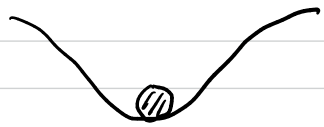
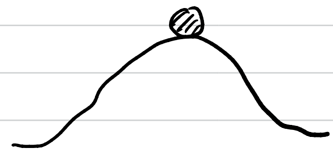
5/10 Stability assessment / Ideal MHD formulation

equilibrium $\vec{j} \times \vec{B} = \vec{\nabla} p + (\rho \vec{v} \cdot \vec{\nabla} \vec{v} + \cdots)$

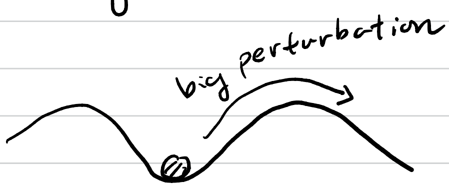
don't know yet if it's stable or not



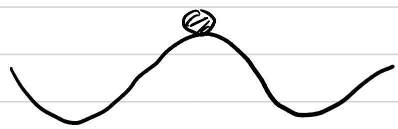
linearly stable



linearly unstable (Kink, ballooning tearing)



nonlinearly unstable (neoclassical tearing mode)



nonlinearly stable (micro turbulence)

* Formulation of linear Stability
  - Adds a small perturbation and see what happens
    $$\vec{A}(\vec{x},t) = \vec{A}_0(\vec{x},0) + \vec{A}_1(\vec{x},t) \quad A_0 \gg A_1$$
  - Assume exponential growth or damping
    $$\vec{A}_1(\vec{x},t) = A_1(\vec{x})\, e^{-i\omega t}$$
  - $\mathrm{Im}(\omega) > 0$ $\begin{cases} \text{Instability (Inhomogeneous media)} \\ \text{waves (Homogeneous media)} \end{cases}$

★ Ideal MHD waves

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\vec{\nabla}\rho + \rho(\vec{\nabla}\cdot\vec{u}) = 0 \qquad\qquad \omega\rho_1 = \rho_0(\vec{k}\cdot\vec{u}_1) \quad \text{(not used)}$$

$$\frac{\partial p}{\partial t} + \vec{u}\cdot\vec{\nabla}p + \gamma p(\vec{\nabla}\cdot\vec{u}) = 0 \qquad\qquad \omega p_1 = \gamma p_0(\vec{k}\cdot\vec{u}_1) \quad (1)$$

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{u}\times\vec{B}) \qquad\qquad \omega\vec{B}_1 = -\vec{k}\times(\vec{u}_1\times\vec{B}_0) \quad (2)$$

$$\mu_0\vec{J} = \vec{\nabla}\times\vec{B} \qquad\qquad \omega\mu_0\vec{J}_1 = -i\vec{k}\times(\vec{k}\times(\vec{u}_1\times\vec{B}_0)) \quad (3)$$

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = \vec{J}\times\vec{B} - \vec{\nabla}p \qquad\qquad \omega\rho_0\vec{u}_1 = i(\vec{J}_1\times\vec{B}_0) + \vec{k}p_1 \quad (4)$$

differential eq $\longrightarrow$ Algebraic eq.

Assume homogeneous media, $\vec{B}_0 = B_0\hat{z}$

$$\vec{J}_0 = 0,\quad \vec{u}_0 = 0,\quad \vec{\nabla}p_0 = 0,\quad \vec{\nabla}\rho_0 = 0$$

Let $A_1(\vec{x},t) = A_1 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

note, $\frac{\partial}{\partial t} \to -i\omega \quad \vec{\nabla} \to i\vec{k}$ for perturbed quantities.

also note, $\vec{u}\cdot\vec{\nabla}\vec{A} = \underbrace{\cancel{\vec{u}_0\cdot\vec{\nabla}}}_{\vec{u}_0=0}\vec{A}_1 + \underbrace{\cancel{\vec{u}_1\cdot\vec{\nabla}\vec{A}_0}}_{\text{homogeneity}} = 0$

Then combine (1) ~ (3) into (4)

$$\omega\vec{B}_1 = -\vec{u}_1(\vec{k}\cdot\vec{B}_0) + \vec{B}_0(\vec{k}\cdot\vec{u}_1)$$

$$\omega\mu_0\vec{J}_1 = -i(\vec{k}\times\vec{u}_1)(\vec{k}\cdot\vec{B}_0) + i(\vec{k}\times\vec{B}_0)(\vec{k}\cdot\vec{u}_1)$$

$$\omega\mu_0\vec{J}_1\times\vec{B}_0 = -i\left[\vec{u}_1(\vec{k}\cdot\vec{B}_0) - \vec{k}(\vec{u}_1\cdot\vec{B}_0)\right](\vec{k}\cdot\vec{B}_0)$$
$$+\, i\left[\vec{B}_0(\vec{k}\cdot\vec{B}_0) - \vec{k}B_0^2\right](\vec{k}\cdot\vec{u}_1)$$

(4) $\longrightarrow$

$$\omega^2\rho_0\mu_0\vec{u}_1 = \left[\vec{u}(\vec{k}\cdot\vec{B}_0) - \vec{k}(\vec{u}_1\cdot\vec{B}_0)\right](\vec{k}\cdot\vec{B}_0)$$
$$-\left[\vec{B}_0(\vec{k}\cdot\vec{B}_0) - \vec{k}B_0^2\right](\vec{k}\cdot\vec{u}_1) + \mu_0\gamma p_0\vec{k}(\vec{k}\cdot\vec{u}_1)$$

Let $V_A^2 = \dfrac{B_0^2}{\mu_0 \rho_0}$ $\quad$ $C_S^2 = \dfrac{\gamma P_0}{\rho_0}$

Alfven $\quad$ Sound

$(4)/B_0^2 \rightarrow (\omega^2 - k_\parallel^2 V_A^2)\vec{u}_1$

$= \vec{k}_\perp (\vec{k}\cdot\vec{u}_1) V_A^2 - \vec{k}(k_\parallel u_{\parallel 1}) V_A^2 + \vec{k}(\vec{k}\cdot\vec{u}_1) C_S^2$

Let $\begin{cases} \vec{k} = k_\perp \hat{y} + k_\parallel \hat{z} & \text{w/o loss of generality} \\ \vec{u}_1 = u_{1x}\hat{x} + u_{1y}\hat{y} + u_{1z}\hat{z} \end{cases}$

$(\omega^2 - k_\parallel^2 V_A^2) u_{1x} = 0$

$(\omega^2 - k_\perp^2 C_S^2 - k^2 V_A^2) u_{1y} - (k_\perp k_\parallel C_S^2) u_{1z} = 0$

$-(k_\perp k_\parallel C_S^2) u_{1y} + (\omega^2 - k_\parallel^2 C_S^2) u_{1z} = 0$

$\omega^2 = k_\parallel^2 V_A^2$ $\quad$ shear Alfven

$\omega^2 = \frac{1}{2} k^2 (V_A^2 + C_S^2)(1 \pm (1-\alpha^2)^{\frac{1}{2}})$ $\quad$ magnetosonic

where, $\alpha^2 = 4\dfrac{k_\parallel^2}{k^2} \cdot \dfrac{C_S^2 V_A^2}{(C_S^2 + V_A^2)^2}$

— $0 < \alpha^2 < 1 \rightarrow \omega^2 > 0$

$\rightarrow \mathrm{Im}(\omega) = 0 \rightarrow$ no instability growth or damping

$\rightarrow$ expected since no free energy source

* magnetosonic wave for low β

$$\beta \sim \frac{C_s^2}{V_A^2} \ll 1$$

$$\boxed{\begin{array}{lll} \omega^2 \approx (K_\perp^2 + K_\parallel^2) V_A^2 & \text{fast.} & \text{compressional Alfven} \\ \omega^2 \approx K_\parallel^2 C_s^2 & \text{slow.} & \text{Sound} \end{array}}$$

(I) Shear Alfven wave

$\omega^2 = K_\parallel^2 V_A^2$ Independent of $K_\perp$

even if $K_\perp \gg K_\parallel$

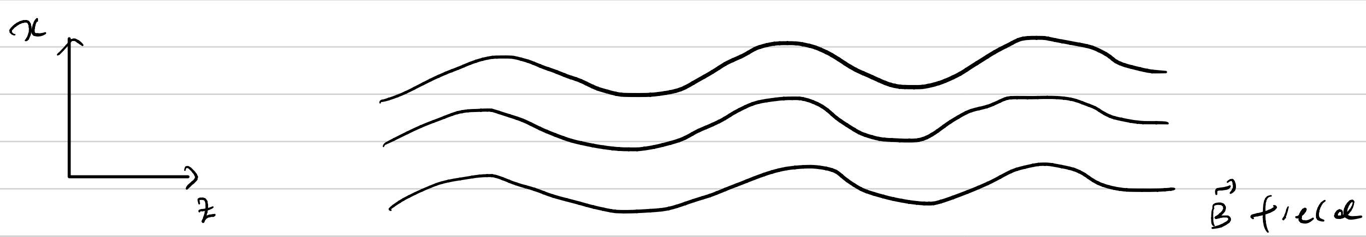
$u_{1x} \neq 0$, $u_{1y} = u_{1z} = 0$ $\quad \vec{\nabla} \cdot \vec{u}_1 = 0 \quad \vec{K} \cdot \vec{u}_1 = 0$

$\rho_1 = P_1 = 0$

note $\quad \omega \vec{B}_1 = -\vec{u}_1 (\vec{K} \cdot \vec{B}_0) + \vec{B}_0 \cancel{(\vec{K} \cdot \vec{u}_1)}$

$B_{1x} \propto u_1 \quad B_{1y} = B_{1z} = 0$

$\vec{u}_1, \vec{B}_1 \perp \vec{K}$ transverse wave



- perpendicular kinetic energy

$\leftrightarrow$ magnetic bending energy

- most dangerous

$$B^2 = (B_0 + B_1)^2 = B_0^2 + \cancel{2\vec{B}_0 \cdot \vec{B}_1} + B_1^2$$

$\rightarrow$ easy to excite

$$|\vec{u}_1| \sim \frac{|\vec{B}_1|}{|\vec{K} \cdot \vec{B}_0|} \qquad \vec{K} \cdot \vec{B}_0 \to 0 \qquad |\vec{B}_1| \to \quad \vec{u}_1 \uparrow \infty$$

(II) compressional Alfven wave

$\omega^2 \approx (k_\perp^2 + k_\parallel^2) V_A^2$ for $\beta \ll 1$

$u_{1x} \simeq 0$, but $u_{1y} \neq 0$ $u_{1z} \neq 0$

$B_{1x} \simeq 0$ $\qquad p_1 \neq 0$, $\rho_1 \neq 0$

- but $\mu_0 p_1 / B_0 B_{1z} \ll 1$

mostly due to compression of magnetic field, not plasma

- note typically $k_\perp \gg k_\parallel$ ($\because 2\pi R_0 \gg 2a$)

major radius minor radius

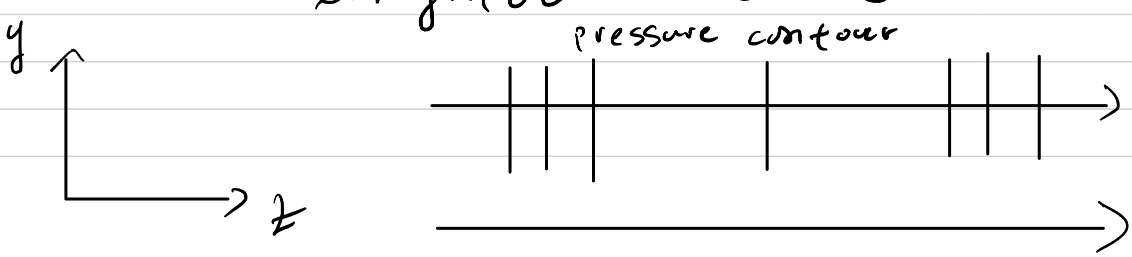
- $u_{1y} \gg u_{1z}$

nearly transverse wave

(III) Sound wave

$\omega^2 = k_\parallel^2 C_s^2$, $u_{1x} \simeq 0$, $u_{1y} \simeq 0$, $\vec{b}_1 \simeq 0$

longitudinal wave



$\star$ $\boxed{k_\perp^2 V_A^2 \gg k_\parallel^2 V_A^2 \gg k_\parallel^2 C_s^2}$

compressional Alfven   Shear Alfven   Sound

Ideal MHD equilibrium / Stability is as fast as at least sound wave

mostly Alfven time scale

5/12 Ideal MHD Stability / energy principle

1. For general geometry, Ideal MHD with $\vec{u}_0 = 0$

If we express $\vec{u}_1 = \frac{\partial \vec{\xi}}{\partial t}$, where $\vec{\xi}(\vec{x}, t)$ plasma displacement

$$\frac{\partial \rho_1}{\partial t} + \vec{u}_1 \cdot \vec{\nabla} \rho_0 + \rho_0 (\vec{\nabla} \cdot \vec{u}_1) = 0$$

$$\frac{\partial P_1}{\partial t} + \vec{u}_1 \cdot \vec{\nabla} P_0 + \gamma P_0 (\vec{\nabla} \cdot \vec{u}_1) = 0$$

$$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla} \times (\vec{u}_1 \times \vec{B}_0)$$

$$\mu_0 \vec{J}_1 = \vec{\nabla} \times \vec{B}_1$$

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} = \vec{J}_1 \times \vec{B}_0 + \vec{J}_0 \times \vec{B}_1 - \vec{\nabla} P_1 \quad (4)$$

$$\rho_1 = -\vec{\xi} \cdot \vec{\nabla} \rho_0 - \rho_0 (\vec{\nabla} \cdot \vec{\xi}) \quad (\text{not used})$$

$$P_1 = -\vec{\xi} \cdot \vec{\nabla} P_0 - \gamma P_0 (\vec{\nabla} \cdot \vec{\xi}) \quad (1)$$

$$\vec{B}_1 = \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0) \quad (2)$$

$$\mu_0 \vec{J}_1 = \vec{\nabla} \times \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0) \quad (3)$$

(1) ~ (3) into (4)

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{1}{\mu_0} \left[\vec{\nabla} \times \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0) \right] \times \vec{B}_0 + \frac{1}{\mu_0} \left[(\vec{\nabla} \times \vec{B}_0) \times \vec{\nabla} \times (\vec{\xi} \times \vec{B}_0) \right] + \vec{\nabla} \left(\vec{\xi} \cdot \vec{\nabla} P_0 + \gamma P_0 (\vec{\nabla} \cdot \vec{\xi}) \right)$$

$$\equiv \vec{F}(\vec{\xi}) \quad : \text{ideal MHD force operator}$$

note one can no longer assume $\vec{\xi} \propto e^{i(\vec{k} \cdot \vec{x} - \omega t)}$ (x)

but still can assume $\vec{\xi} = \vec{\xi}(\vec{x}) e^{-i\omega t}$

$$\omega^2 \rho \vec{\xi} = -\vec{F}(\vec{\xi}) \quad : \text{normal mode equation}$$

→ numerically more amenable than I.V.P

→ Still too much information

→ works only for the system unstable

→ typically, we just want know if the equilibrium is stable or not.

2. Energy principle

(1) note $\vec{F}(\vec{\xi})$ is self-adjoint. meaning

$$\int \vec{\eta} \cdot \vec{F}(\vec{\xi})\, d\vec{x} = \int \vec{\xi} \cdot \vec{F}(\vec{\eta})\, d\vec{x}$$

arbitrary trial fcns.

- expected for conserved system
- can be directly proved with $\vec{F}(\vec{\xi})$

(2) Self-adjointness means

(a) ω^2 is real, ω is purely $\left\langle \begin{array}{l} \text{real} \\ \text{imaginary} \end{array}\right.$

$$\because \quad \omega^2 \int \rho_0 |\vec{\xi}|^2 d\vec{x} = -\int \vec{\xi}^* \cdot \vec{F}(\vec{\xi})\, d\vec{x}$$

$$(\omega^2)^* \int \rho_0 |\vec{\xi}|^2 d\vec{x} = -\int \vec{\xi} \cdot \vec{F}(\vec{\xi}^*)\, d\vec{x}$$

$$\longrightarrow \quad (\omega^2) = (\omega^2)^*$$

$\omega = 0$ at marginal point

(not $Re(\omega) \neq 0,\ Im(\omega) = 0$)

(b) Discrete normal modes are orthogonal.

(weight fcn. ρ_0)

Let $\omega_m^2 \neq \omega_n^2$

$$\int \vec{\xi_n} \cdot \left(-\omega_m^2 \rho_0 \vec{\xi_m} = \vec{F}(\vec{\xi_m}) \right) d\vec{x}$$

$$\int \vec{\xi_m} \cdot \left(-\omega_n^2 \rho_0 \vec{\xi_n} = \vec{F}(\vec{\xi_n}) \right) d\vec{x}$$

$$(\omega_m^2 - \omega_n^2) \int \rho_0 \vec{\xi_m} \cdot \vec{\xi_n}\, d\vec{x} = 0$$

$$\rightarrow \int \rho_0 \vec{\xi_m} \cdot \vec{\xi_n}\, d\vec{x} = 0$$

(3) Variational formulation.

note $\omega^2 = \dfrac{-\frac{1}{2}\int \vec{\xi}^* \cdot \vec{F}(\vec{\xi})\, d\vec{x}}{\frac{1}{2}\int \rho |\vec{\xi}|^2 d\vec{x}} = \dfrac{\delta W(\vec{\xi}^*, \vec{\xi})}{K(\vec{\xi}^* \cdot \vec{\xi})}$

$$\boxed{\text{Then } \vec{\xi}(\vec{x}) \text{ makes } \begin{pmatrix}\text{maximum}\\ \text{minimum}\end{pmatrix} \omega^2 . \quad \text{is an eigenfctn of normal mode problem}}$$

$\because$ At extremum,

$\vec{\xi} \to \vec{\xi} + \delta\vec{\xi}$ then $\omega^2 \to \omega^2 + \cancel{\delta\omega^2}$

$$\omega^2 + \delta\omega^2 = \frac{\delta W(\vec{\xi}^*, \vec{\xi}) + \delta W(\delta\vec{\xi}^*, \vec{\xi}) + \delta W(\vec{\xi}^*, \delta\vec{\xi}) + \mathcal{O}(\delta^2)}{K(\vec{\xi}^*, \vec{\xi}) + K(\delta\vec{\xi}^*, \vec{\xi}) + K(\vec{\xi}^*, \delta\vec{\xi}) + \mathcal{O}(\delta^2)}$$

$$0 = \delta\omega^2 \sim \frac{\delta W(\delta\vec{\xi}^*, \vec{\xi}) + \delta W(\vec{\xi}^*, \delta\vec{\xi}) - \omega^2\left(K(\delta\vec{\xi}^*, \vec{\xi}) + K(\vec{\xi}^*, \delta\vec{\xi})\right)}{K(\vec{\xi}^*, \vec{\xi})}$$

$$\int d\vec{x} \left\{ \delta\vec{\xi}^* \cdot \left[\vec{F}(\vec{\xi}) + \omega^2 \rho \vec{\xi}\right] + \delta\vec{\xi} \cdot \left[\vec{F}(\vec{\xi}^*) + \omega^2 \rho \vec{\xi}^*\right] \right\} = 0$$

hold if $\omega^2 \rho \vec{\xi} = -\vec{F}(\vec{\xi})$

Now, you can just minimize

$$\omega^2 = \frac{\delta W(\vec{\xi}^*, \vec{\xi})}{K(\vec{\xi}^*, \vec{\xi})}$$

to find the most unstable mode

(work when really unstable
not when stable due continuum mode
(Goedbloed & Poedts (2004))

* since one can always K
without changing δW through $\xi_{\parallel}$

(4) Energy principle (Bernstein '58)

$$\boxed{\begin{array}{c}\text{Equilibrium is stable iff } \delta W(\vec{\xi}^*, \vec{\xi}) > 0 \\ \text{for all allowable trial fcns } \vec{\xi} \\ \text{(even if } \vec{\xi} \text{ is not an eigenfcn)}\end{array}}$$

If $\vec{\xi}$ making $\delta W < 0$ exists, it's unstable

∵ If the modes are discrete

any $\vec{\xi} = \sum_n a_n \vec{\xi}_n$

$\delta W(\vec{\xi}, \vec{\xi}^*) = \frac{1}{2} \sum_n |a_n|^2 \omega_n^2$

$\delta W < 0$ means at least one $\omega_n < 0$

∵ Complete proof by Laval '65,

(Freidberg ch 8)

Now one can just minimize δW

3. Standard form of $\delta W(\vec{\xi}^*, \vec{\xi})$ $\quad \vec{\eta} = \vec{\xi}^*$

$$\delta W = -\frac{1}{2}\int \vec{\eta} \cdot \Big[\underbrace{\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}_1)\times\vec{B}}_{1} + \underbrace{\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}_1}_{3} + \vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}P + \underbrace{\gamma\vec{\nabla}\cdot\vec{\xi}}_{2})\Big]\,d\vec{x}$$

Let $\vec{\xi} = \xi_{\parallel}\hat{b} + \vec{\xi}_{\perp}$ $\quad \vec{B}_1 = \vec{\nabla}\times(\vec{\xi}_{\perp}\times\vec{B})$

1 $\int \vec{\eta}\cdot(\vec{\nabla}\times\vec{B}_1)\times\vec{B}_0$

$= \int (\vec{\nabla}\times\vec{B}_1)\cdot(\vec{B}\times\vec{\eta}_{\perp})$

$= \int \vec{\nabla}\cdot\left(\vec{B}_1\times(\vec{B}\times\vec{\eta}_{\perp})\right) + \int \vec{B}_1\cdot\vec{\nabla}\times(\vec{B}\times\vec{\eta}_{\perp})$

$= \int\left(\vec{B}_1\times(\vec{B}\times\vec{\eta}_{\perp})\right)\cdot d\vec{S} - \int \vec{B}_1(\vec{\xi}_{\perp})\cdot\vec{B}_1(\vec{\eta}_{\perp})$

$= -\int \vec{\eta}_{\perp}\cdot d\vec{S}\,(\vec{B}\cdot\vec{B}_1(\vec{\xi}_{\perp})) - \int \vec{B}_1(\vec{\xi}_{\perp})\cdot\vec{B}_1(\vec{\eta}_{\perp})$

② $\int \vec{\eta}_\perp \cdot d\vec{S} \left(\gamma p \vec{\nabla}\cdot\vec{\xi}\right) - \int \gamma p (\vec{\nabla}\cdot\vec{\xi})(\vec{\nabla}\cdot\vec{\eta})$

③ $\vec{B}\cdot\left[\vec{j}\times B_1 + \vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p)\right] = 0$ ✓

①+②+③ leads to, $\vec{\eta}_\perp = \vec{\xi}_\perp^*$

$\delta W = \delta W_F + \delta W_S$

$$\delta W_F = \frac{1}{2}\int \frac{1}{\mu_0}|B_1|^2 + \gamma p|\vec{\nabla}\cdot\vec{\xi}|^2 - \vec{\xi}_\perp^*\cdot\vec{j}\times\vec{B}_1 + (\vec{\nabla}\cdot\vec{\xi}_\perp^*)(\vec{\xi}_\perp\cdot\vec{\nabla}p)\, d\vec{x}$$

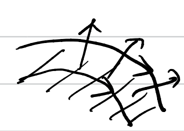
$$\delta W_S = \frac{1}{2}\int \vec{\xi}^*\cdot d\vec{S}\left[\frac{1}{\mu_0}\vec{B}\cdot\vec{B}_1 - \gamma p\vec{\nabla}\cdot\vec{\xi} - \vec{\xi}_\perp\cdot\vec{\nabla}p\right] dS$$

4. Intuitive form, Let $\vec{Q} \equiv \vec{B}_1$

$\vec{Q} = \vec{Q}_\perp + Q_\parallel \hat{b}$, $\vec{j} = \vec{j}_\perp + j_\parallel \hat{b}$

$$\delta W_F = \frac{1}{2\mu_0}\int \Big\{ \underbrace{|\vec{Q}_\perp|^2}_{①} + \underbrace{B^2|\vec{\nabla}\cdot\vec{\xi}_\perp + 2\vec{\xi}_\perp\cdot\vec{\kappa}|^2}_{②} + \underbrace{\mu_0\gamma p|\vec{\nabla}\cdot\vec{\xi}|^2}_{③} - \underbrace{2\mu_0(\vec{\xi}_\perp\cdot\vec{\nabla}p)(\vec{\xi}_\perp^*\cdot\vec{\kappa})}_{④} - \underbrace{\mu_0 j_\parallel(\vec{\xi}_\perp^*\times\hat{b})\cdot\vec{Q}_\perp}_{⑤} \Big\}\, d\vec{x}$$

① : Shear Alfven
② : Compressional Alfven
③ : Sound wave
④ : pressure-driven instability
⑤ : current-driven instability



* minimization of δW → $(\vec{\nabla}\cdot\vec{\xi}) = 0$
by choosing $\xi_\parallel$

5. Self-adjoint form
read Freidberg ch. 8