

* No class on next Monday (9/30).

Newton-Cotes formula

노트 제목

2019-09-25



$$I = \int_a^b f(x) dx = (b-a) \sum_{j=0}^n c_j^n f_j \quad \text{where } c_j^n = \frac{1}{b-a} \int_a^b L_j(x) dx$$

For n=1, $x_0 = a$, $x_1 = b$

$$L_0(x) = \frac{x-b}{a-b}, \quad L_1(x) = \frac{x-a}{b-a}$$

$$c_0^1 = \frac{1}{b-a} \int_a^b L_0(x) dx = \frac{1}{2}, \quad c_1^1 = \frac{1}{b-a} \int_a^b L_1(x) dx = \frac{1}{2}$$

$$I = (b-a) \left(\frac{1}{2} f_0 + \frac{1}{2} f_1 \right) = \boxed{(b-a) \left[\frac{1}{2} f(a) + \frac{1}{2} f(b) \right]}$$

trapezoidal rule

For $n=2$, $x_0 = a$, $x_1 = \frac{1}{2}(a+b)$, $x_2 = b \rightarrow C_0^2, C_1^2, C_2^2$

$$\rightarrow I = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

L_0, L_1, L_2

Simpson's rule

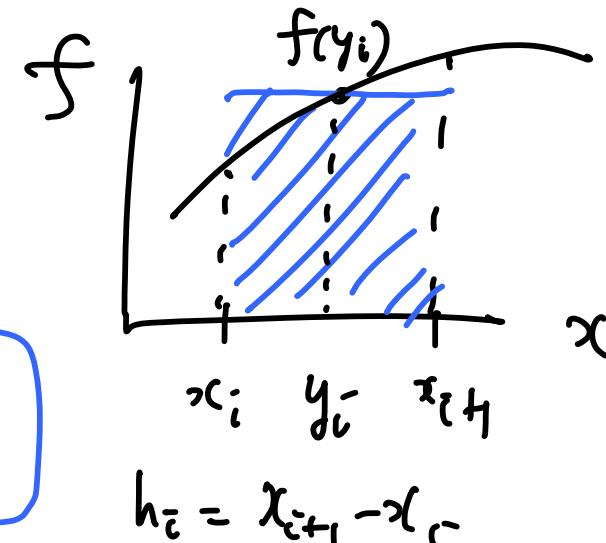
2nd
order
accuracy

n	N	NC_0^n	NC_1^n	NC_2^n	NC_3^n	NC_4^n	NC_5^n	NC_6^n	error accuracy
1	2	1	1						$8.3 \times 10^{-2} \Delta^3 f''$
2	6	1	4	1					$3.5 \times 10^{-4} \Delta^5 f'''$
3	8	1	3	3	1				$1.6 \times 10^{-6} \Delta^5 f'''$
4	90	7	32	12	32	7			$5.2 \times 10^{-7} \Delta^7 f^{(v)}$
5	288	19	25	50	50	25	19		$3.6 \times 10^{-7} \Delta^7 f^{(v)}$
6	840	41	216	27	212	27	216	41	$6.4 \times 10^{-10} \Delta^9 f^{(v,ii)}$

3.2 Error analysis

* Rectangle (or midpoint) rule

$$y_i = \frac{1}{2}(x_i + x_{i+1}) : \text{midpoint}$$



$$\int_{x_i}^{x_{i+1}} f(x) dx = (x_{i+1} - x_i) f(y_i) = h_i f(y_i)$$

$$f(x) = f(y_i) + (x - y_i) f'(y_i) + \frac{1}{2} (x - y_i)^2 f''(y_i) + \dots$$

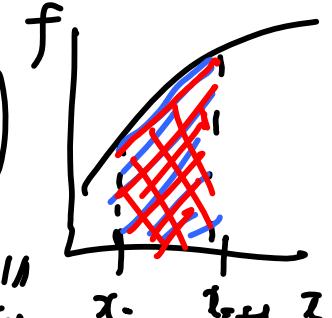
$$\int_{x_i}^{x_{i+1}} f(x) dx = f(y_i) h_i + \frac{1}{2} (x - y_i)^2 \Big|_{x_i}^{x_{i+1}} f'(y_i) + \frac{1}{6} (x - y_i)^3 \Big|_{x_i}^{x_{i+1}} f''(y_i) + \dots$$

$$= \boxed{f(y_i) h_i} + \underbrace{\frac{1}{24} h_i^3}_{\text{midpoint rule}} \cancel{\frac{3}{2}} f''(y_i) + \frac{1}{1920} h_i^5 f''''(y_i) + \dots \cancel{\text{OK}}$$

midpoint rule leading error

3rd-order accurate for one interval

For trapezoidal rule $\int_{x_i}^{x_{i+1}} f(x) dx = \frac{h_i}{2} [f(x_i) + f(x_{i+1})]$



$$\begin{aligned} f(x_i) &= f(y_i) + (x_i - y_i) f'(y_i) + \frac{1}{2} (x_i - y_i)^2 f''(y_i) + \frac{1}{6} (x_i - y_i)^3 f'''(y_i) \\ &= f(y_i) - \frac{h_i}{2} f'(y_i) + \frac{1}{8} h_i^2 f''(y_i) - \frac{1}{68} h_i^3 f'''(y_i) + \dots \end{aligned}$$

$$+ \underbrace{f(x_{i+1})}_{=} = f(y_i) + \frac{h_i}{2} \quad \dots \quad + \quad \dots \quad + \quad \dots \quad + \dots$$

$$\frac{1}{2} (f(x_i) + f(x_{i+1})) = f(y_i) + \frac{1}{8} h_i^2 f''(y_i) + \frac{1}{384} h_i^4 f'''(y_i) + \dots$$

$$\rightarrow f(y_i) = \frac{1}{2} [f(x_i) + f(x_{i+1})] - \frac{1}{8} h_i^2 f''(y_i) - \frac{1}{384} h_i^4 f'''(y_i) + \dots$$

Substitute this into $\textcircled{*}$

$$\rightarrow \int_{x_c}^{x_{c+1}} f(x) dx = \frac{1}{2} h_i [f(x_c) + f(x_{c+1})] - \frac{1}{12} h_i^3 f''(y_c) - \frac{1}{480} h_i^5 f''''(y_c) \dots$$

trapezoidal rule (TR)

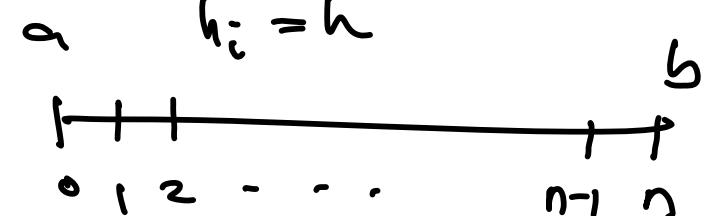
leading error

3rd-order accurate for one interval

The error of TR is twice bigger than that of MR.

- Global interval $[a, b]$

$$I = \int_a^b f(x) dx \quad \text{uniform spacing}$$



$$= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) dx = \sum_{i=0}^{n-1} \left[\frac{1}{2} h (f(x_i) + f(x_{i+1})) - \frac{1}{12} h^3 f''(y_i) - \frac{1}{480} h^5 f''''(y_i) \dots \right]$$

$$= \frac{h}{2} [f(a) + f(b) + 2 \sum_{i=1}^{n-1} f(x_i)] - \frac{1}{12} h^3 \sum_{i=0}^{n-1} f''(y_i) - \frac{1}{480} h^5 \sum_{i=0}^{n-1} f''''(y_i) \dots$$

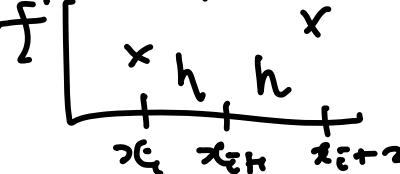
Mean value theorem $\sum_0^n f''(q_i) = n f''(\bar{x})$, where $a \leq \bar{x} \leq b$
 $\sum_0^n f^{iv}(q_i) = n f^{iv}(\xi)$ $a \leq \xi \leq b$

$$\begin{aligned}
 &= \boxed{\frac{h}{2} [f(a) + f(b) + 2 \sum_{i=1}^{n-1} f(x_i)]} - (b-a) \frac{h^2}{12} f''(\bar{x}) - (b-a) \frac{h^4}{480} f^{iv}(\xi) + \dots \\
 &\quad \text{leading error}
 \end{aligned}$$

2nd-order accurate for entire interval

\therefore TR is Second-order accurate.

Simpson's rule



$$\int_{x_i}^{x_{i+2}} f(x) dx = \frac{h}{3} [f(x_i) + 4f(x_{i+1}) + f(x_{i+2})] \equiv S(f)$$

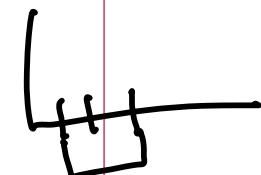
$$\int_{x_i}^{x_{i+2}} f(x) dx = \frac{2h}{3} [f(x_i) + f(x_{i+2})] \equiv T(f)$$

$$\int_{x_i}^{x_{i+2}} f(x) dx = (2h) f(\bar{x}_{i+1}) = R(f)$$

$$\rightarrow S(f) = \frac{2}{3} R(f) + \frac{1}{3} T(f)$$

Recall that the truncation error of TR is twice that of MR with opposite sign.

→ Simpson's rule is 4th order accurate for the entire interval.



$$\left. \begin{aligned} & \int_{x_i}^{x_{i+2}} = \frac{h}{3} (f_i + 4f_{i+1} + f_{i+2}) \\ & \int_{x_{i+2}}^{x_{i+4}} = \frac{h}{3} (f_{i+2} + 4f_{i+3} + f_{i+4}) \\ & \vdots \\ & \text{even number of panels} \quad \text{odd even} \\ & \text{add odd number of grid pts.} \quad \text{odd even} \end{aligned} \right\} \text{4th-order accurate}$$

$$\begin{aligned} I &= \int_a^b f(x) dx \\ &= \frac{h}{3} [f(a) + f(b) + 4 \sum_1^{n-1} f_j + 2 \sum_2^{n-2} f_j] \\ &\quad - \frac{h^4}{180} (b-a) f''(\bar{x}) + \dots \end{aligned}$$

3.3

TR w/ end correction

$$\int_{x_c}^{x_{i+1}} f(x) dx = \frac{h_i}{2} (f_i + f_{i+1}) - \frac{1}{12} h_i^3 f''(y_c) - \frac{1}{480} h_i^5 f^{IV}(y_c) + \dots$$

CD2 // $\frac{f'_{i+1} - f'_i}{h_i} - \frac{1}{6} \left(\frac{h}{2}\right)^2 f''(y_c)$ + --

$$= \frac{h_i}{2} (f_i + f_{i+1}) - \frac{1}{12} h_i^2 (f'_{i+1} - f'_i)$$

Sum over the entire domain. + $\frac{1}{120} h_i^5 f^{IV}(y_c) + \dots$

$$I = \int_a^b f(x) dx = \boxed{\frac{h}{2} \sum_0^n (f_c + f_{c+1}) - \frac{1}{12} h^2 (f'_{(b)} - f'_{(a)})} + \frac{1}{120} (b-a) h^5 f^{IV}(\xi) + \dots$$

6th-order accurate.

TR w/ end correction

ex) $f(x) = e^x \quad \int_0^4 e^x dx = e^4 - 1 = 53.59815 \dots$

4pts $I_{TR} = 54.71015 \quad \text{error} = -1.112$

$I_{SR} = 53.61622 \quad -0.01807$

$I_{TC} = 53.59352 \quad +0.00463 !$

3.4 Romberg integration and Richardson extrapolation

technique for obtaining an accurate sol. by combining two or more less accurate sols.

→ integral method + Richardson extrapolation.

$$\text{TR : } I = \int_a^b f(x) dx = \frac{h}{2} \left[f(a) + f(b) + 2 \sum_{j=1}^{n-1} f_j \right] + c_1 h^2 + c_2 h^4 + \dots$$

$\underbrace{\qquad\qquad\qquad}_{\tilde{I}}$

$$\rightarrow \tilde{I}_1 = I - c_1 h^2 - c_2 h^4 - \dots : \text{2nd-order accurate}$$

Apply TR w/ $h_1 = h/2 \rightarrow$ call this \tilde{I}_2

$$\rightarrow \tilde{I}_2 = I - c_1 \left(\frac{h}{2}\right)^2 - c_2 \left(\frac{h}{2}\right)^4 - \dots$$

$$\text{Idea : } 4\tilde{I}_2 - \tilde{I}_1 = 3I + \frac{3}{4}c_2 h^4 + \dots$$

$$\rightarrow \frac{4\tilde{I}_2 - \tilde{I}_1}{3} = I + \frac{1}{4}c_2 h^4 + \dots : 4^{\text{th}}\text{-order accurate}$$

Combined two 2nd-order estimates of $I \rightarrow 4^{\text{th}}\text{-order accurate estimate.}$

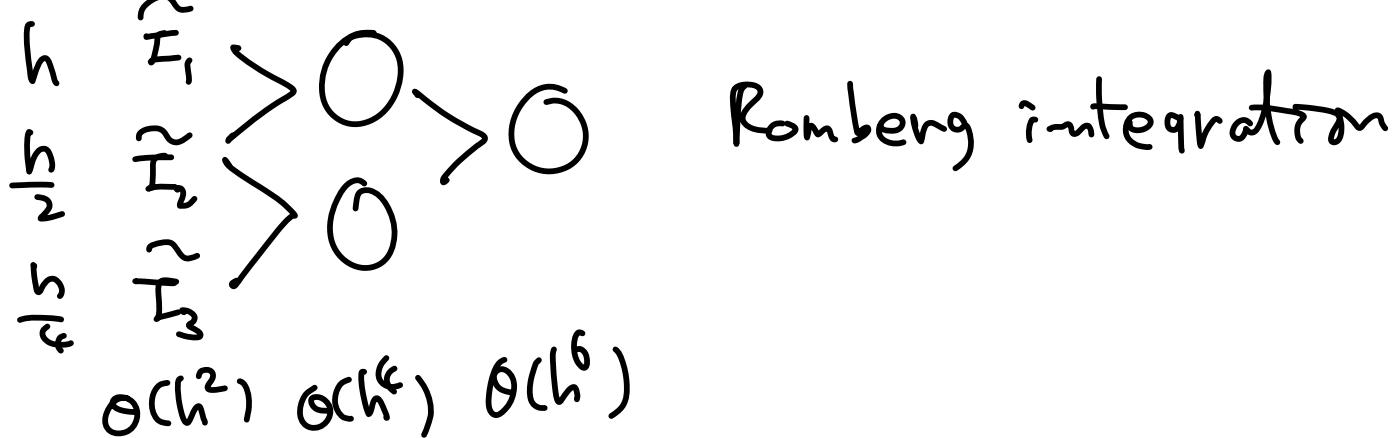
Evaluate I w/ $h_2 = h/4 \rightarrow \tilde{I}_3$

$$\tilde{I}_3 = I - c_1 \left(\frac{h}{4}\right)^2 - c_2 \left(\frac{h}{4}\right)^4 - \dots$$

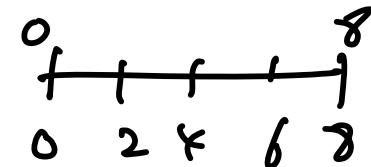
$$\rightarrow \frac{4\tilde{I}_3 - \tilde{I}_2}{3} = I + \frac{1}{64} c_2 h^4 + \frac{5}{1026} c_3 h^6 + \dots$$

$$\Rightarrow \frac{16}{15} \left(\frac{4\tilde{I}_3 - \tilde{I}_2}{3} \right) - \frac{1}{15} \left(\frac{4\tilde{I}_2 - \tilde{I}_1}{3} \right) = I + O(h^6) + \dots$$

: 6th-order accurate



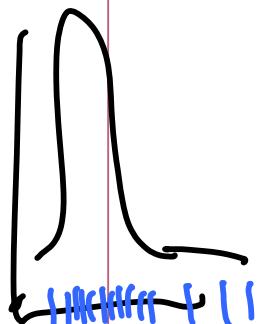
$$\text{Ex) } I = \int_0^8 \left(\frac{5}{8}x^4 - 4x^3 + 2x + 1 \right) dx = 72$$



$$\text{TR } I_1 = \frac{(8-0)}{2} [f(8) + f(0)] = 2120$$

$$h/2 \quad I_2 = \frac{(8-0)}{4} [f(8) + f(0) + 2f(4)] = 712$$

$$h/4 \quad I_3 = \frac{(8-0)}{8} [f(8) + f(0) + 2f(2) + 2f(4) + 2f(6)] = 240$$



$$\begin{array}{ccc} h & 2120 & \xrightarrow{\Delta} \frac{7120 - 2120}{3} = \frac{128}{3} \\ h/2 & 712 & \xrightarrow{\Delta} \dots = \frac{240}{3} \\ h/4 & 240 & \xrightarrow{\Delta} \dots = \frac{240}{3} \end{array}$$

$\propto h^6 f^{(5)}(\bar{x})$

restriction:

points are evenly distributed throughout the interval of integration.