Engineering Mathematics 2

Lecture 9

Yong Sung Park

Previously, we discussed

- Line integral

$$\oint_C f(z)\,dz = 0$$

- Cauchy's integral theorem

Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

Remark: Analytic fn is completely determined by its boundary values.

Show that Cauchy's integral formula is equivalent to the mean value theorem.

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0}$$

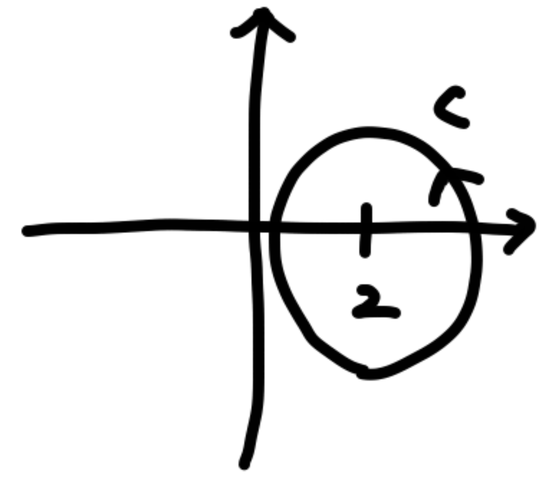
$z - z_0 = \rho e^{it}$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{\cancel{\rho e^{it}}} \cancel{i \rho e^{it}}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it})\, dt$$

Example:

$$\oint_C \frac{e^z}{z-2}\,dz$$



$$= 2\pi i\, e^{2}$$

Example: $\oint_C g(z)\,dz$

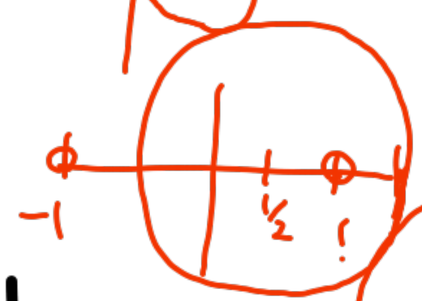
where $g(z) = \dfrac{z^2+1}{z^2-1}$ and C is

$$g(z) = \frac{z^2+1}{(z-1)(z+1)}$$

(a) $|z-1| = 1$

$$\left.\frac{z^2+1}{z+1}\right|_{z=1} = 1$$

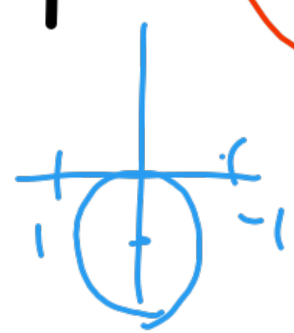
(b) $|z-\frac{1}{2}| = 1$



(c) $|z+1-\frac{i}{2}| = 1$

$$\left.\frac{z^2+1}{z-1}\right|_{-1} = -1$$

(d) $|z-i| = 1$



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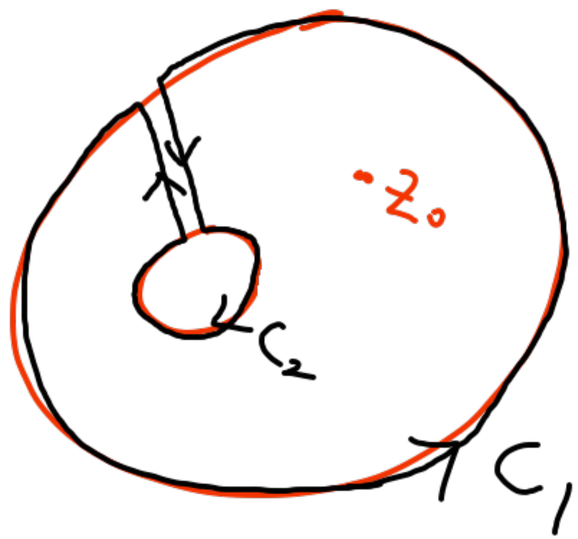
For multiply connected domain



$$C = C_1 + C_2$$

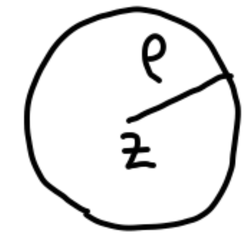
$$2\pi i f(z_0) = \oint_{C_1} \frac{f(z)\,dz}{z - z_0} + \oint_{C_2} \frac{f(z)\,dz}{z - z_0}$$

Derivatives of analytic fns

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)\,dt}{t - z}$$

$$\frac{d^n f}{dz^n} = \frac{n!}{2\pi i} \oint_C \frac{f(t)\,dt}{(t-z)^{n+1}}$$

Cauchy's inequality



$$\left|f^{(n)}(z)\right| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(t)\,dt}{(t-z)^{n+1}}\right|$$

$$\leq \frac{n!}{\cancel{2\pi}}\,\frac{M}{\rho^{n+\cancel{1}}}\,\cancel{2\pi}\cancel{\rho}$$

$$\left|f^{(n)}(z)\right| \leq \frac{n!\,M}{\rho^{n}}$$

Liouville's theorem

If an entire function is bounded in absolute value over whole complex plane, it is a constant.

Morera's theorem

If $f(z)$ is continuous in a simply connected domain D and $\oint_C f(z)dz = 0$ for every closed curve C in D, the $f(z)$ is analytic.