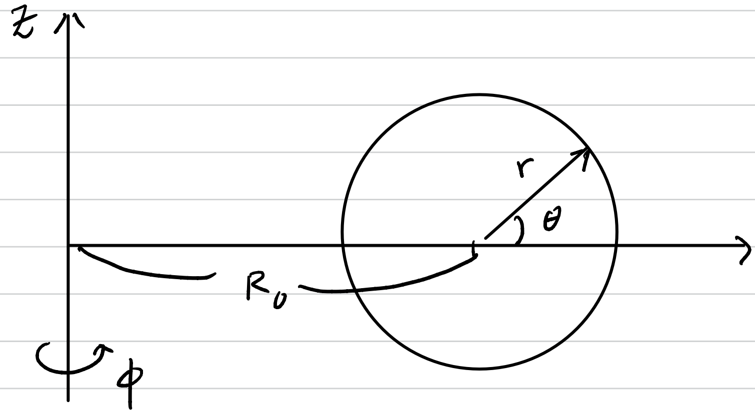
5/3 Tokamak equilibrium / Large-aspect ratio expansion

G-S equation

$$\Delta^* \psi = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{\partial^2 \psi}{\partial Z^2} = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}$$



$R = R_0 + r\cos\theta$

$Z = r\sin\theta$

$\phi = -\varphi$

$(R, \phi, Z) \rightarrow (r, \theta, \varphi)$

Shafranov coordinates

by chain rules,

$$\Delta^* \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} - \epsilon\frac{r}{R}\left(\cos\theta\frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\sin\theta}{r^2}\frac{\partial \psi}{\partial \theta}\right)$$

Assuming $r/R \ll 1$ $\quad \frac{r}{R} \rightarrow \epsilon\frac{r}{R}$

$r/R \rightarrow 0$ it becomes straight cylinder

So, $\psi = \psi_0(r) + \epsilon\psi_1(r, \theta) + \mathcal{O}(\epsilon^2)$

$$\begin{cases} \dfrac{dp}{d\psi} = \dfrac{dp}{d\psi_0} + \epsilon\dfrac{d^2p}{d\psi_0^2}\psi_1 + \mathcal{O}(\epsilon^2) \\ \dfrac{dF^2}{d\psi} = \dfrac{dF^2}{d\psi_0} + \epsilon\dfrac{d^2F^2}{d\psi_0^2}\psi_1 + \mathcal{O}(\epsilon^2) \end{cases}$$

(1) ϵ^0 equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_0}{dr}\right) = -\mu_0 R_0^2\frac{dp}{d\psi_0} - \frac{1}{2}\frac{dF^2}{d\psi_0}$$

$$\begin{cases} B_{\theta 0}(r) = \dfrac{1}{R}\vec{\nabla}\psi \times \hat{\phi} = \dfrac{1}{R_0}\dfrac{d\psi_0}{dr}, \quad \dfrac{d}{d\psi_0} = \dfrac{1}{R_0 B_{\theta 0}}\dfrac{d}{dr} \\ B_{\varphi 0}(r) = -\dfrac{F}{R_0} \end{cases}$$

$$\boxed{\frac{d}{dr}\left(p + \frac{B_{\varphi 0}^2}{2\mu_0}\right) + \frac{B_{\theta 0}}{\mu_0 r}\frac{d}{dr}\left(r B_{\theta 0}\right) = 0}$$

same as the general screw pinch

$$\boxed{\beta_p \equiv \frac{\langle p \rangle_v}{B_{\theta a}^2/2\mu_0} = 1 + \frac{1}{2\mu_0}\left(B_{\varphi a}^2 - \langle B_{\varphi 0}^2\rangle_v\right)}$$

$\beta_p \sim 1$, $\beta_p > 1$ $\qquad$ $\beta_p < 1$

$\langle B_{\varphi 0}^2 \rangle_v < B_{\varphi a}^2$ $\qquad$ $\langle B_{\varphi 0}^2 \rangle_v > B_{\varphi a}^2$

diamagnetism $\qquad$ paramagnetism

(2) ϵ' equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2} - \underline{\frac{1}{R_0}\cos\theta\frac{\partial \psi_0}{\partial r}}$$

$$= -\mu_0 R_0^2 \frac{d^2 p}{d\psi_0^2}\psi_1 - \underline{2\mu_0 R_0 r \cos\theta \frac{dp}{d\psi_0}} - \frac{1}{2}\frac{d^2 F^2}{d\psi_0^2}\psi_1$$

Inhomogeneous terms

$\psi_1(r, \theta) = \bar{\psi}_1(r)\cos\theta$ ✓

$$\psi = \psi_0(r) + \epsilon\psi_1(r)\cos\theta + \epsilon^2\psi_2(r)\sin\theta + \epsilon^3\psi_3(r)\cos 2\theta + \sim$$

aspect-ratio effect toroidal $\qquad$ elongation $\qquad$ triangularity

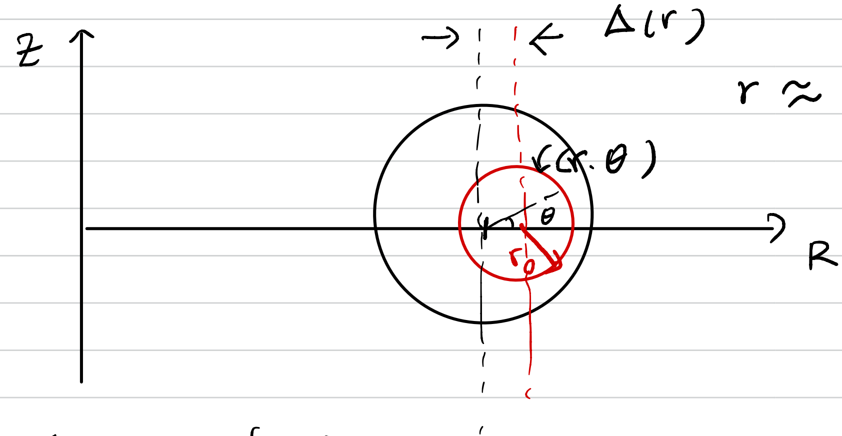
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\bar{\psi}_1}{dr}\right) - \frac{1}{r^2}\bar{\psi}_1 - \frac{1}{R_0}\frac{d\psi_0}{dr} = -2\mu_0 R_0 r\frac{dp}{d\psi_0} + \frac{d}{d\psi_0}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_0}{dr}\right)\right)\psi_1$$

$$\frac{d}{d\psi_0} = \frac{1}{R_0 B_\theta}\frac{d}{dr}$$

$$\boxed{\frac{d}{dr}\left[r B_\theta^2 \frac{d}{dr}\left(\frac{\tilde{\psi}_1}{B_\theta}\right)\right] = r B_\theta^2 - 2\mu_0 r^2 \frac{dp}{dr}}$$

toroidal correction equation

(3) toroidal shafranov shift : Δ



$r \approx r_0 + \Delta \cos\theta$

$\Delta \ll r_0$

Flux surface

$$\psi(r, \theta) \sim \psi_0(r) + \tilde{\psi}_1(r)\cos\theta$$

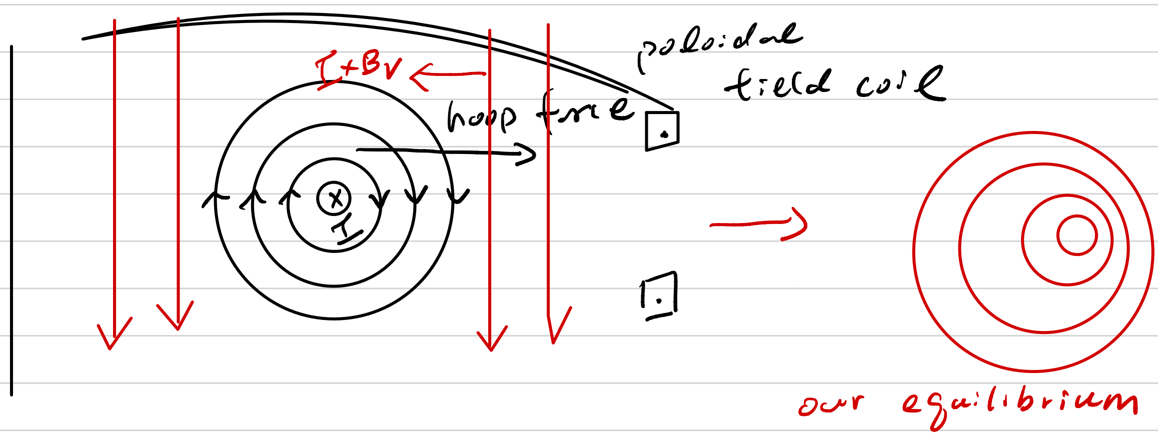
$$\sim \psi_0(r_0) + \psi_0'(r_0)\Delta\cos\theta + \tilde{\psi}_1(r_0)\cos\theta + \sim$$

$$\sim \text{const}$$

To always satisfy this

$$\boxed{\Delta(r) = -\frac{\tilde{\psi}_1(r)}{d\psi_0/dr} = -\frac{\tilde{\psi}_1(r)}{R_0 B_\theta(r)}}$$

(4) Vertical field for toroidal force balance



Integrate ϵ^1 equation up to $r \gg a$

minor radius of plasma

$$\int_0^r \frac{d}{dr}\left(r B_\theta^2 \frac{d}{dr}\left(\frac{\bar{\psi}_1}{B_\theta}\right)\right) dr = r B_\theta^2 \frac{d}{dr}\left(\frac{\bar{\psi}_1}{B_\theta}\right)$$

$$= \underbrace{-2\mu_0 \int_0^a r^2 \frac{dp}{dr} dr}_{①} + \underbrace{\int_0^a r B_\theta^2 dr}_{②} + \underbrace{\int_a^r r B_\theta^2 dr}_{③}$$

① $= a^2 B_{\theta a}^2 \beta_p$

② $= a^2 B_{\theta a}^2 \frac{l_i}{2}$ $\quad l_i$: normalized internal plasma inductance

$$l_i \equiv \frac{L_i / 2\pi R_0}{\mu_0 / 4\pi} = \frac{2 L_i}{\mu_0 R}$$

$$\frac{1}{2} L_i I^2 = \int \frac{B_\theta^2}{2\mu_0} d\vec{r}$$

③ (vacuum, $B_\theta = B_{\theta a} \frac{a}{r}$)

$$= a^2 B_{\theta a}^2 \ln\left(\frac{r}{a}\right)$$

$$\boxed{r B_\theta^2 \frac{d}{dr}\left(\frac{\bar{\psi}_1}{B_\theta}\right) = a^2 B_{\theta a}^2 \left(\beta_p + \frac{l_i}{2} + \ln\left(\frac{r}{a}\right)\right)}$$

now integrate this (a, r) in vacuum

$$\bar{\psi}_1(r) = \frac{1}{2} B_\theta \left[\left(\beta_p + \frac{l_i}{2} - \frac{1}{2}\right)(r^2 - a^2) + r^2 \ln\left(\frac{r}{a}\right) \right] + \bar{\psi}_1(a)$$

using $B_\theta = \frac{\mu_0 I}{2\pi r}$, $r \gg a$

$$\bar{\Psi}_1(r) = \frac{\mu_0 I}{4\pi}\left[\left(\beta_p + \frac{l_i}{2} - \frac{1}{2}\right) + \ln\frac{r}{a}\right] r + \bar{\Psi}_1(a)$$

$$= \begin{pmatrix}\text{flux}\\ \text{field}\end{pmatrix} \text{ by } I + \text{by vertical coil}$$

for $\bar{\Psi}_1(a) \propto \Delta(a) = 0$

magneto statics

$$= -\frac{\mu_0 I}{4\pi}\left[\ln\frac{8R_0}{r} - 1\right] r + \underline{B_V}\, r R_0$$

$$\boxed{B_V = \frac{\mu_0 I}{4\pi R_0}\left[\beta_p + \frac{l_i - 3}{2} + \ln\frac{8R_0}{a}\right]}$$

Shafranov (1966)