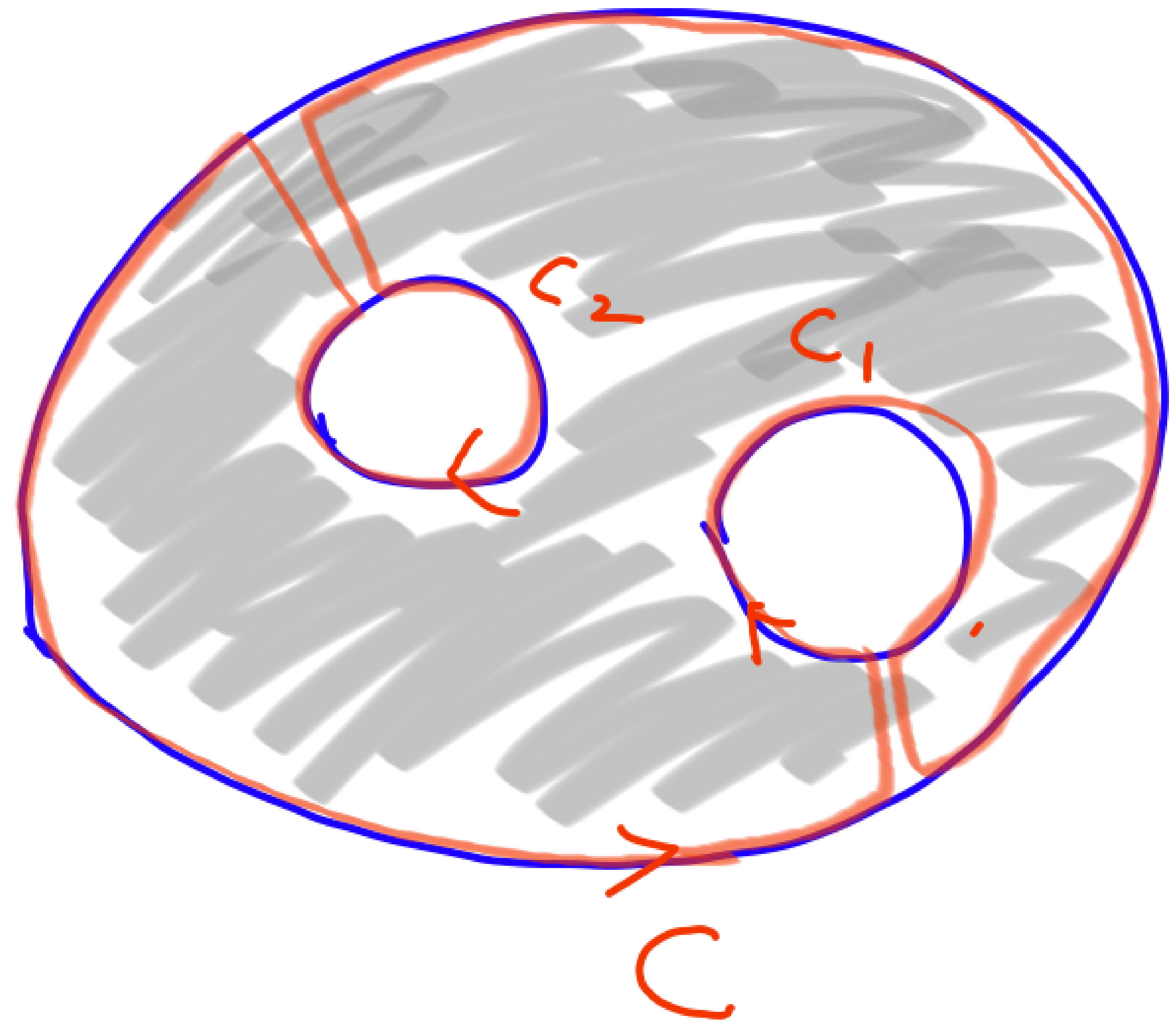
$$\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & , m=-1 \\ 0 & , m \neq -1, \text{ integer} \end{cases}$$

for any simple closed path encompassing z_0



$$\oint_C - \oint_{C_0} = 0$$

$$\oint_C = \oint_{C_0}$$



$$\oint_C - \oint_{C_1} - \oint_{C_2} = 0$$

$$\oint_C = \oint_{C_1} + \oint_{C_2}$$