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457.646 Topics in Structural Reliability
In-Class Material: Class 01

I. Introduction

◎ Uncertainties in Engineering

① **( ) :** Inherent randomness (or physical fluctuation)

e.g. earthquake intensity (PGA, PGV, …), wind velocity, maximum flow rate

⇒ ( ) be reduced

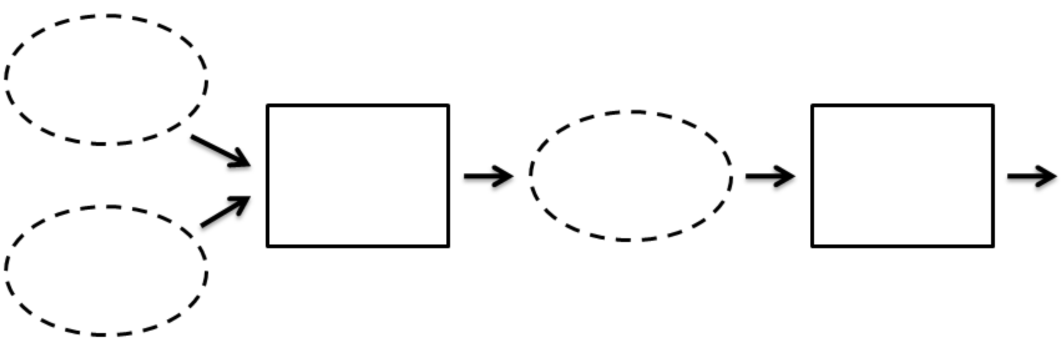
② **( ) :** uncertainty due to insufficient ( )

- ( ) uncertainty: imperfect or simplified model (e.g. 3D→2D)
  missing variables or effects
- ( ) uncertainty: insufficient data
  e.g. "sample mean is not the true mean"

⇒ ( ) be reduced by investing more in knowledge and data

Der Kiureghian, A., and O. Ditlevsen (2009). Aleatory or epistemic? Does it matter? *Structural Safety*, **31**: 105-112

◎ Uncertainty, Risk and Decisions



Decision making under ( ) leads to ( )
Need to quantify ( ) caused by ( )

◎ 457.646 Topics in Structural Reliability (Theory)

- Focus: methods for quantifying risk & applications
- Provide overview and applications of " " reliability methods

  ⇒ The word " " does not refer to physical structures (buildings and bridges, …)

  ⇒ in an ( ) & ( ) manner

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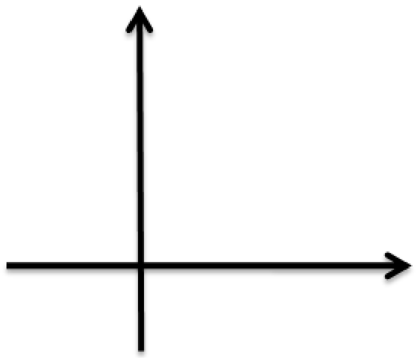
◎ Part 2: Basic theory of probability & statistics (≤ 3 weeks) (ref. A&T textbook)

◎ Part 3: Structural Reliability Analysis (SRA) - Component

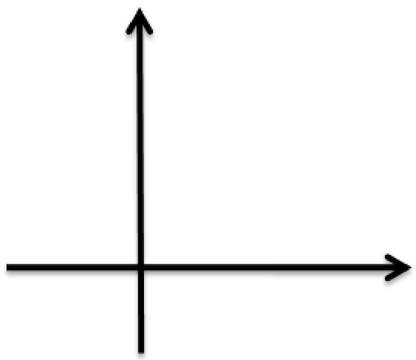


$P_f =$

- Reliability index: $\beta_{MVFOSM}, \beta_{HL}$

- Reliability methods: FORM, SORM, etc. (how to integrate ↖)

◎ Part 4: Structural Reliability Analysis (SRA) - System



$P_f =$

- Reliability methods developed to handle system failure domains

  : "System" reliability methods

◎ Part 5: Structural Reliability under Epistemic Uncertainty

$$P_f = \int_{g(\mathbf{x};\ )\le 0} f_{\mathbf{x}}(\mathbf{x};\ )\, d\mathbf{x}$$

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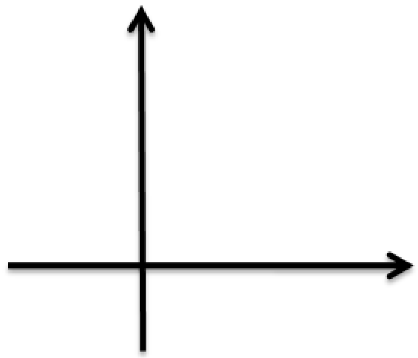
◎ Part 6 : Simulation Methods



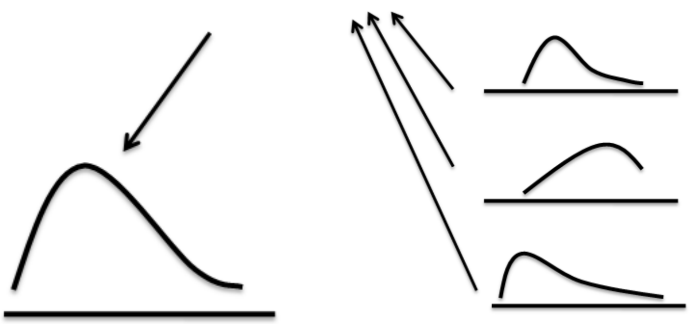
$P_f =$

⇒ Monte Carlo simulations

⇒ Efficient Sampling methods

◎ Part 7: Uncertainty Quantification

$Y = g(\mathbf{x})$



input distributions

Find distributions of output

◎ Part 8: Applications

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II. Basic theory of Probability and Statistics

1. Set Theory

Why do we need ‘set theory’ in uncertainty analysis?

- **Uncertainty:** a (   ) of possible (   )

  e.g.   toss a coin

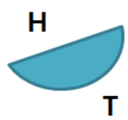
  roll a dice

  weight of a car

- **Probability:** numerical measure of the (   ) of an event (i.e. a group of outcomes) of interest (   ) the other possible outcomes

e.g. “unfair coin”

H:T=   P(H)=



- Uncertainty analysis starts with (   ) the collection of all possible outcomes
- Principles of set theory are essential tool for this task.

2. Definitions

(a) **Sample space** (   ): the set of (   ) possible outcomes

Sample point (   ): an (   ) outcome

e.g.

Criteria	Sample space	Examples
Continuous?	“Discrete”: ( ) quantities	# of typhoons at city A in a year S={ }
	“Continuous”: ( ) quantities	% of congested traffic in Seoul S={ }
Can count sample points?	“Finite” : ( ) ( ) and ( )	S = { }
	“Infinite” : ( ) ( ) or ( )	S = { } S = { }

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(b) **Event** ( ): any collection of sample ( ) or any ( ) of sample space

e.g. Baseball: outcomes of each “at-bat”

S=

discrete or continuous?

infinite or finite?

“A hitter reaches a base”

E=

(c) Some notable events

- ( ) event: E=

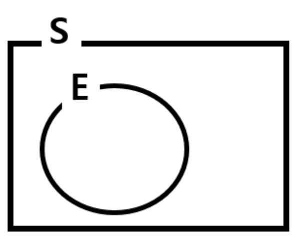
  - Occurs with certainty

- ( ) event: E=

  - cannot occur

- **Complementary** event of E : ( ) or ( )

  - An event that contains ( ) the sample points that are ( ) in E



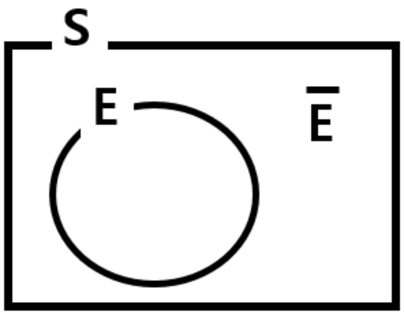
- e.g. “at-bat” outcomes

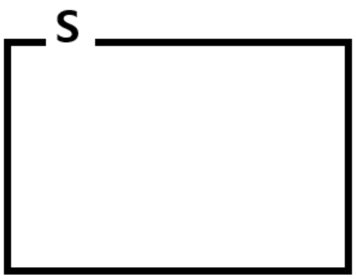
  E: “a hitter reaches a base”

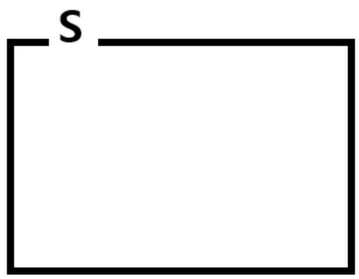
  $\overline{E} =$

- e.g. $\overline{S} =$, $\overline{\phi} =$

(d) **Venn diagram**: (   ) & (   ) representation of the sample space, sample points and events







* GUI-based interactive learning tools for Venn diagrams (and other statistical concepts) are available at http://www.stat.berkeley.edu/~stark/Java/Html/

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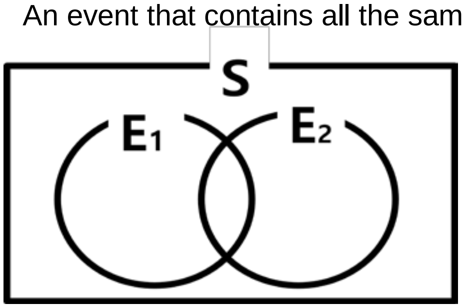
457.646 Topics in Structural Reliability
In-Class Material: Class 02

◎ Set Operations → useful for ( ) reliability analysis

① **"Union" of events:** $E_1 \quad E_2$

- An event that contains all the sample points that are in $E_1 \quad E_2$



e.g., Concrete mixing

- E_1: shortage of water
- E_2: shortage of sand
- E_3: shortage of gravel
- E_4: shortage of cement

E (concrete can't be produced) $=$

$= \quad E_i$

e.g., Wind

- E_1: blown off due to pressure
- E_2: missile-like flying objects

$E = E_1 \quad E_2$

e.g., Bridge pier under EQ

- E_1: reaches displacement capacity
- E_2: reaches shear capacity

$E = E_1 \quad E_2$

※ $A \cup S =$

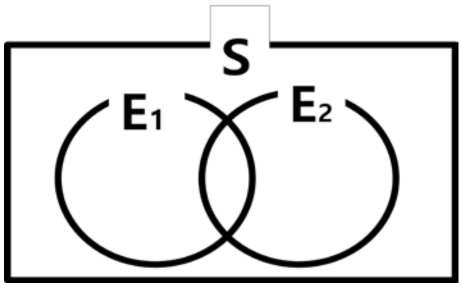
$A \cup \phi =$

$A \cup A =$

If $A \subset B$, then $A \cup B =$

② **"intersection" of events E_1 E_2 or**

: an event that contains all the sample points that are both in E_1 E_2



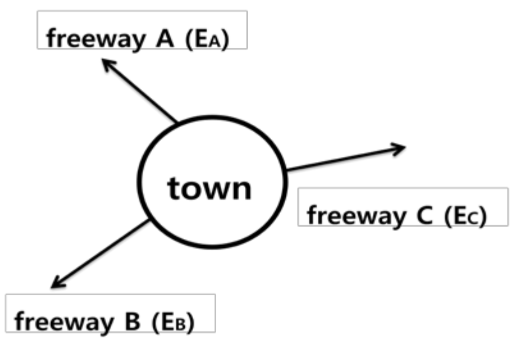
※ $A \cdot S =$

$A \cdot \phi =$

$A \cdot A =$

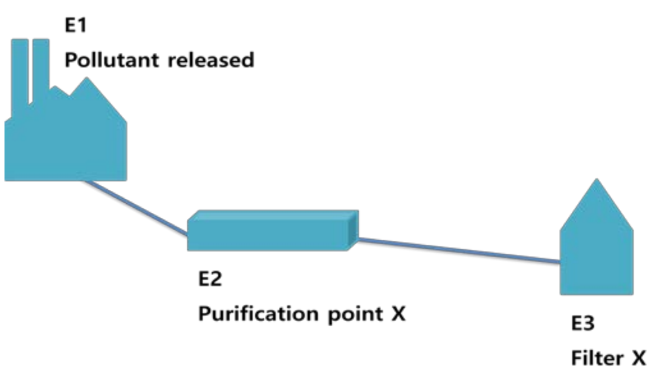
If $A \subset B$, then $AB =$

e.g.,



No evacuation by freeway $E =$

e.g.,



Exposed to pollutant $E =$

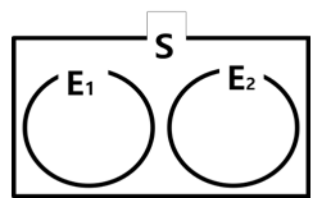
◎ Operation Rules

Commutative Rule	$E_1 \cup E_2 =$ $E_1 E_2 =$
Associative Rule	$(E_1 \cup E_2) \cup E_3 = \qquad\qquad =$ $(E_1 E_2) E_3 = \qquad\qquad =$
Distributive Rule	$(E_1 \cup E_2) E_3 =$ $(E_1 E_2) \cup E_3 =$
De Morgan's Rule	$\overline{(\bigcup_{i=1} E_i)} =$ $\overline{(\bigcap_{i=1} E_i)} =$

◎ Relationship between events

① **Mutually Exclusive events:** $E_1 E_2 =$

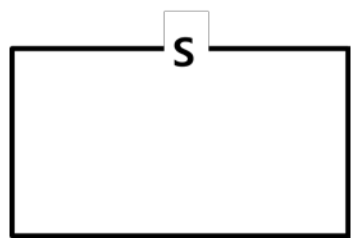
- Cannot occur together



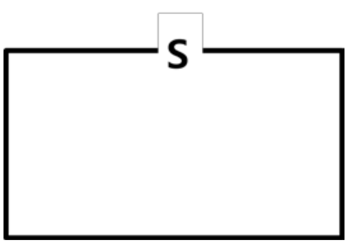
- e.g. E_1 and $\overline{E_1}$
- $E_1 \cdots E_n$ and $\overline{E_i}$, $i \in \{1, \cdots, n\}$

② **Collectively Exhaustive events:** $\bigcup_{i=1}^{n} E_i =$

- The union constitutes the sample space



※ **MECE:**



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2. Mathematics of Probability (measure of likelihood of event)

◎ Four approaches for assigning probability of events

Approach	Description	Example : Prob. (a “Yut” stick shows the flat side)
Notion of **Relative Frequency**	Relative frequency based on empirical data, Prob. = (# of occurrences) / (# of observations)	
On a **Priori** Basis	Derived based on elementary assumptions on likelihood of events	
On **Subjective** Basis	Expert opinion (“degree of belief”)	
Based on **Mixed** Information	Mix the information above to assign probability	

◎ Axioms of Probability

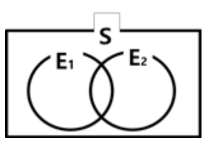
“Axioms”: Statements or ideas which people <u>accept</u> as being the foundation of theory

I. $P(E) \qquad 0$
II. $P(S) \qquad 1$
III. M.E E_1 & $E_2 : P(E_1 \cup E_2) =$

As a result,

① $\quad \le P(E) \le \qquad$ ($\because P(S) = P(\quad \cup \quad) = \qquad + \qquad = \qquad$)
② $P(\phi) = \qquad$ ($\because P(S \cup \phi) = \qquad + \qquad = \qquad$)
③ $P(\overline{E}) = \qquad$ ($\because P(E \cup \overline{E}) = \qquad$)
④ $P(E_1 \cup E_2) = P(E_1) \qquad P(E_2) \qquad P(E_1 E_2)$

“Addition Rule”



- Venn Diagram
- Formal Proof

$$P(E_1 \cup E_2) = P(E_1 \cup \overline{E}_1 E_2) = P\left(E_1\right) + P(\overline{E}_1 E_2)$$
$$P(E_2) = P(E_1 E_2) + P\left(\overline{E_1} E_2\right)$$

“Inclusion-Exclusion Rule”

$$P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum\sum P(E_i E_j) + \sum\sum\sum P(E_i E_j E_k) + \cdots + (-1)^{n-1} \times P(E_1 \cdots E_n)$$

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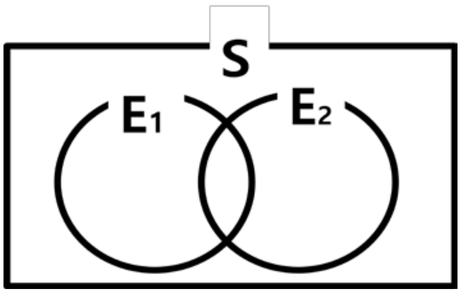
◎ Conditional Probability & Statistical Independence

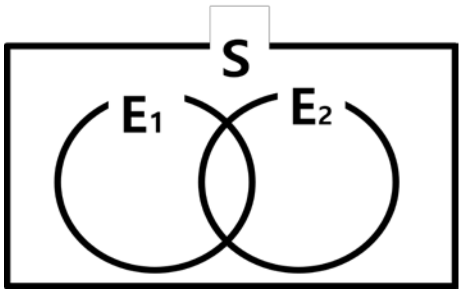
① Conditional Probability

- C.P of given

$P(E_1 | E_2) \equiv$



" "
occurred
⇒



② $P(E_1 | S) =$

③ **"Multiplication Rule":** $P(E_1 E_2) =$

($\because$ $P(E_1 | E_2) =$)

- $P(E_1 E_2 E_3) =$
- $P(E_1 \cdots E_n) =$

④ All the other prob. rules should be applicable to conditional probabilities as long as all the prob. are defined within the same space

- $P(E_1 \cup E_2 | E_3) =$
- $P(E_1 E_2 | E_3) =$
- $P(\overline{E_1} | E_3) =$

⑤ **Statistical Independence:** The occurrence of one event does not affect the likelihood of the other event

- $P(E_1 | E_2) =$
- $P(E_2 | E_1) =$
- $P(E_1 E_2) =$

cf. Mutually Exclusive $P(E_1 E_2) = 0$

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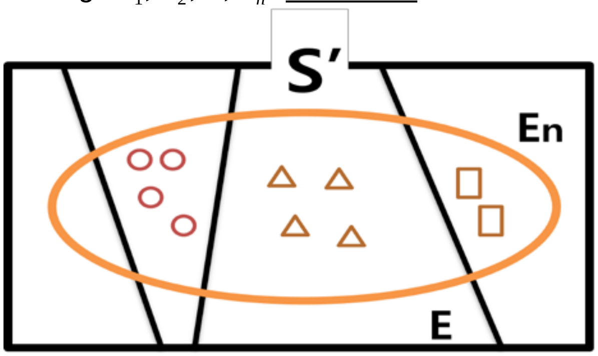
◎ Total Prob. Theorem

Setting: $E_1, E_2, \ldots, E_n$:________ events



$P(E)$ → Not easy to get directly

$P(E \mid E_i)$ → Easier to get

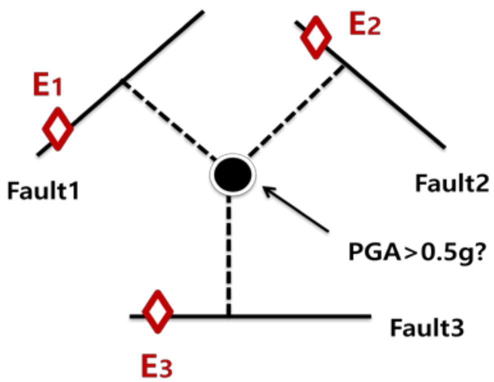
$$P(E) = \sum_{i=1}^{n}$$

Proof:

<u>Examples:</u>

(1) Seismic hazard analysis:

$P(E) =$



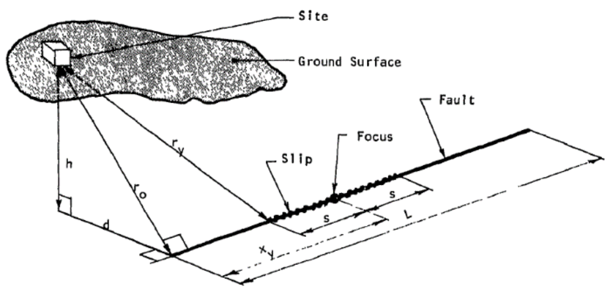


FIG. 3.1 TYPE I SOURCE (BASIC CASE)

Der Kiureghian, A. (1976). *A line source-model for seismic risk analysis*, Ph.D. dissertation, University of Illinois at Urbana-Champaign, Urbana, USA.

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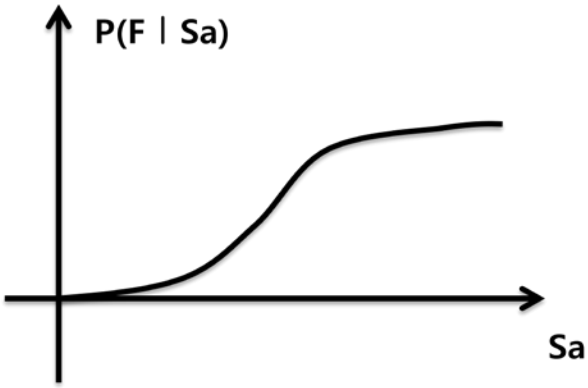
(2) Probability of structural failure under an uncertain input intensity: Fragility



$$P(F) = \sum$$

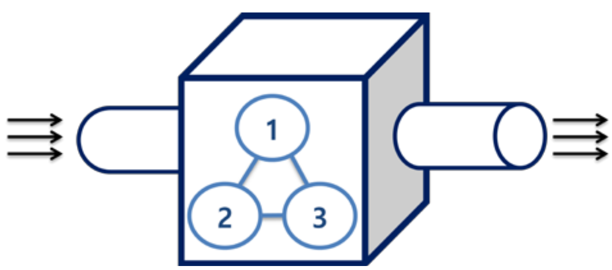
$$= \int$$

◎ Bayes Theorem

$$P(E_i | E) = \frac{P(E | E_i)}{}$$

- Decision making
- Parameter estimation
- Inference

Example)



Contaminated water purified

Measure of cleanness, X (0 : contaminated ~ 100 : clean)

	$P(E_i)$	$P(X \le 20 \| E_i)$
1	0.1	0.9
2	0.3	0.2
3	0.6	0.01

$X \le 20$ $\Rightarrow$ Which one failed?

$$P(E_i | X \le 20) =$$

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457.646 Topics in Structural Reliability
Normal (Gaussian) Distribution

1. **Normal** distribution

- Best known and most widely used. Also known as _______________ distribution.
- According to _________________________, the sum of random variables converges to a normal random variable as the number of the variables increases, no matter what distributions the variables are subjected to.
- Completely defined by the ______________ and the ____________________ of the random variable.

(a) PDF: $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad -\infty < x < \infty$$

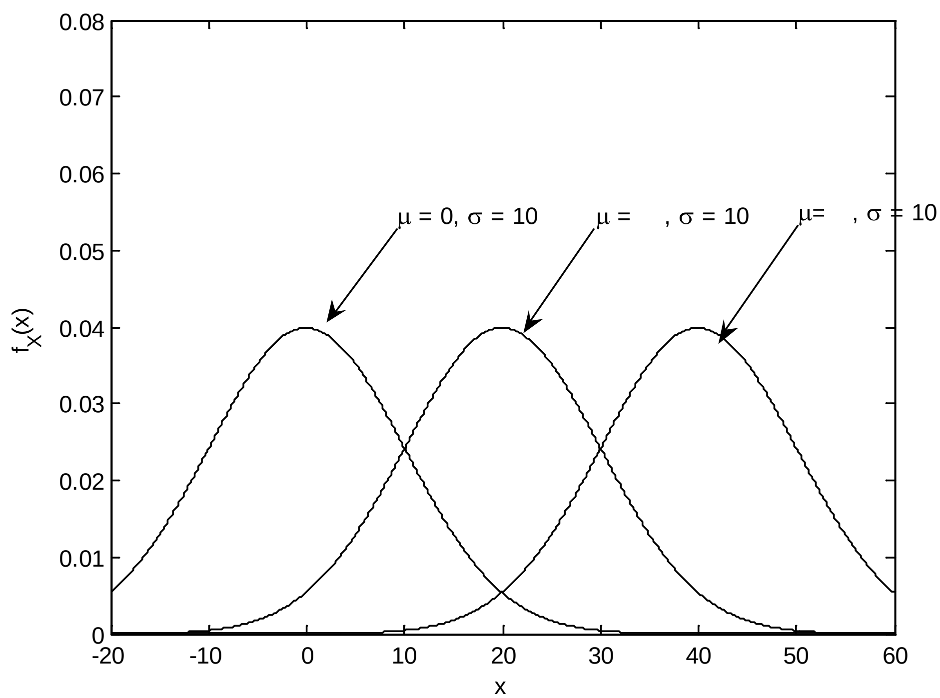

Figure 1. PDF's of normal random variables with different values of μ

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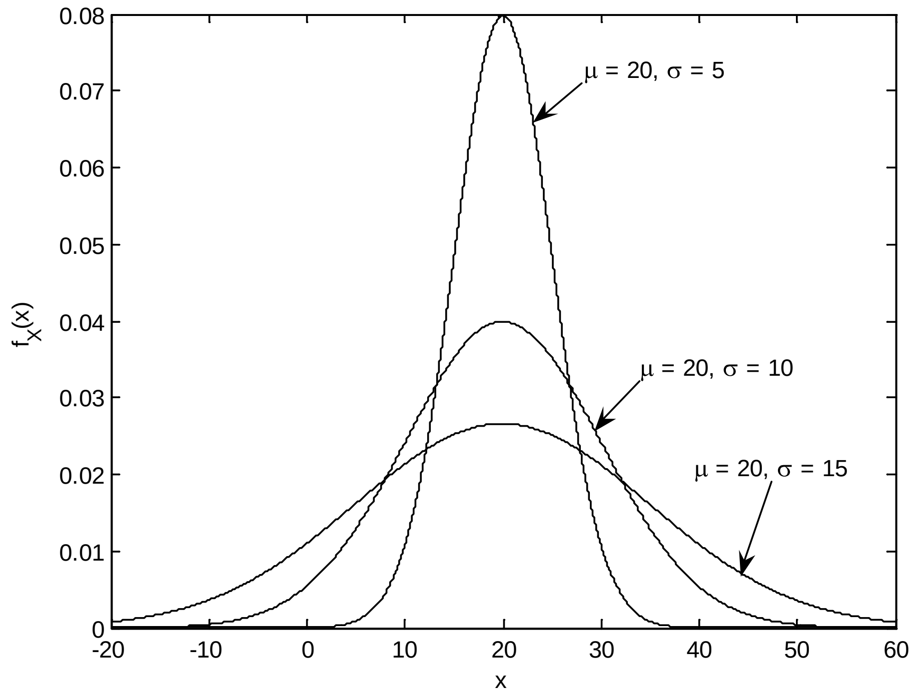

Figure 2. PDF's of normal random variables with different values of σ

(b) CDF: no closed-form expression available

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx\,,\ -\infty < x < \infty$$

(c) Parameters: μ, σ

- μ: __________ of the random variable, i.e. $\mu = \mu_X \equiv \mathrm{E}[X]$
- σ: ______________________ of the random variable, i.e. $\sigma = \sigma_X \equiv \left\{\mathrm{E}[(X-\mu_X)^2]\right\}^{0.5}$

(d) Shape of the PDF plots

- Symmetric around $x =$
- A change in μ_X ________________ the PDF horizontally by the same amount.
- The larger the value of σ_X gets, the more _______________ the PDF becomes around the central axis.

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1a. **Standard** normal distribution

- A special case of the normal distribution: $\mu_X =$, $\sigma_X =$.
- The CDF of the standard normal distribution can be used for computing the CDF of any general normal random variable.

(a) PDF: $U \sim N(\quad, \quad^2)$

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right), \ -\infty < u < \infty$$

(b) CDF:

$$\Phi(u) = \int_{-\infty}^{u} \varphi(u)du \,, \quad -\infty < u < \infty$$

→ no closed-form expression available, but <u>the table of the standard normal CDF</u> $\Phi(\cdot)$ can be found in books or computer software (e.g. See Appendix A of A&T)

(c) Inverse CDF of standard normal distribution: $\Phi^{-1}(\cdot)$

$$\Phi(u_p) = p \quad \Leftrightarrow \quad u_p = \Phi^{-1}(p)$$

(d) Symmetry around $u =$:

$$\Phi(-u) = 1 - \Phi(u)$$
$$u_{1-p} = -u_p$$

→ The table of the standard normal CDF is often provided for positive *u* values only, but using the symmetry one can find the CDF for negative values as well.

(e) One can compute the CDF of a general normal random variable $X \sim N(\mu, \sigma^2)$ by use of the CDF of the standard normal random variable $U \sim N(0, 1^2)$ as follows.

$$\begin{aligned}
F_X(a) &= P(X \le a) \\
&= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx \\
&= \int_{-\infty}^{\left(\frac{a-\mu}{\sigma}\right)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}u^2\right) \sigma du \\
&= \Phi\left(\frac{\qquad}{}\right)
\end{aligned}$$

Hence, $P(a < X \le b) = F_X(\quad) - F_X(\quad) = \Phi\left(\frac{\qquad}{}\right) - \Phi\left(\frac{\qquad}{}\right)$

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Example 1: Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $u = 1.84$

(b) between $u = -1.97$ and $u = 0.86$

Example 2: The drainage demand during a storm (in mgd: million gallons/day): $X \sim N(1.2, 0.4^2)$. The maximum drain capacity is 1.5 mgd.

(a) Probability of flooding?

(b) Probability that the drainage demand during a storm will be between 1.0 and 1.6 mgd?

(c) The 90-percentile drainage demand?

x	PHI(x)	x	PHI(x)	x	PHI(x)	x	PHI(x)	x	PHI(x)
0.00	0.5	0.90	0.81593987	1.80	0.96406968	2.70	0.99653303	3.60	0.999840891
0.01	0.50398936	0.91	0.81858875	1.81	0.96485211	2.71	0.99663584	3.61	0.999846901
0.02	0.50797831	0.92	0.82121362	1.82	0.9656205	2.72	0.9967359	3.62	0.999852698
0.03	0.51196647	0.93	0.82381446	1.83	0.96637503	2.73	0.99683328	3.63	0.999858289
0.04	0.51595344	0.94	0.82639122	1.84	0.96711588	2.74	0.99692804	3.64	0.999863681
0.05	0.51993881	0.95	0.82894387	1.85	0.96784323	2.75	0.99702024	3.65	0.99986888
0.06	0.52392218	0.96	0.83147239	1.86	0.96855724	2.76	0.99710993	3.66	0.999873892
0.07	0.52790317	0.97	0.83397675	1.87	0.96925809	2.77	0.99719719	3.67	0.999878725
0.08	0.53188137	0.98	0.83645694	1.88	0.96994596	2.78	0.99728206	3.68	0.999883383
0.09	0.53585639	0.99	0.83891294	1.89	0.97062102	2.79	0.9973646	3.69	0.999887873
0.10	0.53982784	1.00	0.84134475	1.90	0.97128344	2.80	0.99744487	3.70	0.9998922
0.11	0.54379531	1.01	0.84375235	1.91	0.97193339	2.81	0.99752293	3.71	0.99989637
0.12	0.54775843	1.02	0.84613577	1.92	0.97257105	2.82	0.99759882	3.72	0.999900389
0.13	0.55171679	1.03	0.848495	1.93	0.97319658	2.83	0.9976726	3.73	0.99990426
0.14	0.55567	1.04	0.85083005	1.94	0.97381016	2.84	0.99774432	3.74	0.99990799
0.15	0.55961769	1.05	0.85314094	1.95	0.97441194	2.85	0.99781404	3.75	0.999911583
0.16	0.56355946	1.06	0.8554277	1.96	0.9750021	2.86	0.99788179	3.76	0.999915043
0.17	0.56749493	1.07	0.85769035	1.97	0.97558081	2.87	0.99794764	3.77	0.999918376
0.18	0.57142372	1.08	0.85992891	1.98	0.97614824	2.88	0.99801162	3.78	0.999921586
0.19	0.57534543	1.09	0.86214343	1.99	0.97670453	2.89	0.99807379	3.79	0.999924676
0.20	0.57925971	1.10	0.86433394	2.00	0.97724987	2.90	0.99813419	3.80	0.999927652
0.21	0.58316616	1.11	0.86650049	2.01	0.97778441	2.91	0.99819286	3.81	0.999930517
0.22	0.58706442	1.12	0.86864312	2.02	0.97830831	2.92	0.99824984	3.82	0.999933274
0.23	0.59095412	1.13	0.87076189	2.03	0.97882173	2.93	0.99830519	3.83	0.999935928
0.24	0.59483487	1.14	0.87285685	2.04	0.97932484	2.94	0.99835894	3.84	0.999938483
0.25	0.59870633	1.15	0.87492806	2.05	0.97981778	2.95	0.99841113	3.85	0.999940941
0.26	0.60256811	1.16	0.8769756	2.06	0.98030073	2.96	0.9984618	3.86	0.999943306
0.27	0.60641987	1.17	0.87899952	2.07	0.98077383	2.97	0.998511	3.87	0.999945582
0.28	0.61026125	1.18	0.88099989	2.08	0.98123723	2.98	0.99855876	3.88	0.999947772
0.29	0.61409188	1.19	0.8829768	2.09	0.9816911	2.99	0.99860511	3.89	0.999949878
0.30	0.61791142	1.20	0.88493033	2.10	0.98213558	3.00	0.9986501	3.90	0.999951904
0.31	0.62171952	1.21	0.88686055	2.11	0.98257082	3.01	0.99869376	3.91	0.999953852
0.32	0.62551583	1.22	0.88876756	2.12	0.98299698	3.02	0.99873613	3.92	0.999955726
0.33	0.62930002	1.23	0.89065145	2.13	0.98341419	3.03	0.99877723	3.93	0.999957527
0.34	0.63307174	1.24	0.8925123	2.14	0.98382262	3.04	0.99881711	3.94	0.999959259
0.35	0.63683065	1.25	0.89435023	2.15	0.98422239	3.05	0.99885579	3.95	0.999960924
0.36	0.64057643	1.26	0.89616532	2.16	0.98461367	3.06	0.99889332	3.96	0.999962525
0.37	0.64430875	1.27	0.89795768	2.17	0.98499658	3.07	0.99892971	3.97	0.999964064
0.38	0.64802729	1.28	0.89972743	2.18	0.98537127	3.08	0.998965	3.98	0.999965542
0.39	0.65173173	1.29	0.90147467	2.19	0.98573788	3.09	0.99899922	3.99	0.999966963
0.40	0.65542174	1.30	0.90319952	2.20	0.98609655	3.10	0.9990324	4.00*	3.16712E-05
0.41	0.65909703	1.31	0.90490208	2.21	0.98644742	3.11	0.99906456	4.05	2.56088E-05
0.42	0.66275727	1.32	0.90658249	2.22	0.98679062	3.12	0.99909574	4.10	2.06575E-05
0.43	0.66640218	1.33	0.90824086	2.23	0.98712628	3.13	0.99912597	4.15	1.66238E-05
0.44	0.67003145	1.34	0.90987733	2.24	0.98745454	3.14	0.99915526	4.20	1.33457E-05
0.45	0.67364478	1.35	0.91149201	2.25	0.98777553	3.15	0.99918365	4.25	1.06885E-05
0.46	0.67724189	1.36	0.91308504	2.26	0.98808937	3.16	0.99921115	4.30	8.53991E-06
0.47	0.68082249	1.37	0.91465655	2.27	0.98839621	3.17	0.99923781	4.35	6.80688E-06
0.48	0.6843863	1.38	0.91620668	2.28	0.98869616	3.18	0.99926362	4.40	5.41254E-06
0.49	0.68793305	1.39	0.91773556	2.29	0.98898934	3.19	0.99928864	4.45	4.29351E-06
0.50	0.69146246	1.40	0.91924334	2.30	0.98927589	3.20	0.99931286	4.50	3.39767E-06
0.51	0.69497427	1.41	0.92073016	2.31	0.98955592	3.21	0.99933633	4.55	2.68230E-06
0.52	0.69846821	1.42	0.92219616	2.32	0.98982956	3.22	0.99935905	4.60	2.11245E-06
0.53	0.70194403	1.43	0.92364149	2.33	0.99009692	3.23	0.99938105	4.65	1.65968E-06
0.54	0.70540148	1.44	0.9250663	2.34	0.99035813	3.24	0.99940235	4.70	1.30081E-06
0.55	0.70884031	1.45	0.92647074	2.35	0.99061329	3.25	0.99942297	4.75	1.01708E-06
0.56	0.71226028	1.46	0.92785496	2.36	0.99086253	3.26	0.99944294	4.80	7.93328E-07
0.57	0.71566115	1.47	0.92921912	2.37	0.99110596	3.27	0.99946226	4.85	6.17307E-07
0.58	0.71904269	1.48	0.93056338	2.38	0.99134368	3.28	0.99948096	4.90	4.79183E-07
0.59	0.72240468	1.49	0.93188788	2.39	0.99157581	3.29	0.99949906	4.95	3.71068E-07
0.60	0.72574688	1.50	0.9331928	2.40	0.99180246	3.30	0.99951658	5.00	2.86652E-07
0.61	0.7290691	1.51	0.93447829	2.41	0.99202374	3.31	0.99953352	5.10	1.69827E-07
0.62	0.73237111	1.52	0.93574451	2.42	0.99223975	3.32	0.99954991	5.20	9.96443E-08
0.63	0.73565271	1.53	0.93699164	2.43	0.99245059	3.33	0.99956577	5.30	5.79013E-08
0.64	0.7389137	1.54	0.93821982	2.44	0.99265637	3.34	0.99958111	5.40	3.33204E-08
0.65	0.74215389	1.55	0.93942924	2.45	0.99285719	3.35	0.99959594	5.50	1.89896E-08
0.66	0.74537309	1.56	0.94062006	2.46	0.99305315	3.36	0.99961029	5.60	1.07176E-08
0.67	0.7485711	1.57	0.94179244	2.47	0.99324435	3.37	0.99962416	5.70	5.99037E-09
0.68	0.75174777	1.58	0.94294657	2.48	0.99343088	3.38	0.99963757	5.80	3.31575E-09
0.69	0.75490291	1.59	0.9440826	2.49	0.99361285	3.39	0.99965054	5.90	1.81751E-09
0.70	0.75803635	1.60	0.94520071	2.50	0.99379033	3.40	0.99966307	6.00	9.86588E-10
0.71	0.76114793	1.61	0.94630107	2.51	0.99396344	3.41	0.99967519	6.10	5.30342E-10
0.72	0.7642375	1.62	0.94738386	2.52	0.99413226	3.42	0.99968689	6.20	2.82316E-10
0.73	0.76730491	1.63	0.94844925	2.53	0.99429687	3.43	0.99969821	6.30	1.48823E-10
0.74	0.77035	1.64	0.94949742	2.54	0.99445738	3.44	0.99970914	6.40	7.76885E-11
0.75	0.77337265	1.65	0.95052853	2.55	0.99461385	3.45	0.99971971	6.50	4.01600E-11
0.76	0.77637271	1.66	0.95154277	2.56	0.99476639	3.46	0.99972991	6.60	2.05579E-11
0.77	0.77935005	1.67	0.95254032	2.57	0.99491507	3.47	0.99973977	6.70	1.04210E-11
0.78	0.78230456	1.68	0.95352134	2.58	0.99505998	3.48	0.99974929	6.80	5.23093E-12
0.79	0.78523612	1.69	0.95448602	2.59	0.9952012	3.49	0.99975849	6.90	2.60014E-12
0.80	0.7881446	1.70	0.95543454	2.60	0.99533881	3.50	0.99976737	7.00	1.27987E-12
0.81	0.79102991	1.71	0.95636706	2.61	0.99547289	3.51	0.99977595	7.10	6.23834E-13
0.82	0.79389195	1.72	0.95728378	2.62	0.99560351	3.52	0.99978423	7.20	3.01092E-13
0.83	0.79673061	1.73	0.95818486	2.63	0.99573076	3.53	0.99979222	7.30	1.43885E-13
0.84	0.79954581	1.74	0.95907049	2.64	0.9958547	3.54	0.99979994	7.40	6.80567E-14
0.85	0.80233746	1.75	0.95994084	2.65	0.99597541	3.55	0.99980738	7.50	3.18634E-14
0.86	0.80510548	1.76	0.9607961	2.66	0.99609297	3.56	0.99981457	7.60	1.47660E-14
0.87	0.8078498	1.77	0.96163643	2.67	0.99620744	3.57	0.99982151	7.70	6.77236E-15
0.88	0.81057035	1.78	0.96246202	2.68	0.99631889	3.58	0.9998282	7.80	3.10862E-15
0.89	0.81326706	1.79	0.96327304	2.69	0.9964274	3.59	0.99983466	7.90	0.00000E+00

* Note: For x>=4.0, 1-PHI(x) is given instead.

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Probability Distribution Models in Matlab® Statistics Toolbox

Full Name	Short	Parameters	Probability Density/Mass Function	Mean	Variance
Binomial	*bino*	$0<p<1$ n integer	$\binom{n}{x}p^x(1-p)^{(n-x)},\quad x=0,1,\ldots,n$	np	$np(1-p)$
Geometric	*geo*	$0<p<1$	$p(1-p)^x,\quad x=0.1,2,\ldots$	$(1-p)/p$	$(1-p)/p^2$
Hypergeometric	*hyge*	$0<K,N\le M$ K,N,M integers	$\binom{K}{x}\binom{M-K}{N-x}\binom{M}{N}^{-1},\quad K+N-M\le x\le K$	$\frac{NK}{M}$	$N\frac{K}{M}\frac{M-K}{M}\frac{M-N}{M-1}$
Negative Binomial	*nbin*	$0<p<1$ r integer	$\binom{r+x-1}{x}p^r(1-p)^x,\quad x=0,1,\ldots$	$r(1-p)/p$	$r(1-p)/p^2$
Poisson	*poiss*	$0<\lambda$	$\frac{\lambda^x}{x!}e^{-\lambda},\quad x=0,1,\ldots$	λ	λ
Beta	*beta*	$0<a,b$	$B(a,b)^{-1}x^{a-1}(1-x)^{b-1},\quad 0\le x\le 1$	$a/(a+b)$	$ab/(a+b+1)/(a+b)^2$
Chisquare	*chi2*	$0<\nu$	$x^{(\nu-2)/2}e^{-x/2}2^{-\nu/2}\Gamma(\nu/2)^{-1},\quad 0<x$	ν	2ν
Exponential	*exp*	$0<\mu$	$\mu^{-1}e^{-x/\mu},\quad 0<x$	μ	μ^2
F	*f*	$0<\nu_1,\nu_2$	$\frac{\Gamma[(v_1+v_2)/2](v_1/v_2)^{v_1/2}x^{v_1/2-1}}{\Gamma(v_1/2)\Gamma(v_2/2)[1+(v_1/v_2)x]^{(v_1+v_2)/2}},\quad 0<x$	$v_2/(v_2-2)$	$\frac{2v_2^2(v_1+v_2-2)}{v_1(v_2-2)^2(v_2-4)}$
Gamma	*gam*	$0<a,b$	$b^{-a}\Gamma(a)^{-1}x^{a-1}e^{-x/b},\quad 0<x$	ab	ab^2
Lognormal	*logn*	$\lambda, 0<\zeta$	$x^{-1}\xi^{-1}(2\pi)^{-1/2}\exp[-(\ln x-\lambda)^2/2\xi^2],\quad 0<x$	$e^{(\lambda+0.5\xi^2)}$	$e^{(2\lambda+2\xi^2)}-e^{(2\lambda+\xi^2)}$
Normal	*norm*	$\mu, 0<\sigma$	$\sigma^{-1}(2\pi)^{-1/2}\exp[-(x-\mu)^2/2\sigma^2]$	μ	σ^2
Rayleigh	*rayl*	$0<b$	$xb^{-2}\exp(-x^2/2b^2),\quad 0<x$	$b\sqrt{\pi/2}$	$(4-\pi)b^2/2$
T	*t*	$0<\nu$	$(v\pi)^{-1/2}\Gamma((v+1)/2)\Gamma(v/2)^{-1}(1+x^2/v)^{-(v+1)/2}$	0	$v/(v-2)$
Uniform	*unif*	$a<b$	$(b-a)^{-1},\quad a\le x\le b$	$(a+b)/2$	$(b-a)^2/12$
Weibull	*weib*	$0<a,b$	$abx^{b-1}e^{-ax^b},\quad 0<x$	$a^{-1/b}\Gamma(1+b^{-1})$	$a^{-2/b}[\Gamma(1+2b^{-1})-\Gamma^2(1+b^{-1})]$

Use *shortname***pdf**( ) to compute the probability density/mass function; *shortname***cdf**( ) to compute cumulative distribution function; *shortname***fit**( ) to estimate parameters from data; *shortname***rnd**( ) to generate random numbers; *shortname***stat**( ) to compute mean and variance for specified parameters; and *shortname***inv**( ) to compute the inverse cumulative probability. Use Matlab® help to learn more about these commands.

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457.646 Topics in Structural Reliability
In-Class Material: Class 03

3. Random Variables, Prob. Functions & Partial Descriptors:
- Tools to associate uncertain q_________ with probabilities

◎ Random variables

: a ~~variable~~ _________ that takes on one of the values in a specified set according to the assigned probabilities

Example: X = the random number one can get from throwing a fair dice

Specified set:

Assigned probabilities:

◎ Prob. Functions (mapping b/w &)

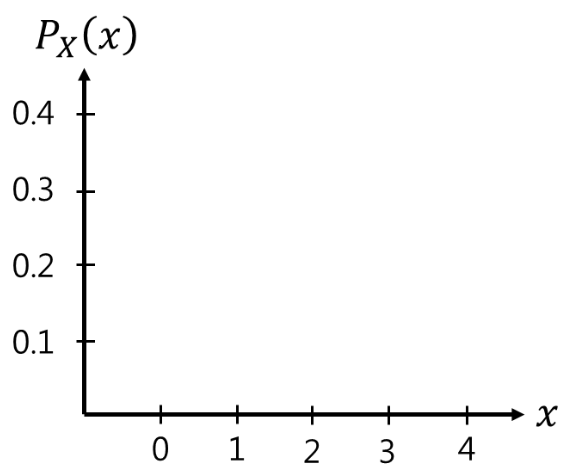
Functions for discrete random variables

① Probability ________ Function ( ) of X

$P_X(x) \equiv$ $\qquad\qquad$ $x \rightarrow \boxed{P_X(\ )} \rightarrow$

e.g. # of land falls of hurricanes/year

x	$P_X(x)$
0	0.10
1	0.40
2	0.30
3	0.15
4	0.05



※ $\qquad \le P_X(x) \le$

$\sum_{\text{all } x} P_X(x) =$

$$P(a < X \le b) = \sum$$

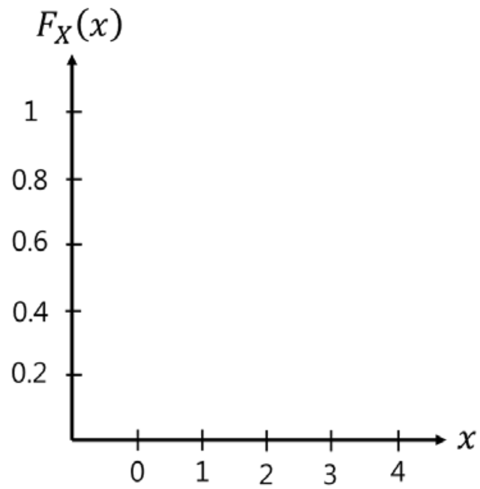
e.g. $P(0 < X \le 2) =$

② Cumulative ________ Function ( ) of X

$$F_X(x) \equiv \qquad\qquad = \sum$$

$$x \rightarrow \boxed{F_X(\;)} \rightarrow$$

x	$P_X(x)$	$F_X(x)$
0	0.10	
1	0.40	
2	0.30	
3	0.15	
4	0.05	



※ $F_X(a) = \sum$

$F_X(-\infty)$

$F_X(\infty)$

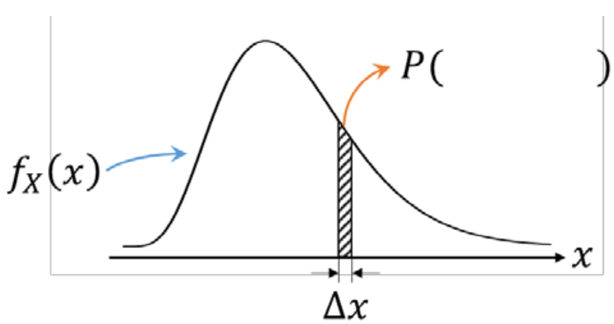
$P(a < X \le b) = \qquad\qquad -$

Functions for continuous r.v.

③ Probability ________ Function ( ) of X

$$f_X(x) = \lim_{\Delta x \to 0}$$

"Density" of Probability at $X = x$



Normalization rule Area =

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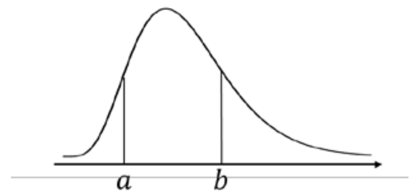
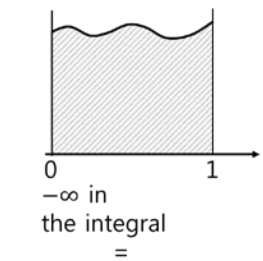
※ $\leq f_X(x)$

$\int f_X(x)dx = P(\quad\quad) =$

$P(a < X \leq b) = \int f_X(x)dx$





④ Cumulative ___________ Function (   ) of X

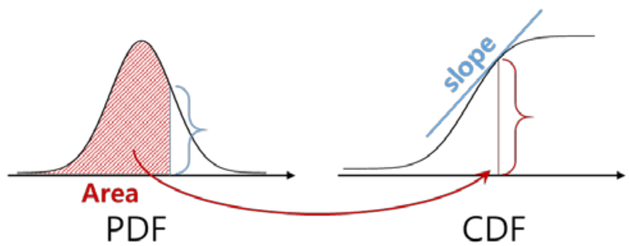
$F_X(x) \equiv P(X \quad x) = \int \quad dx$

※ $\dfrac{dF_X(x)}{dx} =$

non-   ing

$F_X(-\infty)$

$F_X(\infty)$



◎ Partial Descriptors of a r.v. :

(a) "Complete" description by probability functions:

(b) "Partial" descriptors: measures of key characteristics; can derive from (   )

Note:

- Expectation: $E[\cdot] = \int_{-\infty}^{\infty} (\cdot) f_X(x)dx$ (continuous) or $\sum_{\text{all } x} (\cdot) p_X(x)$ (discrete)
- Moment: $E[\cdot X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$ or $\sum_{\text{all } x} x^n p_X(x)$
- Central Moment, $E[(X-\mu_X)^n] = \int_{-\infty}^{\infty} (x-\mu_X)^n f_X(x)dx$ or $\sum_{\text{all } x} (x-\mu_X)^n p_X(x)$

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`	Name	Definition	Meaning (PDF/CDF)
Measure of Central Location	Mean, μ_X	First moment, $\mathrm{E}[X]$	Location of the ( ) of an area underneath ( )
	Median, $x_{0.5}$	$F_X(x_{0.5}) = 0.5$ $F_X^{-1}(0.5)$	The value of a r.v. at which values above and below it are _______lly probable. If symmetric?
	Mode, $\tilde{x}$	$\arg\max_x f_X(x)$	The outcome that has the _______est probability mass or density
Measure of Dispersion	Variance, σ_X^2	Second-order central moment $\mathrm{E}[(X-\mu_X)^2]$ $= \mathrm{E}[X^2] - \mathrm{E}[X]^2$	Average of squared deviations
	Standard Deviation, σ_X	$\sqrt{\sigma_X^2}$	Radius of ( )
	Coefficient of Variation (C.O.V.), δ_X	$\dfrac{\sigma_X}{\lvert\mu_X\rvert}$	__________ed radius of ( )
Asymmetry	Coefficient of Skewness, γ_X	Third-order central moment normalized by σ_X^3, $\dfrac{\mathrm{E}[(X-\mu_X)^3]}{\sigma_X^3}$	Behavior of two tails > 0 = 0 < 0
Flatness	Coefficient of Kurtosis, κ_X	Fourth-order central moment normalized by σ_X^4, $\dfrac{\mathrm{E}[(X-\mu_X)^4]}{\sigma_X^4}$	"Peakedness" - more of the variance is due to infrequent extreme deviations, as opposed to frequent modestly-sized deviations.

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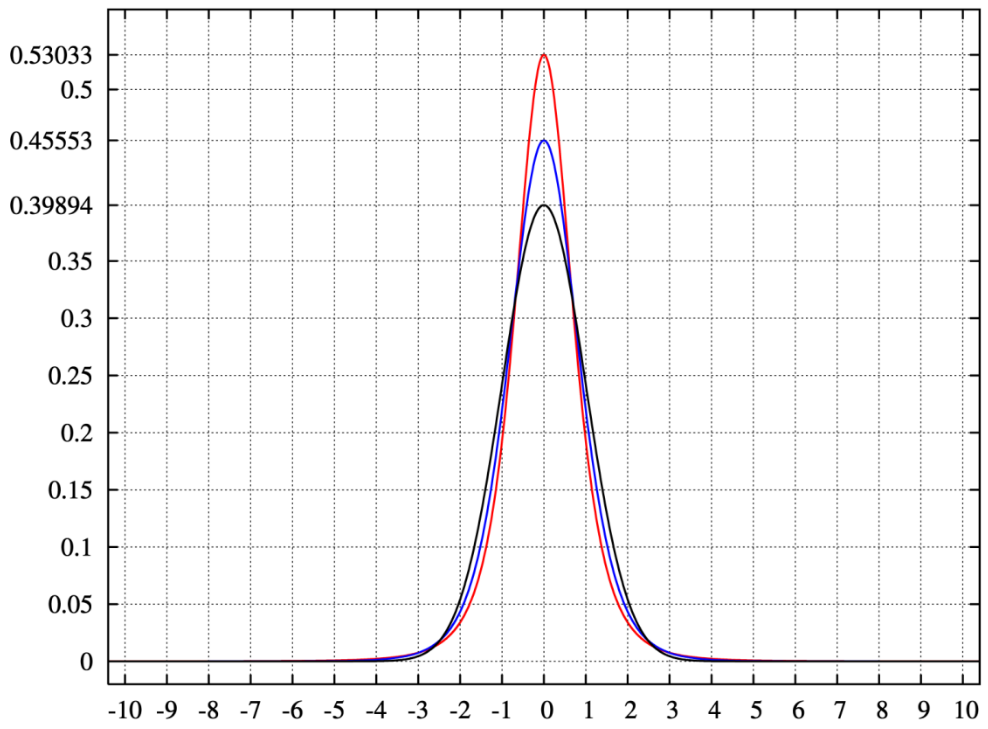
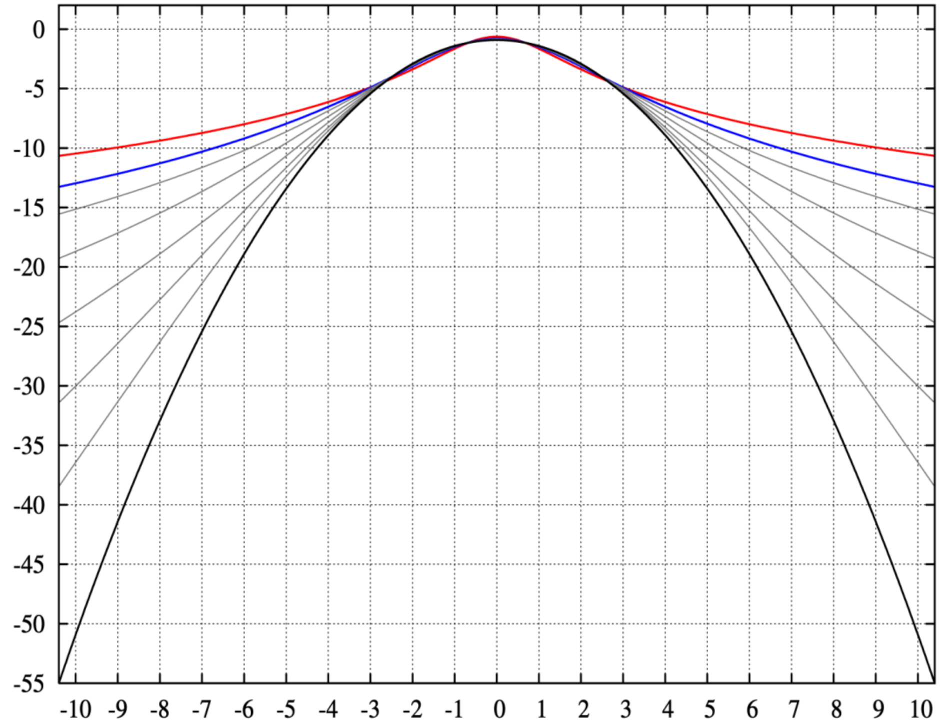
Example: PDF and Log-PDF of Pearson type VII distribution with kurtosis of infinity (red), 2 (blue), and 0 (black) (source: Wikipedia)





4. Probability Distribution Models

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457.646 Topics in Structural Reliability
Lognormal Distribution

1. **Lognormal** distribution

- Closely related to the ________ distribution.
- Defined for _________ values only.

(a) PDF: $X \sim LN(\lambda, \zeta^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right], \quad 0 < x < \infty$$

(b) CDF:

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx, \quad 0 < x < \infty$$

→ no closed-form expression available, but can be computed by use of <u>the table of the standard normal CDF</u> $\Phi(\cdot)$ (as shown below)

(c) Parameters: λ, ζ

- λ: mean of _____________, i.e. $\lambda = \lambda_X \equiv \mathrm{E}[\ln X]$
- ζ: standard deviation of______________, i.e. $\zeta^2 = \zeta_X^2 = \sigma_{\ln X}^2$

(d) Shape of the PDF plots

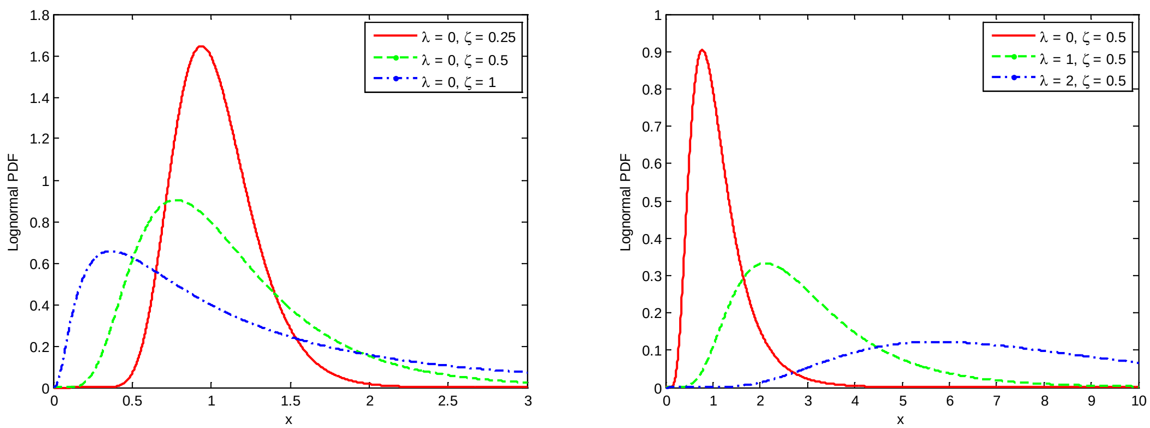

Figure 3. PDF's of lognormal random variables.

(e) Relationship between normal and lognormal distribution:

"The logarithm of a ____________ random variable is a ___________ random variable."

$$X \sim LN(\lambda, \zeta^2) \Rightarrow \ln X \sim N(\lambda, \zeta^2)$$

(f) Can obtain the CDF of lognormal $X \sim LN(\lambda, \zeta^2)$ from the CDF of standard normal:

$$\begin{aligned} F_X(a) &= P(X \le a) \\ &= P(\ln X \le \ln a) \qquad \text{Since } \ln X \sim N(\lambda, \zeta^2), \\ &= \Phi\left(\frac{\ln a - \lambda}{\zeta}\right) \end{aligned}$$

(g) "The exponential function of a ______________ random variable is a _______________ random variable."

$$\begin{tikzcd} \boxed{X \sim LN(\lambda, \zeta)} \arrow[r, bend left, "Y = \ln X"] & \boxed{Y \sim N(\lambda, \zeta)} \arrow[l, bend left, "X = e^Y"] \end{tikzcd}$$

(h) $(\lambda, \zeta) \to (\mu, \delta)$: Find the mean and c.o.v. from the distribution parameters

$$\mu = \mathrm{E}[X] = \exp(\lambda + 0.5\zeta^2)$$
$$\delta = \sigma/\mu = \sqrt{\exp(\zeta^2) - 1} \quad (\cong \zeta \text{ for } \zeta << 1)$$

(i) $(\mu, \delta) \to (\lambda, \zeta)$: Find the distribution parameters from the mean and c.o.v.

$$\zeta = \sqrt{\ln(1 + \delta^2)} \quad (\cong \delta \text{ for } \delta << 1)$$
$$\lambda = \ln \mu - 0.5 \ln(1 + \delta^2)$$

(j) $(x_{0.5}) \leftrightarrow (\lambda)$: Relationship between the median and λ

$$\lambda = \ln x_{0.5}, \quad x_{0.5} = e^{\lambda}$$

(k) $(\mu, \delta) \to (x_{0.5})$: Find the median from the mean and c.o.v.

$$x_{0.5} = \frac{\mu}{\sqrt{1 + \delta^2}}$$

Note: $x_{0.5} < \mu$ for the lognormal distribution.

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Example 1: The drainage demand during a storm (in mgd: million gallons/day) is assumed to follow the lognormal distribution with the same mean and standard deviation as Example 1 (mean 1.2, standard deviation 0.4). The maximum drain capacity is 1.5 mgd.

(a) Distribution parameters, i.e. λ and ζ?

(b) Probability of the flooding?

(c) Probability that the drainage demand during a storm will be between 1.0 and 1.6 mgd?

(d) The 90-percentile drainage demand?

Example 2: Consider a bridge whose uncertain capacity against "complete damage" limit-state caused by earthquake events is defined in terms of peak ground acceleration (PGA; unit: g) that the bridge can sustain. Suppose the median of the capacity is 1.03g and the coefficient of variation is 0.50. It is assumed that the capacity follows a lognormal distribution.

(a) Distribution parameters of the lognormal distribution, i.e. λ and ζ?

(b) The mean and standard deviation of the uncertain capacity, i.e. μ and σ?

(c) Suppose the peak ground acceleration from an earthquake event is 0.5g. What is the probability that the structure will exceed "complete damage" limit state?

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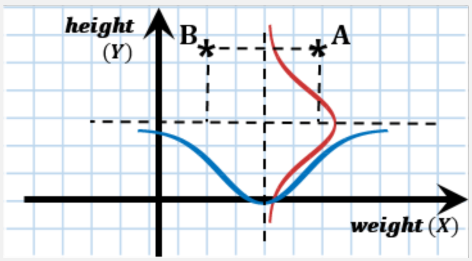


Question: Which one more likely?

Case A: Heavy & Tall

Case B: Light & Tall

Ⅱ-5. Multiple Random Variables

◎ "Joint" Probability Functions

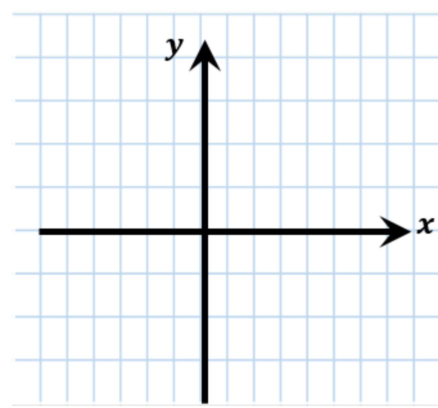
e.g. $P(X \le 20)$ = $\int \qquad dx$

=

$P(\quad \cap \quad) = ?$

Need more information than ( ) and ( )

① **Joint Cumulative Distribution Function (CDF) (Discrete/Continuous) ↔ cf. __________ CDF**



$F_{XY}(x,y) \equiv P(\qquad)$

- $F_{XY}(-\infty,-\infty) =$
- $F_{XY}(\infty,\infty) =$
- $F_{XY}(-\infty,y)$
- $F_{XY}(\infty,y) = P(\quad \cap \quad) = P(\quad)$
  $=$

② **Joint Probability Mass Function (discrete r.v's) ↔ cf. __________ PMF**

(a) Definition : $P_{XY}(x,y) \equiv P(\quad , \quad)$

(b) $F_{XY}(a,b) = \sum$

(c) Conditional PMF

$P_{X|Y}(x|y) \equiv \qquad = \frac{\qquad}{\qquad} =$

(d) $P_{XY}(x,y) \rightarrow P_X(x),\ P_Y(y)?$

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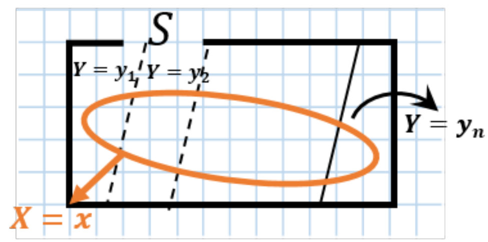
$P_X(x) = \sum$

$= \sum$



$\Rightarrow$( ) rule

(e) If X & Y are statically independent,

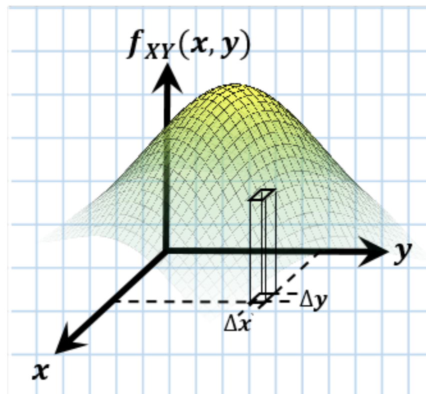
$P_{X|Y}(x|y) =$

$\Leftrightarrow P_{Y|X}(y|x) \qquad P_Y(y)$

$\Leftrightarrow P_{XY}(x,y)$

* In-class material on Joint PMF

③ **Joint PDF (continuous r.v's)**



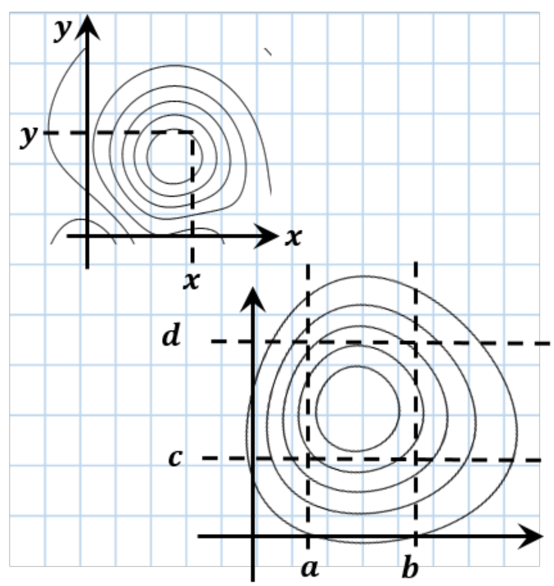
$f_{XY}(x,y) = \lim_{\Delta x, \Delta y \to 0} \rule{4cm}{0.4pt}$

(a) Joint cumulative distribution function (CDF)

$F_{XY}(x,y) \equiv P(X \le x, Y \le y)$

$= \int$

$f_{XY}(x,y) =$



(b) $P(a < X \le b, c < Y \le d) =$

(c) Conditional PDF

$f_{X|Y}(x|y)$

$= \lim_{\Delta x \to 0} \dfrac{P(x < X \le x + \Delta x \qquad\qquad)}{\Delta x}$

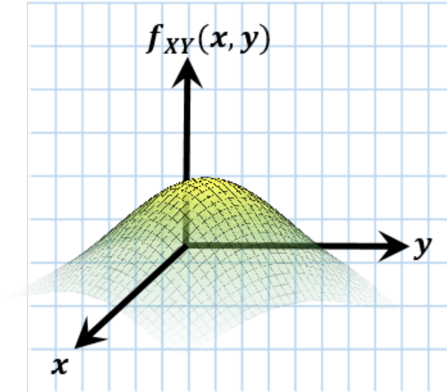
Can show

$=$

※ Multiplication rule $f_{XY}(x,y) =$

(s.i $f_{XY}(x,y) =$)

(d) Joint PDF→marginal PDF?



$$f_X(x) = \int$$

$$= \int$$

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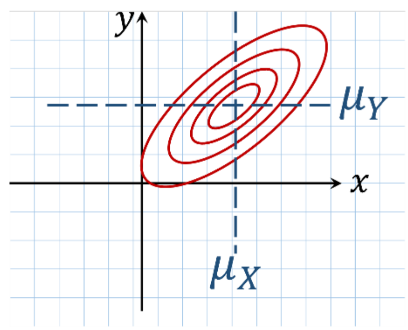
※ **See supplementary material on bivariate normal joint PDF**

◎ Covariance & Correlation Coefficient

– Partial descriptors or measures for __________ dependence

① Covariance



(a) Definition:

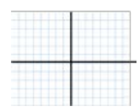
$$Cov[X,Y] \equiv \mathrm{E}[\qquad\qquad\qquad\qquad]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \qquad\qquad f_{XY}(x,y)dydx$$

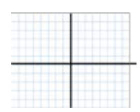
c.f. c.o.v. $\delta =$

(b) $Cov[X,Y] = \qquad\qquad -$

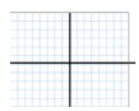
(c) $Cov[X,Y] \;\; > 0$ _______ linear dependence



$= 0$ _______ linear dependence



< 0 _______ linear dependence



⇒ **Not useful to measure/compare the strength of the linear dependence. Why?**

② Correlation Coefficient

(a) Dimensionless measure of linear dependence

$$\rho_{XY} \equiv \overline{\qquad\qquad\qquad}$$

(b) $\qquad \le \rho_{XY} \le$

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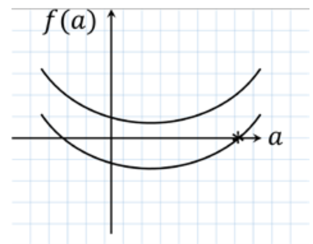
Proof: Consider



$$f(a) = \iint [a(x-\mu_X) - (y-\mu_Y)]^2 f_{XY}(x,y)dxdy$$

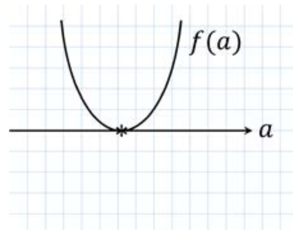
$$= a^2 Var[X] - 2a \cdot Cov[X,Y] + Var[Y] \qquad 0$$

$$\therefore D/4 = (Cov[X,Y])^2 - Var[X] \cdot Var[Y] \qquad 0$$

$$\therefore \frac{[Cov(X,Y)]^2}{Var[X] \cdot Var[Y]} \le$$

$$\le \rho_{XY} \le$$

(c) What does $\rho_{XY} =$ & $\rho_{XY} =$ mean?



Consider the case D=

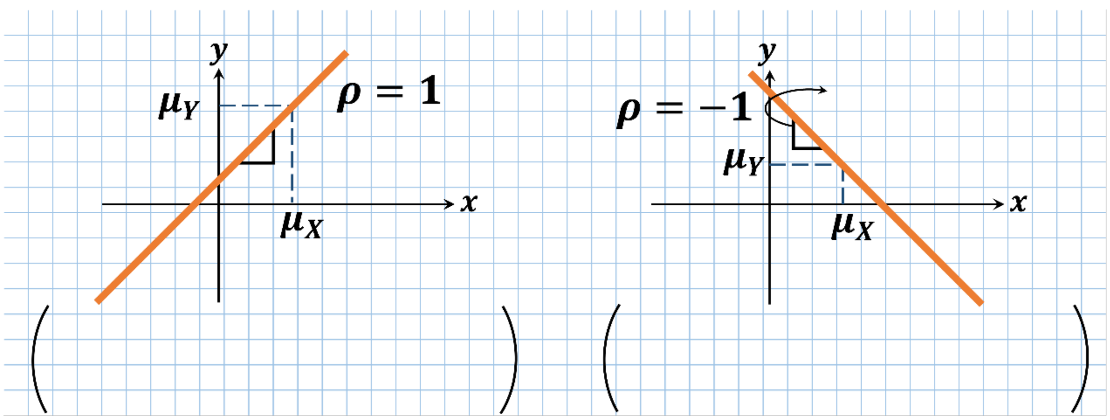
$$f(a) = Var[X]\left(a - \frac{Cov[X,Y]}{Var[X]}\right)^2 + \ldots$$

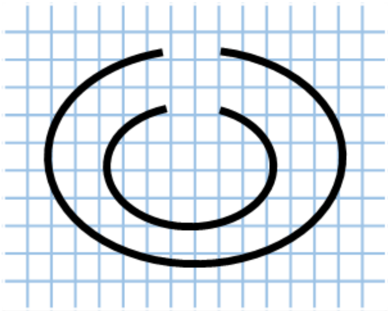
$$f(a) = 0 \text{ at } a = \frac{Cov[X,Y]}{Var[x]} = a^*$$

Substituting this into $f(a)$,

$$f(a^*) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\qquad (x-\mu_X) - (y-\mu_Y)]^2 f_{XY}(x,y)dxdy = 0$$

$\therefore$ for $\forall(x,y)$, the following (deterministic/probabilistic) and (linear/nonlinear) relationship between X and Y holds:





(d) $\rho_{XY} = 0 \quad \Leftrightarrow \quad Cov[X,Y] = 0$

"No linear dependence"
"Un   "

(e) "Uncorrelated" vs "Statistical Independence"

$$\begin{array}{lc} \rho_{XY} = 0 & \rightarrow \\ (E[XY] = \qquad\quad) & \leftarrow \end{array} \quad f_{XY}(x,y) =$$

$\rightarrow ?$

Suppose $Y = X^2$ and X has a symmetric distribution in $[-a, a]$

$E[XY] =$

$E[X] =$

$Cov[X,Y] =$

$\leftarrow ?$

※ Vector/matrix formulation for multiple RVs

$$\mathbf{X} = \begin{Bmatrix} X_1 \\ \vdots \\ X_n \end{Bmatrix} \qquad \boldsymbol{\mu}_{\mathrm{X}} = \begin{Bmatrix} \mu_{X_n} \\ \vdots \\ \mu_{X_n} \end{Bmatrix} \qquad \boldsymbol{\Sigma}_{\mathrm{XX}} = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \vdots \\ sym & & \cdots & \sigma_n^2 \end{bmatrix}$$

(   ) vector   (   ) vector$= \mathrm{E}[\mathbf{X}]$ (   ) matrix

$$\boldsymbol{\Sigma}_{\mathrm{XX}} = \mathrm{E}[(\mathbf{X} - \mathbf{M}_{\mathrm{X}})(\mathbf{X} - \mathbf{M}_{\mathrm{X}})^{\mathrm{T}}] = \mathrm{E}[\mathbf{X}\mathbf{X}^{\mathrm{T}}] - \mathbf{M}_{\mathrm{X}}\mathbf{M}_{\mathrm{X}}^{\mathrm{T}}$$
$$= \mathbf{DRD}$$

where

$$\mathbf{D} = \begin{bmatrix} & & \\ & \ddots & \\ & & \end{bmatrix} = [\quad]$$ diagonal matrix of ___________

$$\mathbf{R} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \\ & 1 & & \\ & & \ddots & \\ sym & & & 1 \end{bmatrix} = [\quad]$$ ________________ matrix

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※ $\boldsymbol{\Sigma}_{XX}$ and $\mathbf{R}_{XX}$ are _______ and _______ ________

- $\mathbf{a}^{\mathrm{T}}\boldsymbol{\Sigma}_{XX}\mathbf{a} > 0$ $(\forall \mathbf{a} \neq \mathbf{0})$ If no perfect linear dependence
  (a simple proof: $Y = \mathbf{a}^{\mathrm{T}}\mathbf{X}$, $\sigma_y^2 = \mathbf{a}^{\mathrm{T}}\boldsymbol{\Sigma}_{XX}\mathbf{a} > 0$)

- $\mathbf{a}^{\mathrm{T}}\boldsymbol{\Sigma}_{XX}\mathbf{a} = \mathbf{0}$ for $\exists \mathbf{a}$ if there exist linear dependence among **X**

  e.g. $X_1 = 2X_2$, $Y = 1\cdot X_1 - 2X_2 = \begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$

  $\sigma_Y^2 = \mathbf{a}^{\mathrm{T}}\boldsymbol{\Sigma}_{XX}\mathbf{a} = 0$

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457.646 Topics in Structural Reliability

In-Class Material: Class 06

II-6. Functions of Random Variables (See Supp. 03)

Consider $\mathbf{Y} = \mathbf{g}(\mathbf{X})$

(1) For input $\mathbf{X}$: distribution model $f_{\mathbf{X}}(\mathbf{x})$ or expectations $(\mathbf{M}_{\mathbf{X}},\ \mathbf{\Sigma}_{\mathbf{XX}})$ available

(2) For output $\mathbf{Y}$: distribution model ( ) or expectations (,) ?

<u>Examples:</u>

(1) Regional/inventory loss: $L = \sum_{i=1}^{n} V_i D_i$ → linear function

(2) Wind-induced pressure: $P = \frac{1}{2} C_\rho \rho V^2$

◎ Mathematical expectation of linear functions

$$Y_k = a_{k,0} + \sum_{i=1}^{n} a_{k,i} X_i, \quad k = 1, \ldots, m$$

① Algebraic formula $(n \leq 3)$: See supp.

② Matrix formula:

For $\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}\mathbf{X}$

where

$$\mathbf{Y} = \begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{Bmatrix}, \quad \mathbf{A}_0 = \begin{Bmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{m,0} \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \text{ and } \mathbf{X} = \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix}$$

$\mathbf{M}_{\mathbf{Y}} =$

$\mathbf{\Sigma}_{\mathbf{YY}} =$

❖ Proof of Positive-definiteness of $\mathbf{\Sigma}_{\mathbf{XX}}$

Consider $Y = \mathbf{a}^{\mathrm{T}}\mathbf{X}$ $\quad (\mathbf{A}_0 = \qquad , \ \mathbf{A} = \qquad)$

Using the formula above,

$\mathbf{\Sigma}_{\mathbf{YY}} = \sigma_Y^2 =$

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❖ **Linear transformation for standardization, i.e., &**

Suppose **X** has and

Find $\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}\mathbf{X}$

such that $\mathbf{M}_\mathbf{Y} =$ and $\mathbf{\Sigma}_\mathbf{YY} =$

$\mathbf{M}_\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}\mathbf{M}_\mathbf{X} =$ (1)

$\mathbf{\Sigma}_\mathbf{YY} = \mathbf{A}\mathbf{\Sigma}_\mathbf{XX}\mathbf{A}^\mathrm{T} =$ (2)

Since $\mathbf{\Sigma}_\mathbf{XX}$ is positive semi-definite, $\mathbf{\Sigma}_\mathbf{XX} = \mathbf{L}_\Sigma \mathbf{L}_\Sigma^\mathrm{T}$ (e.g. by ____________ decomposition)

Therefore, $= \mathbf{I}$ and

$\mathbf{A} =$ → Substitute to ( )

$\mathbf{A}_0 =$

In summary,

$\mathbf{Y} =$

$=$

Alternatively,

$$\mathbf{\Sigma}_\mathrm{XX} = \mathbf{D}_\mathrm{X}\mathbf{R}_\mathrm{XX}\mathbf{D}_\mathrm{X}$$

$$=$$

$$= \mathbf{L}_\Sigma \mathbf{L}_\Sigma^\mathrm{T}$$

Therefore, $\mathbf{L}_\Sigma =$ and $\mathbf{L}_\Sigma^{-1} =$

$\mathbf{Y} =$

→ This version is preferred because of numerical stability in decomposition ($|\rho| \le 1$).

◎ **Mathematical expectation of nonlinear functions**

$Y_k = g_k(x),\ k = 1, \cdots, m$

Taylor series expansion around the mean point, $\mathbf{x} = \mathbf{M}_\mathbf{X}$

$$Y_k \cong g_k(\boldsymbol{M}_X) + \left.\frac{\partial g_k}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{M}_\mathbf{X}} (\mathbf{x} - \mathbf{M}_\mathbf{X}) + \cdots$$

Matrix form

$$\mathbf{Y} \cong \mathbf{g}(\mathbf{M}_\mathrm{X}) + \mathbf{J}_{\mathrm{Y,X}}\big|_{\mathrm{x}=\mathbf{M}_\mathrm{X}} (\mathbf{X} - \mathbf{M}_\mathrm{X})$$

① First-order approximation

(Scalar: See supp.)

$$\mathbf{M}_{\mathbf{Y}}^{FO} = \mathbf{g}(\qquad)$$
$$\boldsymbol{\Sigma}_{\mathbf{YY}}^{FO} =$$

② Second-order approximation

⇒ Can use 2nd order approximation from Taylor series expansion

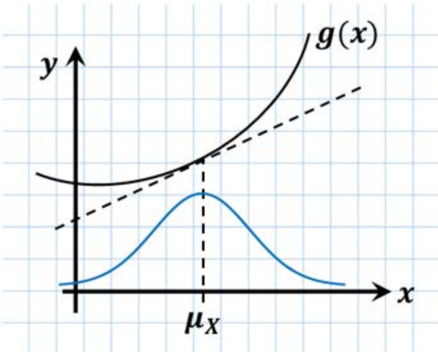
⇒ Not useful because higher-order moments are needed $(\gamma, \kappa, \cdots)$

③ Accuracy of FO/SO approximation

Sources of large errors in approx.

- σ_X
- Nonlinearity in $g(x)$

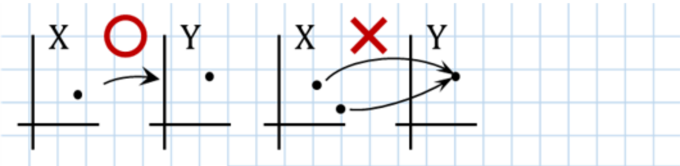
Example : $\mathbf{U} = \mathbf{K}^{-1}\mathbf{P}$ (Frame structure)



◎ Derived Distribution of Functions

Consider $\mathbf{Y} = \mathbf{g}(\mathbf{X})$ where $\mathbf{Y} = \{Y_1, \cdots, Y_m\}$ and $\mathbf{X} = \{X_1, \cdots, X_n\}$

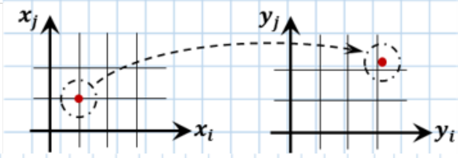
Given: $f_{\mathbf{X}}(\mathbf{x})$ → $f_{\mathbf{Y}}(\mathbf{y})$?



① $m = n$, one-to-one mapping

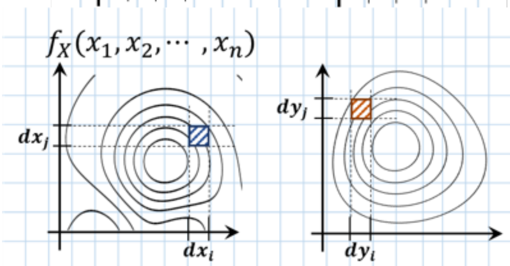
a) Discrete

$P_{\mathbf{Y}}(y_1, \cdots, y_n) \qquad P_{\mathbf{X}}(x_1, \cdots, x_n)$



b) Continuous

$f_{\mathbf{Y}}(y_1, \cdots, y_n) \qquad f_{\mathbf{X}}(x_1, \ldots, x_n)$



$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \cdot \left|\det \qquad\right|$$
$$= f_{\mathbf{X}}(\mathbf{x}) \cdot \left|\det \qquad\right|^{-1}$$

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"Jacobian" $\mathbf{J}_{\mathbf{y},\mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$

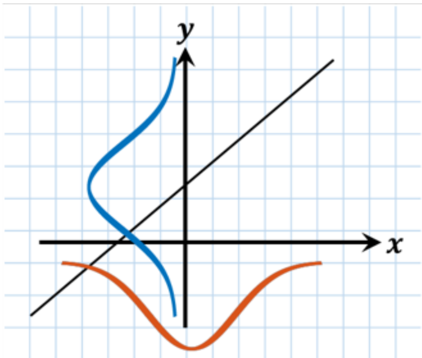
Consider $\mathbf{y} = \mathbf{g}(\mathbf{x}),\ \mathbf{x} = \mathbf{h}(\mathbf{y})$

※ $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{h}(\mathbf{y}))\left|\det \mathbf{J}_{\mathbf{y},\mathbf{x}}(\mathbf{h}(\mathbf{y}))\right|^{-1}$

※ $m = n = 1$

$$f_Y(y) = f_X(x) \qquad\qquad = f_X(\quad)\left|\frac{dh(y)}{dy}\right|$$

Example: $X \sim N(0, 1^2)$



a) $Y = g(X) = aX + b$

One-to-one mapping?

$f_Y(y) = f_X(x)\cdot$

$=$

$=$

__________ Distribution

$\mu_Y =$

$\sigma_Y =$

b) T_1, T_2 ~ exponential r.v.'s (See supplement on "Other Distribution Models")

$f_{T_1}(t_1) = \alpha \cdot \exp(-\alpha t_1),\ t_1 > 0$

$f_{T_2}(t_2) = \beta \cdot \exp(-\beta t_2),\ t_2 > 0$

T_1, T_2 : statistically independent

Joint PDF of $\begin{cases} Y_1 = T_1 + T_2 \\ Y_2 = T_1 - T_2 \end{cases}$?

$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{T}}(\mathbf{t})\left|\det \mathbf{J}_{y,t}\right|^{-1}$

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$$\mathbf{J}_{\mathbf{y},\mathbf{t}} = \begin{bmatrix} \frac{\partial y_1}{\partial t_1} & \frac{\partial y_1}{\partial t_2} \\ \frac{\partial y_2}{\partial t_1} & \frac{\partial y_2}{\partial t_2} \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\left|\det \mathbf{J}_{\mathbf{y},\mathbf{t}}\right|^{-1} =$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) =$$

Inverse relationship

$$\begin{cases} T_1 = \frac{1}{2}(Y_1 + Y_2) \\ T_2 = \frac{1}{2}(Y_1 - Y_2) \end{cases}$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) = \frac{\alpha\beta}{2}\exp[-\frac{\alpha+\beta}{2}y_1 - \frac{\alpha-\beta}{2}y_2], \quad y_1 > 0, -y_1 < y_2 < y_1$$

- Range of **Y** derived from the condition $t_1, t_2 > 0 \;\; \& \;\; \mathbf{t} = \mathbf{h}(\mathbf{y})$

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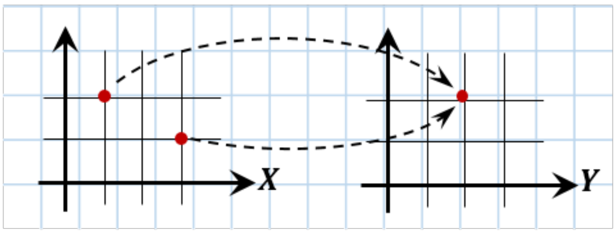
II-6. Functions of Random Variables (contd.)

◎ Derived Distribution of Functions (contd.)



② $m = n$, **but NOT one-to-one mapping**

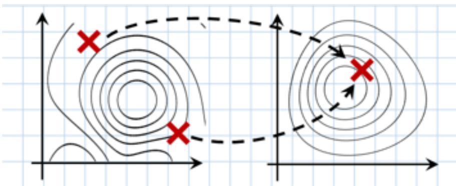
a) Discrete

$$P_{\mathbf{Y}}(y_1, \cdots, y_n) =$$

$$P_{\mathbf{X}}(x_1, \cdots, x_n)$$

roots of $\mathbf{y} = \mathbf{g}(\mathbf{x})$

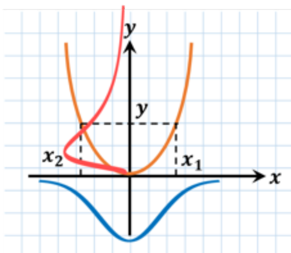
b) Continuous



$$f_Y(y) = \sum_{\substack{\text{all roots of} \\ \mathbf{y}=\mathbf{g}(\mathbf{x})}}$$

Example c)

$$Y = g(X) = X^2, \qquad X \sim N(0, 1^2)$$



$$\begin{cases} x_1 = h_1(y) = \\ x_2 = h_2(y) = \end{cases}$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_1^2\right) \qquad + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_2^2\right) \qquad =$$

③ $m < n$, **one-to one mapping**

$$\mathbf{Y}' \begin{cases} \mathbf{Y} \begin{cases} Y_1 = g_m(X_1, \cdots, X_n) \\ \vdots \qquad \vdots \\ Y_m = g_m(X_1, \cdots, X_n) \end{cases} \\ Y_{m+1} = \\ \vdots \\ Y_n = \end{cases}$$

$$\mathbf{Y' = g'(X)}$$

Discrete

$$P_{\mathbf{Y}'}(\mathbf{y}') = P_{\mathbf{X}}(\mathbf{x})$$

Then,

$$P_{\mathbf{Y}}(\mathbf{y}) = \sum \cdots \sum P_{\mathbf{X}}(\mathbf{x})$$

a) Continuous

$$f_{\mathbf{Y}'}(\mathbf{y}')dy_1 \cdots dy_m \qquad = f_{\mathbf{X}}(\mathbf{x})dx_1 \cdots dx_m dx_{m+1} \cdots dx_n$$

$$f_{\mathbf{Y}'}(\mathbf{y}') = f_{\mathbf{X}}(\mathbf{x})\left|\det J_{\mathbf{Y}',\mathbf{X}}\right|^{-1}$$

$$= f_{\mathbf{X}}(\mathbf{x})\left|\det J_{\mathbf{Y},\mathbf{X}}\right|_{mxm}^{-1}$$

$$J_{Y',X} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_m}{\partial x_1} & & \cdots & \frac{\partial y_m}{\partial x_m} \\ & & & \end{bmatrix}$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) = \int_{x_{m+1}} \cdots \int_{x_n} f_{\mathbf{X}}(\mathbf{x})\left|\det J_{\mathbf{Y},\mathbf{X}}\right|_{m\times m}^{-1} dx_{m+1} \cdots dx_n$$

Example d)

$Y = T_1 + T_2$ **←contd. From Example b)**

$f_Y(y)$?

$$\mathbf{Y}' \begin{cases} Y_1 = T_1 + T_2 \\ Y_2 = \quad T_2 \end{cases}$$

$$f_{\mathbf{Y}'}(\mathbf{y}') = f_{\mathbf{T}}(\mathbf{t})\left|\det J_{\mathbf{Y}',\mathbf{T}}\right|^{-1} \qquad \left|\det J_{\mathbf{Y},\mathbf{T}}\right|_{1\times 1}^{-1} =$$

$$= f_{\mathbf{T}}(\mathbf{t})\left|\det J_{\mathbf{Y}',\mathbf{T}}\right|_{1\times 1}^{-1}$$

$=$

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$$f_{\mathbf{Y}}(\mathbf{y}) = f_{Y_1}(y_1) = \int \qquad dt_2$$

$$= \int f_{T_1}(\qquad) f_{T_2}(\qquad) dt_2$$

$$= \frac{\alpha\beta}{\alpha-\beta}[\exp(-\beta y) - \exp(-\alpha y)],\ y > 0$$

When $\alpha = \beta$, using l'Hopitals rule,

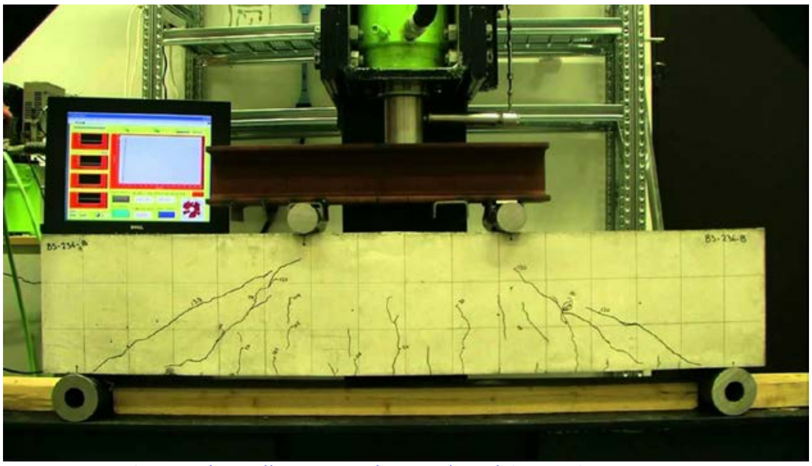
$$\lim_{\beta\to\alpha} f_Y(y) = \lim_{\beta\to\alpha} \frac{\frac{\partial(\ )}{\partial\beta}}{\frac{\partial(\ )}{\partial\beta}} = \alpha^2 y \exp(-\alpha y),\ y > 0$$

④ $m < n$ **, NOT one-to one mapping**

Ⅲ. Structural Reliability (Component)

◎ Structural Reliability Analysis (contd.)

e.g. Shear failure of RC beam w/o stirrups



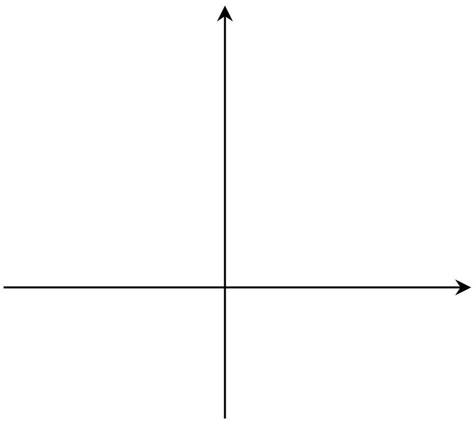
Source: https://www.youtube.com/watch?v=DPQIpT1ZvXY

"Limit-state" function

$$g(\mathbf{X}) = V_c - V_d$$

$$= \frac{1}{6}\sqrt{f_c^{'}}b_w d + \varepsilon - V_d \leq 0$$

where $X = \{f_c^{'}, b_w, d, \varepsilon, V_d, \cdots\}$ random variables



Failure Probability

$$P_f = P(g(\mathbf{x}) \qquad)$$

$$=$$

"<u>Structural</u> Reliability Analysis"

(Anatomical + Systematic)

Three important tasks for structural reliability analysis:

1)

2)

3)

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◎ Joint Probability Distribution Models

① Joint Normal $\mathbf{X} \sim N(\mathbf{M}_{\mathrm{X}}, \mathbf{\Sigma}_{\mathrm{XX}})$

a) Joint PDF

$$\boxed{f_{\mathrm{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{\Sigma}}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{M}_{\mathrm{X}})^{\mathrm{T}} \mathbf{\Sigma}_{\mathrm{XX}}^{-1} (\mathbf{x}-\mathbf{M}_{\mathrm{X}})\right]}$$

$n=1 \quad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ Uni-variate normal PDF (See supp.)

$n=2 \quad f_{X_1X_2}(x_1, x_2) = f(\qquad\qquad\qquad)$ Bi-variate normal PDF (See supp.)

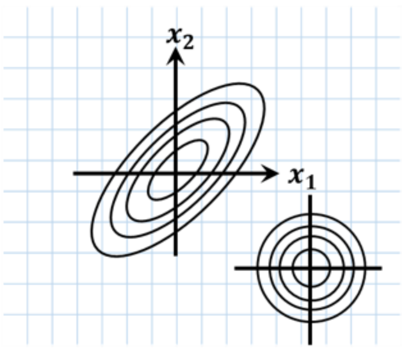
b) Properties

- Joint distribution completely defined by
- All lower order distribution are
- $\mathbf{X} = \left\{ \quad \right\} \quad \mathbf{M}_{\mathrm{X}} = \left\{ \quad \right\} \quad \sum_{\mathrm{XX}} = \left\{ \qquad \right\}$

Given $\mathbf{X}_2 = \mathbf{x}_2$, then $\mathbf{X}_1 \sim N(\mathbf{M}_{1|2}, \mathbf{\Sigma}_{1,1|2})$

Conditional mean and covariance

$$\begin{cases} \mathbf{M}_{1|2} = \mathbf{M}_1 + \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{x}_2 - \mathbf{M}_2) \\ \mathbf{\Sigma}_{1,1|2} = \mathbf{\Sigma}_{1,1} - \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1} \end{cases}$$



e.g. $n=2$, i.e. $\mathbf{X} = \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$

$X_1 \sim N(\mu_{1|2}, \sigma_{1|2}^2)$ | if $\rho = 0$ ("　　　　")

$\mu_{1|2} = \mu_1 + \rho\sigma_1\left(\frac{x_2 - \mu_2}{\sigma_2}\right)$ | $\mu_{1|2} =$

$\sigma_{1|2}^2 = \sigma_1^2(1-\rho^2)$ | $\sigma_{1|2}^2 =$

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- Uncorrelated (   ) s.i for jointly normal
  (in general, $\rho = 0 \,\square\quad s.i$)

- Linear functions of $\mathbf{X} \sim N(\mathbf{M}, \boldsymbol{\Sigma})$ $\rightarrow$ follow ______________

  $$\mathbf{Y} = \mathbf{AX} + \mathbf{A_0}$$

  $$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \cdot \qquad\qquad J_{\mathbf{Y},\mathbf{X}} = \qquad\qquad \therefore \det =$$

  $$f_{\mathbf{Y}}(\mathbf{y}) \propto \exp\left[-\frac{1}{2}(\mathbf{x} - \mathbf{M}_{\mathbf{x}})^{\mathrm{T}} \boldsymbol{\Sigma}_{\mathbf{xx}}^{-1} (\mathbf{x} - \mathbf{M}_{\mathbf{x}})\right]$$

  In summary, $\mathbf{X} \sim N(\mathbf{M}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{XX}})$

  $$\Rightarrow \mathbf{Y} \sim N(\mathbf{M}_{\mathbf{Y}}, \boldsymbol{\Sigma}_{\mathbf{YY}})$$

  $$\mathbf{M}_{\mathbf{Y}} =$$

  $$\boldsymbol{\Sigma}_{\mathbf{YY}} =$$

c) Standard Normal

For univariate, 'standard normal' means, $\mu =$ $\qquad , \sigma =$

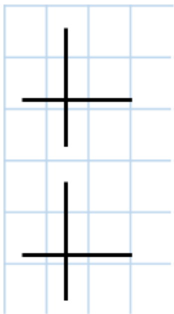
$\therefore$ For jointly normal,

$$\mathbf{M}_{\mathbf{X}} =$$

$$\boldsymbol{\Sigma}_{\mathbf{XX}} =$$

$$\mathbf{Z} \sim N(\mathbf{0}, \quad) \qquad \varphi_n(\mathbf{z}, \quad) = \frac{1}{(2\pi)^{n/2}\sqrt{\det}} \cdot \exp\left[-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{R}_{\mathbf{XX}}\mathbf{z}\right]$$

$$\mathbf{U} \sim N(\mathbf{0}, \quad) \qquad \varphi_n(\mathbf{u}, \quad) = \frac{1}{(2\pi)^{n/2}\sqrt{\det}} \cdot \exp\left[-\frac{1}{2} \qquad \right]$$



$$= \prod_{i=1}^{n}$$

U used for FORM/SORM

For normal,

$$\begin{cases} \mathbf{x} = \mathbf{DLu} + \mathbf{M} \\ \mathbf{u} = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{x} - \mathbf{M}) \end{cases}$$

457.646 Topics in Structural Reliability
In-Class Material: Class 08

III. Structural Reliability (Component)

◎ Joint Probability Distribution Models

② Joint Lognormal

$X_1, \cdots, X_n$ are jointly lognormal if $\ln X_1, \cdots, \ln X_n$ are jointly ________

a) Parameters

$$\lambda_i = E[\qquad] = \ln \mu_i - 0.5\ln(1+\delta_i^2)$$

$$\xi_i^2 = Var[\qquad] = \ln(1+\delta_i^2)\ \ (\cong \delta_i^2 \text{ for } \delta \quad 1)$$

$$\rho_{\ln X_i, \ln X_j} = \frac{1}{\xi_i \xi_j}\ln(1+\rho_{ij}\delta_i\delta_j)$$

b) Properties

- Completely defined in terms of ( ) & ( )
- All lower order distribution are jointly
- Conditional distribution are jointly
- Uncorrelated ⇄ S.I.
- Product / Quotient of jointly lognormal r.v.'s follows
- $\rho_{X_i, \ln X_j} = \dfrac{1}{\xi_i}\delta_j \rho_{ij}$

③ General Joint Distribution Forms

e.g. Johnson & Kotz (1976)

$\Rightarrow$ on multivariate prob. distribution models

④ Joint Distribution by conditioning (e.g. Bayesian Networks)

$$f(x_1, \cdots, x_n) = f(x_n | x_1, \cdots, x_{n-1}) \times$$

⑤ Joint Distribution model with :

Prescribed marginals: $\quad, i = 1, \cdots, n$ **and**

correlation coefficient matrix :

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- Read CRC Ch.14
- **See Liu & Kiureghian (1986)** a) Morgenstern

  b) Nataf

※ "Copula": formula to construct joint PDF with marginal distributions (Review by Jongmin Park (SNU): Term Project Report in 2014)

a) Morgenstern distribution

$$\boxed{F_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} F_{X_i}(x_i) \cdot \left\{1 + \sum_{i<j} \alpha_{ij}[1 - F_{X_i}(x_i)][1 - F_{X_j}(x_j)]\right\}}$$

Q) Can we derive $F_{X_i}(x_i)$ from $F_{\mathbf{X}}(\mathbf{x})$?

i.e. $x_2, x_3, \cdots, x_n \to$ then $F_{\mathbf{X}}(\mathbf{x}) =$?

Q) Can we describe dependence using α_{ij} ?

$F_{X_iX_j}(x_i, x_j) =$

$f_{X_iX_j}(x_i, x_j) = \rule{6cm}{0.4pt}$

$$= f_{X_i}(x_i) \cdot f_{X_j}(x_j) \cdot \{1 + \alpha_{ij}[1 - 2F_{X_i}(x_i)][1 - 2F_{X_j}(x_j)]\}$$

$\Rightarrow \qquad \le \alpha_{ij} \le$

$$\begin{cases} \alpha_{ij} = 0 \\ \alpha_{ij} \neq 0 \end{cases}$$

Therefore, α_{ij} is a parameter that represents (corr coeff.)

But α_{ij} ρ_{ij}

Lin & Der Kiureghian (1986) showed

$$\rho_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{x_i - \mu_i}{\sigma_i}\right)\left(\frac{x_j - \mu_j}{\sigma_j}\right) f_{X_iX_j}(x_i, x_j)dx_i dx_j$$

$$= 4\alpha_{ij} Q_i Q_j \qquad \Rightarrow \qquad |\rho_{ij}| \le 0.30$$

Where $Q_i = \int_{-\infty}^{\infty} \left[\left(\frac{x_i - \mu_i}{\sigma_i}\right) F_{X_i}(x_i)\right] f_{X_i}(x_i) dx_i \approx 0.28$

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Table 1: selected distribution

Table 2: Q_i

Table 3 : maximum $|\rho_{ij}|$

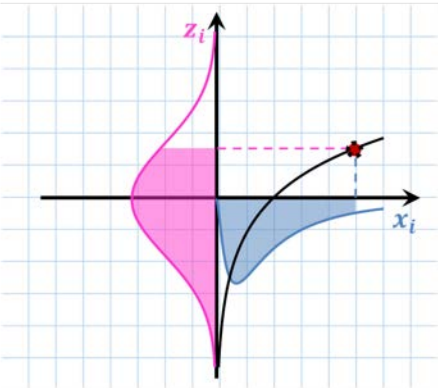
$\Rightarrow$ In summary, using Morgenstern's model, you cannot describe X_i, X_j whose $|\rho_{ij}| > 0.30$

b) Nataf model (Nataf, 1962) ("Gaussian Copula")

$$\boxed{\begin{array}{l}\mathbf{X} \sim F_{X_i}(x_i),\ i = 1,\cdots,n \\ \quad \mathbf{R} = [\rho_{ij}]\end{array}} \underset{\substack{(2) \\ Nataf}}{\overset{(1)}{\rightleftarrows}} \boxed{\begin{array}{l}\mathbf{Z} \sim N(\mathbf{0}, \mathbf{R'}) \\ \quad \mathbf{R'} = [\rho'_{ij}]\end{array}}$$

Transformation to $\mathbf{Z}$

$$\boxed{Z_i = \qquad\qquad\qquad\qquad}$$



Why?

$$f_{Z_i}(z_i) = f_{X_i}(x_i) \cdot \left| \frac{dx_i}{dz_i} \right|$$

$$f_{Z_i}(z_i) \cdot \qquad = f_{X_i}(x_i) \cdot$$

$$\Phi(\quad) = F_{X_i}(\quad)$$

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In-Class Material: Class 09

III. Structural Reliability (Component) - continued

◎ Joint Probability Distribution Models

⑤ Joint distribution models with marginal & corr. coeff (contd.)

a) Morgenstern: $F_{X_i}(x_i),\ i=1,\ldots,n\ \&\ \alpha_{ij}$ but $|\rho_{ij}|<0.30$

b) Nataf model (Nataf, 1962)

★ Joint PDF by Nataf model

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{Z}}(\mathbf{z})\cdot\left|\det J_{\mathbf{Z},\mathbf{X}}\right|$$

$$= \varphi_n(\mathbf{z};\mathbf{R}')\underline{\qquad\qquad}$$

$$J_{\mathbf{Z},\mathbf{X}} = \begin{bmatrix} \qquad\qquad \end{bmatrix}$$

$$\boxed{= \left[\prod_{i=1}^{n} f_{X_i}(x_i)\right]\cdot\underline{\qquad\qquad}}$$

Note:

$$F_{X_i}(x_i) = \Phi(z_i)$$

$$f_{X_i}(x_i)dx_i = \varphi(z_i)dz_i$$

★ ρ'_{ij} (corr. coeff. b/w Z_i and Z_j)?

$$\rho_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\underline{\qquad}\right)\left(\underline{\qquad}\right) f_{X_iX_j}(x_i,x_j)dx_idx_j$$

$$\therefore \rho_{ij} = \iint\left(\underline{\qquad}\right)\left(\underline{\qquad}\right)\varphi_2(z_i,z_j;\rho'_{ij})\underline{\qquad\qquad}\ \underline{\qquad}\ \underline{\qquad} dz_idz_j$$

In general, $|\rho'_{ij}| \qquad |\rho_{ij}|$

$\therefore |\rho_{ij}| \le A < 1$ may not cover the whole range of ρ_{ij}

$\rho'_{ij} \cong F\ \cdot\rho_{ij}$ Liu & ADK (Table 4~6) for pairs of selected distribution types

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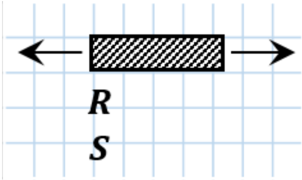
Table 9: Range of ρ_{ij} ~ wider (than Morgenstern)

Later used for transformation of dependent RVs into $\mathbf{U} \sim N(\mathbf{0}, \mathbf{I})$

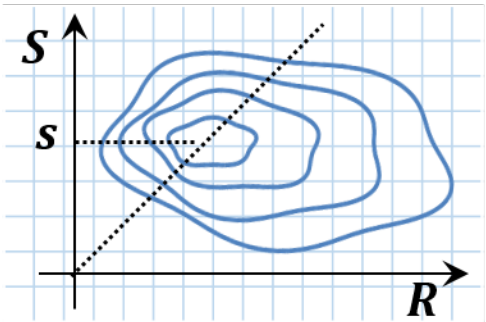
$\mathbf{X}$ $\mathbf{Z}$ $\mathbf{U}$

◎ Elementary Structural Reliability Problem

Describe the failure event in terms of __________ & ___________



① **Failure :** $g(\mathbf{x}) = g(\quad,\quad) = \qquad \le 0$



② **Failure probability :** $P_f = P(\qquad \le 0)$

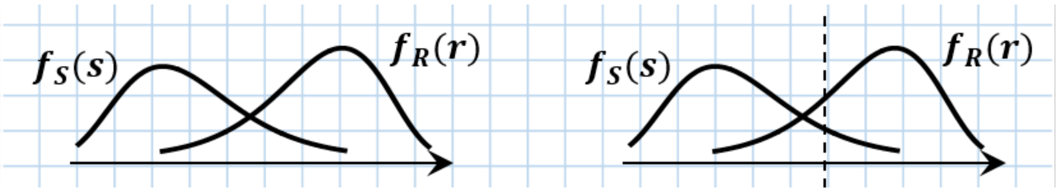
$$P_f = \iint f_{R,S}(r,s)drds$$

$$= \iint f_{R|S}(r|s) \cdot f_s(s)drds$$

$$= \iint f_{R|S}(r|s)\ dr\ f_s(s)ds$$

$$= \int \qquad f_s(s)ds$$

If R&S are s.i $P_f = \int \qquad ds$



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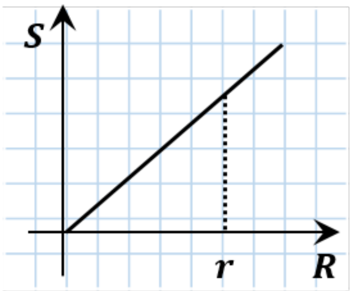
OR



$$P_f = \iint_{r \le s} f_{S|R}(s|r) f_R(r) ds dr$$

$$= \iint f_{S|R}(s|r) ds f_R(r) dr$$

$$= \int [\qquad] f_R(r) dr$$

if s.i $= \int \qquad dr$

③ **Reliability Index by "Safety Margin,"** β_{SM}

$\boxed{M = \qquad}$: Safety Margin

Failure : $\{R - S \le 0\} \Leftrightarrow \{ \qquad \le 0\}$

$\Leftrightarrow \{U_M \le \qquad \}$

※ Standardization

$$U_M = \frac{M}{\qquad} \quad \begin{cases} E[U_M] = \\ Var[U_M] = \end{cases}$$

For n RVs, $\mathbf{U} = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{X} - \mathbf{M})$

$$\therefore P_f = P(U_M \le \qquad) = F_{U_M}(\qquad)$$

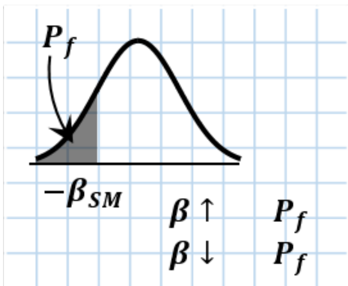
$$= F_{U_M}(\qquad)$$

β_{SM} : reliability index by safety margin

$$= \frac{\qquad}{\qquad}$$

$$= \frac{\qquad}{\qquad}$$

$$= \frac{r-1}{\sqrt{r^2\delta_R^2 + \delta_s^2 - 2r\delta_R\delta_S\rho_{RS}}}, \quad r = \mu_R / \mu_S$$



F_{U_M} : depends on distribution of R and S

e.g. special case ~ R and S are jointly normal

Then U_M ~

Therefore $P_f = F_{U_M}(-\beta_{SM}) =$

※ A. Cornell (1968. ACI codes)

Assumed R&S are jointly normal & used β_{SM} to compute P_f

④ Reliability Index by "Safety Factor"

$$F = \ln \frac{\quad}{\quad} \qquad = \ln \qquad - \ln$$

Failure : $\{ \qquad\qquad\qquad \leq 0\}$ (※ used for LRFD $\phi R_n \geq \sum \gamma_k Q_k$)

$$\Leftrightarrow \{ \qquad \leq 0\}$$

$$\Leftrightarrow \{u_F \leq -\frac{\quad}{\quad}\}$$

$$\therefore \beta_{SF} = \frac{\qquad}{\qquad}$$

$\mu_F =$

$\sigma_F^2 =$

$$P_f = F_{u_F}(- \qquad)$$

⇒ special case: R & S are jointly lognormal

$U_F \sim$

$\therefore P_f = \Phi(\qquad)$

$\mu_F^{(LN)} =$

$\sigma_F^{(LN)} =$

$$\beta_{SF}^{(LN)} = \frac{\ln\left(r \cdot \sqrt{\frac{1+\delta_S^2}{1+\delta_R^2}} \right)}{\sqrt{\ln(1+\delta_R^2) - 2\ln(1+\rho_{RS}\delta_R\delta_S) + \ln(1+\delta_S^2)}}, \quad r = \frac{\mu_R}{\mu_s}$$

Safety factor-based reliability-index when R & S are jointly lognormal

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457.646 Topics in Structural Reliability

In-Class Material: Class 10

◎ Second moment reliability index β_{MVFOSM}

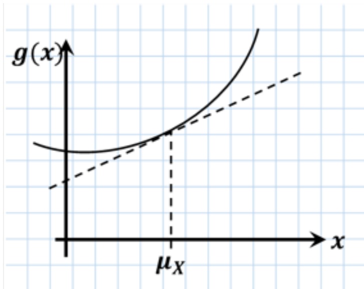
M V F O S M

- Failure : $g(\mathbf{x}) \le 0$ (NOT "elementary")
- Use ( ) & ( ) only. Therefore, can't compute P_f (index, not method)

- Ang & Cornell (1974) ASCE Journal of Structural Engineering

Use ( ) order approximation to estimate & of $g(\mathbf{x})$

$P_f = P(g \le 0) = P(u_g \le \quad)$



μ_g^{FO} & σ_g^{FO} ?

$g(\mathbf{x}) \simeq g(\quad) + \sum_{i=1}^{n} \qquad (x_i - \mu_{X_i})$

$$\begin{cases} \mu_g^{FO} = \\ \sigma_g^{2FO} = \sum \qquad\qquad + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \end{cases}$$

$=$

$$\boxed{\therefore \ \beta_{MVFOSM} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \underline{\qquad\qquad\qquad}}$$

If we assume $u_g \sim N(0,1)$

$P_f \cong \Phi(\quad)$

⇒ Popular for a while

⇒ But problem

i.e. equivalent limit-state functions could give different β_{MVFORM}

$$\left.\begin{aligned} g_1(x) &= X_1^2 + 3X_2 < 0 \\ g_2(x) &= g_1(X)/X_1^2 = 1 + 3\frac{X_2}{X_1^2} < 0 \end{aligned}\right\}$$ equivalent ⇒ the same β_{MVFORM} ?

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Example: lack of invariance of second order reliability methods

Consider a structural reliability problem with two random variables X_1 and X_2. The mean vector and the covariance matrix of X_1 and X_2 are

$$\mathbf{M}_{\mathrm{X}} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad \mathbf{\Sigma}_{\mathrm{XX}} = \begin{bmatrix} 4 & 5 \\ 5 & 25 \end{bmatrix}$$

<u>Case 1:</u> $g(X_1, X_2) = X_1^2 + 3X_2$

Gradient $\nabla g = [2X_1 \quad 3]$. At the mean point $\mathbf{X} = \mathbf{M}_{\mathrm{X}}$, $\nabla g = [10 \quad 3]$.

First order approximation on μ_g and σ_g^2:

$$\mu_g \cong 5^2 + 3 \times 10 = 55$$

$$\sigma_g^2 \cong \nabla g \mathbf{\Sigma}_{\mathrm{XX}} \nabla g^{\mathrm{T}} = 925$$

$$\beta_{MVFOSM} = \frac{\mu_g}{\sigma_g} = \frac{55}{\sqrt{925}} = 1.81$$

$$P_f = \Phi(-1.81) = 0.0351$$

<u>Case 2:</u> $g(X_1, X_2) = 1 + \frac{3X_2}{X_1^2}$

$\nabla g = [-6X_2X_1^{-3} \quad 3X_1^{-2}]$.

At the mean point $\mathbf{X} = \mathbf{M}_{\mathrm{X}}$, $\nabla g = [-0.48 \quad 0.12]$.

$$\mu_g \cong 1 + 3 \times 10 / 25 = 2.20$$

$$\sigma_g^2 \cong \nabla g \mathbf{\Sigma}_{\mathrm{XX}} \nabla g^{\mathrm{T}} = 0.706$$

$$\beta_{MVFOSM} = \frac{\mu_g}{\sigma_g} = \frac{2.20}{\sqrt{0.706}} = 2.62$$

$$P_f = \Phi(-2.62) = 0.00440$$

Although the two limit-state functions are equivalent ones with the same failure domains, the second order reliability method yields different reliability indices and failure probability estimates.
///

Summary:

$$\beta_{SM} = \frac{\mu_M}{\sigma_M} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2 - 2\sigma_R\sigma_S\rho_{RS}}}$$

$$\beta_{SF} = \frac{\mu_F}{\sigma_F}, \text{ for LN } \beta_{SF} = \frac{\lambda_R - \lambda_S}{\sqrt{\zeta_R^2 + \zeta_S^2 - 2\zeta_R\zeta_S\rho_{\ln R \ln S}}}$$

$$\beta_{MVFOSM} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \frac{g(\mathbf{M}_{\mathrm{X}})}{\nabla g(\mathbf{M}_{\mathrm{X}})\mathbf{\Sigma}_{\mathrm{XX}}\nabla g(\mathbf{M}_{\mathrm{X}})^{\mathrm{T}}} \quad \text{(Oct1974)}$$

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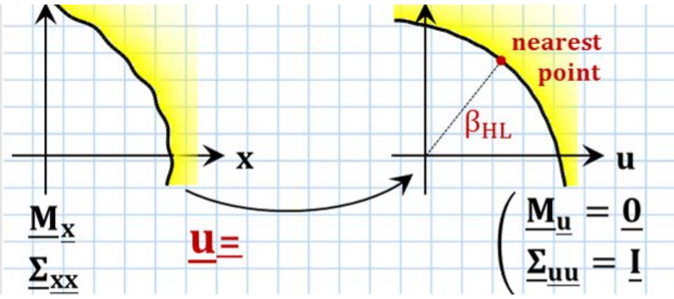
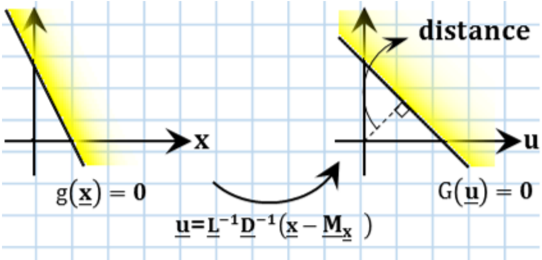
◎ Hasofer-Lind Reliability Index, β_{HL} (JEM, May 1974)

(or β_{AFOSM} "Advanced" FOSM)



Linear Limit-State Function

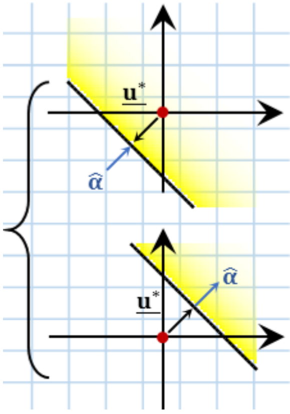
$$g(\mathbf{x}) = a_0 + \mathbf{a}^T\mathbf{x}$$
$$= a_0 + \mathbf{a}^T(\qquad\qquad)$$
$$= a_0 + \mathbf{a}^T\mathbf{M} + \mathbf{a}^T\mathbf{DLu}$$
$$= b_0 + \mathbf{b}^T\mathbf{u} = G(\mathbf{u})$$



$$\boxed{\beta = \frac{\mu_G}{\sigma_G} = \frac{b_0}{\|\mathbf{b}\|}} \quad \text{VS} \quad \text{distance} = \frac{|b_0|}{\|\mathbf{b}\|}$$

Can have +/– sign   always positive

For $G(\mathbf{u}) = b_0 + \mathbf{b}^T\mathbf{u}$



$b_o = G(\mathbf{0}) < 0$

(in failure domain)

$\beta < 0$

$b_o = G(\mathbf{0}) > 0$

(in safe domain)

$\beta > 0$

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i. $\hat{\boldsymbol{\alpha}} = -\dfrac{\nabla G}{\|\nabla G\|}$: "Negative normalized gradient vector"

: Unit row vector pointing toward the _______ domain

e.g. linear function : $\hat{\boldsymbol{\alpha}} = -\dfrac{\mathbf{b}^{\mathrm{T}}}{\|\mathbf{b}\|}$

ii. $\mathbf{u}^*$: "Design point"

"Most probable failure point (MPP)"

"Beta point"

e.g. linear function : $\mathbf{u}^* \equiv -b_0 \dfrac{\mathbf{b}}{\|\mathbf{b}\|^2}$

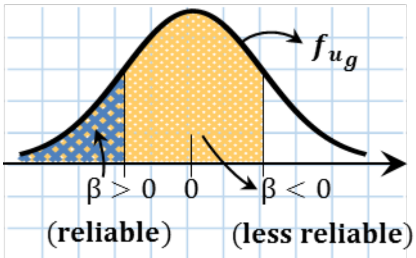
iii.

$$\boxed{\beta_{HL} \equiv \hat{\boldsymbol{\alpha}}\mathbf{u}^*}$$

Hasofer-Lind Reliability Index

$$\begin{cases} |\beta_{HL}| : \text{distance between origin and } \mathbf{u}^* \\ \text{sign : directions of } \hat{\boldsymbol{\alpha}} \text{ and } \mathbf{u}^* \end{cases}$$

e.g. linear function : $\beta_{HL} = \dfrac{b_0}{\|\mathbf{b}\|}\left(= \dfrac{\mu_G}{\sigma_G} \right)$



$P_f = F_{u_g}(-\beta_{HL})$

What if $\mathbf{X} \sim N(\mathbf{M}_{\mathrm{X}}, \sum_{\mathrm{XX}})$ and $g(\mathbf{x})$ linear?

$\Rightarrow G, g \sim N$

$P_f = \Phi(-\beta_{HL})$

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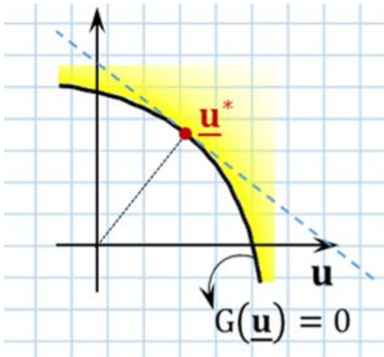
In-Class Material: Class 11

◎ Hasofer-Lind Reliability Index, β_{HL} (contd.)

② Nonlinear Limit-State Function



Transform $g(\quad)$ to $G(\quad)$ by

$$\begin{cases} \mathbf{X} = \\ \mathbf{u} = \end{cases}$$

- suspose one can find $\mathbf{u}^*$
- Linearize $G(\mathbf{u})$ at $\mathbf{u} =$

→ $G(u) \overset{FO}{\square} G(\quad) + \qquad (\qquad)$

$= \qquad\qquad = 0$

Reliability index

Try $\dfrac{\mu_G}{\sigma_G} \cong \dfrac{\mu_G^{FO}}{\sigma_G^{FO}}$?

$\mu_G^{FO} =$

$\sigma_G^{2\,FO} = \qquad\qquad = \|\qquad\|^2$

$\therefore \dfrac{\mu_G^{FO}}{\sigma_G^{FO}} = \qquad = \qquad =$

In summary, the "distance" between the origin and the design point $\mathbf{u}^*$ in $\mathbf{u}$-space gives reliability index based on first-order approximation

★ **Note!** $\begin{cases} MVFOSM & \dfrac{\mu^{FO}}{\sigma^{FO}} \text{ at } \mathbf{x} = \\ HL & \dfrac{\mu^{FO}}{\sigma^{FO}} \text{ at } \mathbf{u} = \end{cases}$

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※ Procedure : i) Transform $g(\mathbf{x})$ to $G(\mathbf{u})$ using $\mathbf{x} =$

ii) Find

iii) Find at

iv) $\beta_{HL} =$

※ Description of β_{HL} in $\mathbf{x}$ space?

$$\boxed{\nabla G(\mathbf{u}^*)(\mathbf{u}-\mathbf{u}^*) = 0} \quad \underset{\mathbf{x}=}{\overset{\mathbf{u}=}{\rightleftarrows}} \quad \boxed{\nabla g(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*) = 0}$$

Approx. Limit state space in $\mathbf{u}$

Proof :

$\nabla_{\mathbf{x}} g(\mathbf{x}^*) = \nabla_{\mathbf{u}} G(\mathbf{u}^*) \times$

$=$

$\mathbf{x}^* =$

$\mathbf{x} =$

$\therefore g^{FO} = \nabla g(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*)$

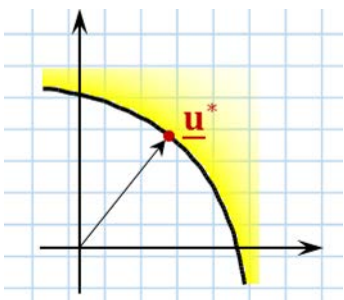
$$\boxed{\beta_{HL} = \frac{\mu_g^{FO}}{\sigma_g^{FO}} = \frac{\quad}{\sqrt{\quad}}}$$

FO at $\underline{\mathbf{x}} =$

Cf. $\beta_{MVFOSM} = \dfrac{\mu_g^{FO}}{\sigma_g^{FO}} = \dfrac{g(\mathbf{M}_\mathbf{x})}{\sqrt{\nabla g(\mathbf{M}_\mathbf{x}) \sum_{\mathbf{xx}} \nabla g(\mathbf{M}_\mathbf{x})^{\mathrm{T}}}}$

FO at $\underline{\mathbf{x}} =$

③ Finding the design point $\mathbf{u}^*$



$\mathbf{u}^* = \mathrm{argmin}\{ \quad | \quad \}$

Then evaluate $\hat{\boldsymbol{\alpha}} =$ at

And compute $\beta_{HL} = \hat{\boldsymbol{\alpha}}\mathbf{u}^*$

$\Rightarrow$ constrained nonlinear optimization problem

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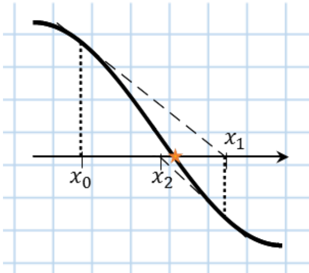
Reviews on optimization algorithm of finding $\mathbf{u}^*$

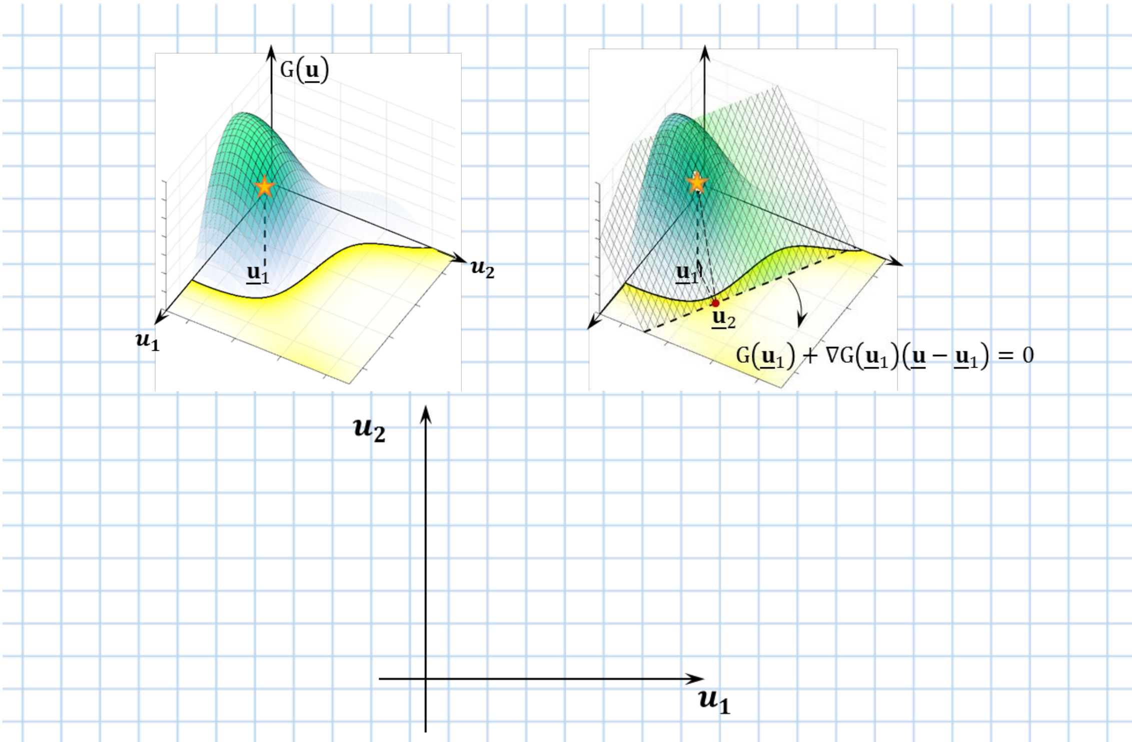
- Liu & ADK (1990)
- Papaioannou et al. (2010)

HL-RF, SQP, GP, DFO



a) HL-RF algorithm (Rackwitz & Fissler 1978)

"Newton-Raphson-like algorithm" solve $f(x)=0$ for $x = x^*$?



$\mathbf{u}_1$: initial point (e.g $\mathbf{u}_1 =$ $\mathbf{M}_{\mathbf{u}} = \mathbf{0}$)

$\mathbf{u}_2 = (\qquad\qquad\qquad) \times (\qquad\qquad\qquad)$

$=$

$=$

$\mathbf{u}_{i+1} =$

To update $\mathbf{u}_i$ to , $\mathbf{u}_{i+1}$, one needs

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$$G(\mathbf{u}_i) =$$

$$\nabla_{\mathbf{u}} G(\mathbf{u}_i) =$$

Iterate until 1)

2)

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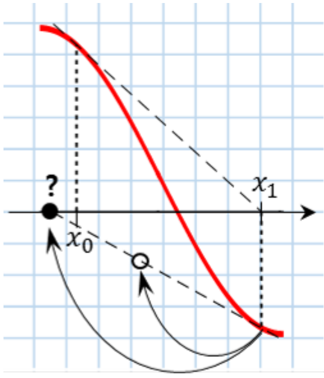
See Supplement, **"HL-RF Algorithm for HL Reliability Index and FORM/SORM"**

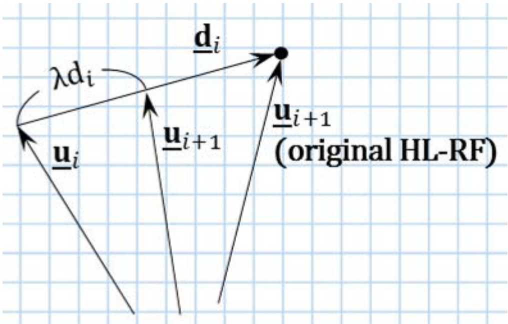
☆ **Convergence Issue**



Solution: Does not go full step, i.e. "step size" control

- Modified HL-RF (Liu & ADK 1990)
- Improved HL-RF (Zhang & ADK 1995)

$$\begin{cases} u_{i+1} = u_i + \lambda d_i \quad (\lambda,\ \text{stepsize}<1) \\ \mathbf{d}_i = \left(\hat{\boldsymbol{\alpha}}_i \mathbf{u}_i + \dfrac{G(\mathbf{u}_i)}{\|\nabla G(\mathbf{u}_i)\|} \right) \hat{\boldsymbol{\alpha}}_i^{\mathrm{T}} - \mathbf{u}_i \end{cases}$$



How? "Merit" function $m(\mathbf{u})$ is defined such that $m(\mathbf{u})$ is minimum at $\mathbf{u} =$

Then, select λ at each step such that $m(\mathbf{u})$ d____________

e.g. 1) Modified HL-RF: $m(\mathbf{u}) = \dfrac{1}{2}\left\| \mathbf{u} - \hat{\boldsymbol{\alpha}}\mathbf{u}\hat{\boldsymbol{\alpha}}^{\mathrm{T}} \right\|^2 + \dfrac{1}{2} c \cdot G(\mathbf{u})^2$

($m(\mathbf{u})$ can have minima that are not solution)

2) Improved HL-RF: $m(\mathbf{u}) = \dfrac{1}{2}\|\mathbf{u}\|^2 + c\left|G(\mathbf{u})\right|$

Select λ such that $m(\mathbf{u}_{i+1})$ $m(\mathbf{u}_i)$ because the direction vector is a descent direction in terms of merit function

as long as $c > \dfrac{\|\mathbf{u}_{i+1}\|}{\|\nabla G(\mathbf{u}_{i+1})\|}$

※ Zhang & ADK(1995) proved this based on so-called "Armijo's rule" and provided detailed updating rule for *c* (but FERUM uses a simple rule)

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Example: β_{HL} by improved HL-RF algorithm

Limit-state function $g(X_1, X_2) = 0.5X_1^2 - X_2 + 3\sin(2X_1)$

Mean vector and covariance matrix of X_1 and X_2:

$$\mathbf{M} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 4 & 5 \\ 5 & 25 \end{bmatrix}$$

Gradient $\nabla g = [X_1 + 6\cos(2X_1) \quad -1]$

Preparation:

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{L}\mathbf{L}^{\mathrm{T}} \quad \text{(Cholesky decomposition):} \quad \mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.87 \end{bmatrix}, \quad \mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 \\ -0.58 & 1.15 \end{bmatrix}$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{x} - \mathbf{M}_{\mathbf{x}}); \quad \mathbf{x}(\mathbf{u}) = \mathbf{DLu} + \mathbf{M}$$

$$\mathbf{J}_{\mathbf{u},\mathbf{x}} = \mathbf{L}^{-1}\mathbf{D}^{-1} = \begin{bmatrix} 0.5 & 0 \\ -0.29 & 0.23 \end{bmatrix}; \quad \mathbf{J}_{\mathbf{x},\mathbf{u}} = \mathbf{DL} = \begin{bmatrix} 2 & 0 \\ 2.5 & 4.33 \end{bmatrix} \quad \text{(constant since linear)}$$

Initialization:

$i = 1; \ \varepsilon_1 = \varepsilon_2 = 10^{-3}$

Starting point: $\mathbf{x}_1 = \mathbf{M} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}; \ \mathbf{u}_1 = \mathbf{u}(\mathbf{x}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Scale parameter: $G_0 = g(\mathbf{M}) = 0.5 \cdot 5^2 - 3 + 3 \cdot \sin(2 \cdot 5) = 7.87$

Computation (1st step):

$$G(\mathbf{u}_1) = g(\mathbf{x}_1) = 7.8679$$

$$\nabla G(\mathbf{u}_1) = \nabla g(\mathbf{x}_1)\mathbf{J}_{\mathbf{x},\mathbf{u}} = [-0.03 \quad -1]\begin{bmatrix} 2 & 0 \\ 2.5 & 4.33 \end{bmatrix} = [-2.57 \quad -4.33]$$

$$\hat{\alpha}_1 = -\frac{[-2.57 \quad -4.33]}{(2.57^2 + 4.33^2)^{1/2}} = [0.51 \quad 0.86]$$

Convergence check (1st step): Skipped.

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Update (1st→2nd):

$$c_1 \geq \frac{\|\mathbf{u}_1\|}{\|\nabla G(\mathbf{u}_1)\|} = 0\text{; Set } \quad c_1 = 10$$

Current value of the merit function:

$$m(\mathbf{u}_1) = 0.5\|\mathbf{u}_1\|^2 + c_1|G(\mathbf{u}_1)| = 0.5(0)^2 + 10(7.87) = 78.7$$

$$\mathbf{d}_1 = \left[\hat{\alpha}_1\mathbf{u}_1 + \frac{G(\mathbf{u}_1)}{\|\nabla G(\mathbf{u}_1)\|}\right]\hat{\alpha}_1^{\mathrm{T}} - \mathbf{u}_1$$

$$= \left\{[0.51 \quad 0.86]\begin{bmatrix}0\\0\end{bmatrix} + \frac{7.87}{5.03}\right\}\begin{bmatrix}0.51\\0.86\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix}$$

$$= \begin{bmatrix}0.80\\1.34\end{bmatrix}$$

Try a step size: $\lambda = 1$ (original HL-RF)

$$\mathbf{u}_2 = \mathbf{u}_1 + \lambda\mathbf{d}_1$$

$$= \begin{bmatrix}0\\0\end{bmatrix} + (1)\begin{bmatrix}0.80\\1.34\end{bmatrix} = \begin{bmatrix}0.80\\1.34\end{bmatrix}$$

Check $m(\mathbf{u}_2) < m(\mathbf{u}_1)$

$$\mathbf{x}_2 = \mathbf{x}(\mathbf{u}_2) = \begin{bmatrix}6.59\\10.81\end{bmatrix}$$

$$G(\mathbf{u}_2) = g(\mathbf{x}_2) = 0.5\cdot 6.59^2 - 10.81 + 3\sin(2\cdot 6.59) = 12.68$$

$$m(\mathbf{u}_2) = 0.5(6.59^2 + 10.81^2) + 10(12.68) = 126.82 > 78.7 \quad \textbf{N.G. (reject: } \lambda = 1\textbf{)}$$

Try a step size: $\lambda = 0.5$

$$\mathbf{u}_2 = \mathbf{u}_1 + \lambda\mathbf{d}_1$$

$$= \begin{bmatrix}0\\0\end{bmatrix} + (0.5)\begin{bmatrix}0.80\\1.34\end{bmatrix} = \begin{bmatrix}0.40\\0.67\end{bmatrix}$$

Check $m(\mathbf{u}_2) < m(\mathbf{u}_1)$

$$\mathbf{x}_2 = \mathbf{x}(\mathbf{u}_2) = \begin{bmatrix}5.80\\6.91\end{bmatrix}$$

$$G(\mathbf{u}_2) = g(\mathbf{x}_2) = 0.5\cdot 5.08^2 - 6.91 + 3\sin(2\cdot 5.08) = 7.42$$

$$m(\mathbf{u}_2) = 0.5(0.40^2 + 0.67^2) + 10(7.42) = 74.60 < 78.7 \quad \textbf{O.K. (accept: } \lambda = 0.5\textbf{)}$$

Computation (2nd step):

$$\nabla g = [X_1 + 6\cos(2X_1) \quad -1]$$

$$G(\mathbf{u}_2) = 7.42$$

$$\nabla G(\mathbf{u}_2) = [9.18 \quad -1]\begin{bmatrix}2 & 0\\2.5 & 4.33\end{bmatrix} = [15.86 \quad -4.33]$$

$$\hat{\alpha}_2 = [-0.97 \quad 0.26]$$

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Convergence check (2nd step):

$$\left|G(\mathbf{u}_2)/G_0\right| = \frac{7.42}{7.67} = 0.94 > \varepsilon_1 \quad \textbf{N.G.}$$

$$\left\|\mathbf{u}_2 - \hat{\alpha}_2\mathbf{u}_2\hat{\alpha}_2^{\mathrm{T}}\right\| = 0.75 > \varepsilon_2 \quad \textbf{N.G.}$$

Update (2nd→3rd):

$$c_2 \geq \left\|\mathbf{u}_2\right\| / \left\|\nabla G(\mathbf{u}_2)\right\| = 0.05\text{; set } c_2 = 10$$

$$\vdots$$

Repeat until the convergence criteria are satisfied.

Note: If $m(\mathbf{u}_{i+1}) \geq m(\mathbf{u}_i)$, reduce the value of λ until you satisfy $m(\mathbf{u}_{i+1}) < m(\mathbf{u}_i)$

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☆ **Santos, Matioli & Beck (2012)**

New optimization algorithms for structural reliability Analysis

⇒ provides a good review on HLRF, mHLRF and iHLRF
⇒ proposes nHLRF and two Lagrangian methods
⇒ nHLRF→ as efficient as iHLRF & more robust
⇒ Lagrangian→ Less efficient than HLRF's but more general and probably more suitable than HLRFs for large no. of rvs

◎ Reliability Indices VS Reliability Methods

$(\beta_{SM}, \beta_{SF}, \beta_{MVFOSM}, \beta_{HL})$ (P_f)

Reliability indices

- Use partial & (i.e. ∇)
- Do not provide a framework to consider type of of input r.v's
- P_f could be estimated for special cases only

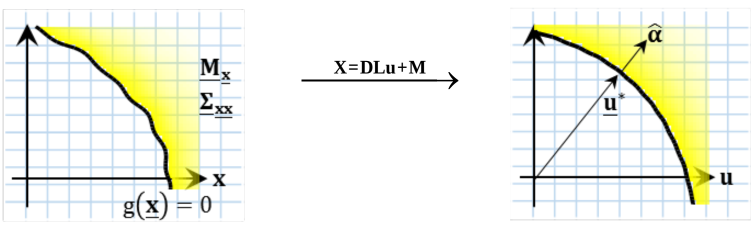
  (e.g., $P_f = \Phi(-\beta_{SM})$ when R, S ~ Normal)

→ Therefore, cannot be considered as reliability _________

cf. FORM/SORM ~ reliability methods

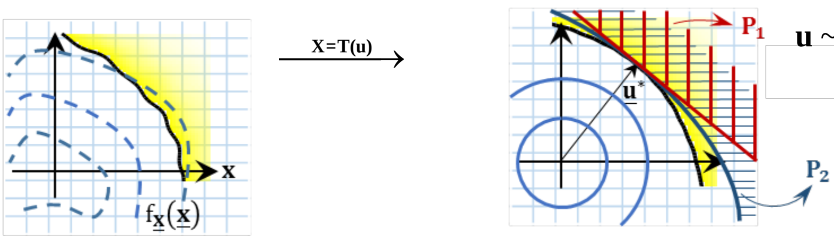
$$= \begin{array}{c}\text{design}\\ \text{point}\\ \text{concept}\\ (\text{e.g. } \beta_{HL})\end{array} + \begin{cases} \text{1) transformation to} \\ \quad \text{achieve } \mathbf{u} \sim N(\mathbf{0},\mathbf{I}) \\ \\ \text{2) procedure to get} \end{cases}$$

β_{HL} approach $\beta_{HL} = \hat{\boldsymbol{\alpha}}\mathbf{u}^*$



$$\begin{cases} \mathbf{M}_{\mathbf{u}} = \mathbf{0} \\ \boldsymbol{\Sigma}_{\mathbf{uu}} = \mathbf{I} \end{cases}$$

FORM/SORM



$\mathbf{u} \sim N(\mathbf{0},\mathbf{I})$

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◎ Probability in the Uncorrelated Standard Normal Space

$\mathbf{u} \sim N(\mathbf{0},\mathbf{I})$ (cf. $\mathbf{Z} \sim \mathrm{N}(\mathbf{0},\mathbf{R})$)

Joint PDF

$$\varphi(\mathbf{u}) = \frac{1}{(2\pi)^{n/2}}\exp(-\frac{1}{2}\|\mathbf{u}\|^2)$$

$$= \prod_{i=1}^{n}\varphi(u_i)$$

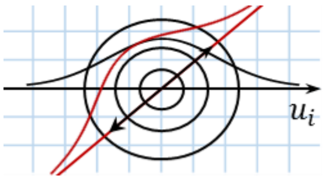
where $\varphi(u_i) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}u_i^2)$

① Rotational Symmetry

~the probability density is completely defined by from origin

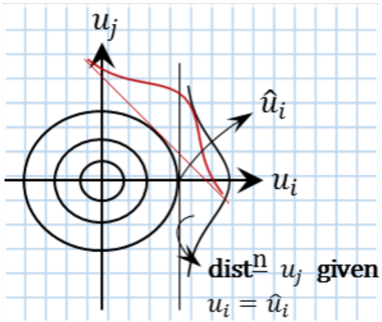
② Exponential Decay of Density

In r__________ direction



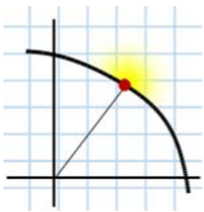
③ Exponential Decay of Density

In t__________ direction



$\mathbf{u}^*$: Richest point in terms of prob. density

Therefore, approximation around $\mathbf{u}^*$ should be good



④ FORM : First Order Reliability Method

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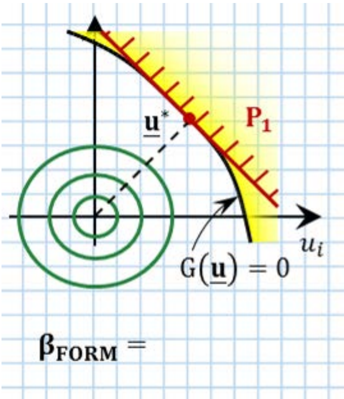
In-Class Material: Class 13

④ First-order reliability method (FORM)

$P_f \cong$ Probability in the linear half space determined by

FO approximation of failure domain at $\mathbf{u} =$

$= \ p_1$



i)

$G(\mathbf{u}) \quad\quad +$

$= \quad\quad\quad\quad \leq 0$

Divide by $\|\nabla G(\mathbf{u}^*)\|$

$(\mathbf{u} - \mathbf{u}^*) \leq 0$

$(\mathbf{u} - \mathbf{u}^*) \leq 0$

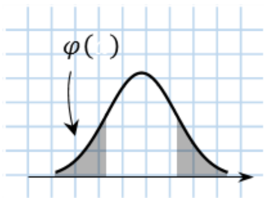
$\therefore\ p_1 = P(\quad\quad\quad \leq 0)$

Consider $Z = \hat{\boldsymbol{\alpha}}\mathbf{u} = \quad + \quad + \cdots +$

i) Type $\rightarrow$ (_____ function of)

ii) $\mu_Z =$

iii) $\sigma_Z^2 =$



In summary $Z \sim \ (\quad)$

$P_1 = P(\beta_{FORM} - \quad \leq 0)$

$\therefore \quad = P(\quad \geq \beta_{FORM})$

$= \Phi(\quad)$

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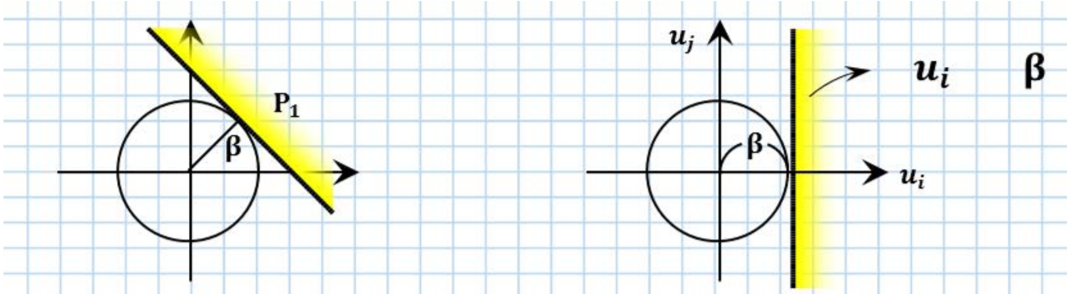
or

ii) From rotational symmetry



$$P_1 = P(u_i \quad \beta \quad) = \Phi(\quad)$$

※ SORM (p_2): later

◎ Probabilistic Transformation & Jacobian (to achieve $\mathbf{U} \sim$)

∴ Transformation (&Jacobian) depends on _____

cf. $\beta_{HL} \begin{cases} \mathbf{x}(\mathbf{u}) = \mathbf{DLu} + \mathbf{M} \\ \mathbf{J}_{\mathbf{x},\mathbf{u}} = \mathbf{DL} \end{cases}$

☆ Why do we need $\mathbf{X}(\mathbf{u})$ and $\mathbf{J}_{\mathbf{X},\mathbf{u}}$

$$\left.\begin{cases} G(\mathbf{u}_i) = g(\quad) \\ \nabla_{\mathbf{u}} G(\mathbf{u}_i) = \nabla_{\mathbf{x}} g(\quad) \end{cases}\right\} \rightarrow \text{need} \begin{cases} \mathbf{x}_i = \mathbf{x}(\mathbf{u}_i) \\ \mathbf{J}_{\mathbf{x},\mathbf{u}} = \mathbf{J}_{\mathbf{u},\mathbf{x}}^{-1} \text{ at} \end{cases}$$

⇒ Four cases

S.I	Dependent
①	② ③ ④

① **X** ~ statistically independent of each other

Each follows general distribution ($F_{X_i}(x_i)$ or $f_{X_i}(x_i)$)

$f_{\mathbf{X}}(\mathbf{x}) =$

$\Rightarrow$ Transformation $\boxed{u_i = \Phi^{-1}[\qquad\qquad]}$

Check $f_{\mathbf{U}}(\mathbf{u}) = \varphi_n(\mathbf{u};\quad) = \prod_{i=1}^{n}\qquad$?

$f_{\mathbf{u}}(\mathbf{u}) = f_{\mathbf{X}}(\mathbf{x}) \cdot \left|\qquad\qquad\right|$

$= \qquad\qquad \times \overline{\qquad\qquad}$ $\qquad J_{\mathbf{x,u}} = \left[\qquad\qquad\right]$

$=$

$\therefore \mathbf{u} \sim$

$\Rightarrow$ Jacobian $J_{\mathbf{x,u}}$

$J_{ii} = \frac{dx_i}{du_i} = \overline{\qquad\qquad}$; Ratio of PDFs

Note $F_{X_i}(x_i) = \Phi(u_i)$

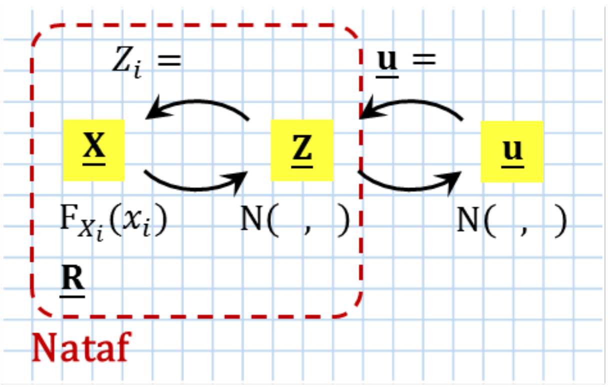
$f_{X_i}(x_i)dx_i = \varphi(u_i)du_i$

② **X** ~ Correlated Normal, $\mathbf{N(M,\Sigma)}$

$\Rightarrow$ Transform $\begin{cases} \mathbf{x} = \\ \mathbf{u} = \end{cases}$ $\qquad \begin{matrix} \mathbf{X} \\ N(\mathbf{M},\mathbf{\Sigma}) \end{matrix} \quad \square \quad \begin{matrix} \mathbf{u} \\ N(\mathbf{0},\mathbf{I}) \end{matrix}$

$\Rightarrow$ Jacobian $J_{\mathbf{x,u}}$ $\qquad$ therefore β_{HL} $\quad \beta_{FORM}$ for $\mathbf{X} \sim N(\quad)$

③ **X** ~ Nataf distribution : & available



note $\begin{cases} \mathbf{X(U) = DLU + M} \\ \mathbf{Z} = \end{cases}$

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$\Rightarrow$ Transform $\mathbf{u} =$ $\quad$ $\mathbf{z} =$ $\left\{ \quad \right\}$

$\Rightarrow$ Jacobian $\begin{cases} J_{\mathbf{u},\mathbf{x}} = J \quad \cdot J \quad = \\ J_{\mathbf{x},\mathbf{u}} = \end{cases}$

④ Non-normal, non-Nataf, dependent RVs

e.g. Hohenbichler & Rackwitz 1981 (named, Rosenblatt's transformation)

Transformation for non-normal, non-Nataf, dependent random variables

>> Rosenblatt's transformation (Rosenblatt 1952; Hohenbichler & Rachwitz 1981)

Given:

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_n}(x_n \mid x_1, \ldots, x_{n-1}) f_{X_{n-1}}(x_{n-1} \mid x_1, \ldots, x_{n-2}) \cdots f_{X_2}(x_2 \mid x_1) f_{X_1}(x_1)$$

~ conditional PDFs are available.

Transformation: triangular transformation

$$\begin{aligned} u_1 &= \Phi^{-1}\left[F_{X_1}(x_1)\right] \\ u_2 &= \Phi^{-1}\left[F_{X_2}(x_2 \mid x_1)\right] \\ &\vdots \\ u_n &= \Phi^{-1}\left[F_{X_n}(x_n \mid x_1, \ldots, x_{n-1})\right] \end{aligned}$$

** **Proof:** $\mathbf{U} \sim N(\mathbf{0}, \mathbf{I})$?

$$\begin{aligned} f_{\mathbf{U}}(\mathbf{u}) &= f_{\mathbf{X}}(\mathbf{x}) \left|\det \mathbf{J}_{\mathbf{u},\mathbf{x}}\right|^{-1} \\ &= f_{\mathbf{X}}(\mathbf{x}) \left[\prod_{i=1}^{n} J_{i,i}\right]^{-1} \quad (\because \mathbf{J}_{\mathbf{u},\mathbf{x}} \text{ lower triangular matrix}) \\ &= f_{\mathbf{X}}(\mathbf{x}) \frac{\varphi(u_1)}{f_{X_1}(x_1)} \frac{\varphi(u_2)}{f_{X_2}(x_2 \mid x_1)} \cdots \frac{\varphi(u_n)}{f_{X_n}(x_n \mid x_1, \ldots, x_{n-1})} \\ &= \prod_{i=1}^{n} \varphi(u_i) \quad \text{(uncorrelated standard normal)} \end{aligned}$$

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Jacobian: $\mathbf{J}_{\mathbf{u},\mathbf{x}} = [J_{ij}]$ where

$$J_{ij} = \begin{cases} \dfrac{f_{X_1}(x_1)}{\varphi(u_1)} & i = j = 1 \\ \dfrac{f_{X_i}(x_i \mid x_1,\ldots,x_{i-1})}{\varphi(u_i)} & i = j > 1 \\ \dfrac{1}{\varphi(u_i)} \dfrac{\partial F_{X_i}(x_i \mid x_1,\ldots,x_{i-1})}{\partial x_j} & i > j \\ 0 & i < j \end{cases}$$

** **What does** $F_{X_i}(x_i \mid x_1,\ldots,x_{i-1})$ **mean?**

First of all, $F_{X_i}(x_i \mid x_1,\ldots,x_{i-1}) \neq \dfrac{F_{X_1\ldots X_i}(x_1,\ldots,x_i)}{F_{X_1\ldots X_{i-1}}(x_1,\ldots,x_{i-1})}$. It is rather the conditional probability that $X_i \leq x_i$ given $X_1 = x_1, X_2 = x_2, \ldots, X_{i-1} = x_{i-1}$ that is,

$$\begin{aligned} F_{X_i}(x_i \mid x_1,\ldots,x_{i-1}) &= P(X_i \leq x_i \mid X_1 = x_1,\ldots,X_{i-1} = x_{i-1}) \\ &= \int_{-\infty}^{x_i} f_{X_i}(x_i \mid x_1,\ldots,x_{i-1})dx_i \\ &= \int_{-\infty}^{x_i} \frac{f(x_1,\ldots,x_i)}{f(x_1,\ldots,x_{i-1})}dx_i \\ &= \frac{1}{f(x_1,\ldots,x_{i-1})}\int_{-\infty}^{x_i} \frac{\partial^i F(x_1,\ldots,x_i)}{\partial x_1 \cdots \partial x_i}dx_i \\ &= \frac{1}{f(x_1,\ldots,x_{i-1})}\frac{\partial^{i-1} F(x_1,\ldots,x_i)}{\partial x_1 \cdots \partial x_{i-1}} \end{aligned}$$

For example,

$$F_{X_2}(x_2 \mid x_1) = \frac{1}{f_{X_1}(x_1)}\frac{\partial F(x_1,x_2)}{\partial x_1}, \quad F_{X_3}(x_3 \mid x_1,x_2) = \frac{1}{f(x_1,x_2)}\frac{\partial^2 F(x_1,x_2,x_3)}{\partial x_1 \partial x_2}$$

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457.646 Topics in Structural Reliability

In-Class Material: Class 14

FERUM: Finite Element Reliability Using Matlab®

FERUM (URL: http://www.ce.berkeley.edu/FERUM) is an open source Matlab® toolbox for structural reliability analysis, created by Dr. Terje Haukaas during his Ph.D. study at UC Berkeley (currently at the University of British Columbia).

- **FERUMcore** contains the core algorithms to perform FORM, SORM, Monte Carlo simulations and importance sampling.
- **FERUMlinearfecode** is a simple finite element code provided with FERUM to enable linear finite element reliability analysis with truss, beam or quad4 elements. Limit-state functions can be defined in terms of displacement reponse from this code. Gradients can be computed either by direct differentiation (DDM) or by a forward finite difference scheme.
- **FERUMnonlinearfecode** is an add-on to FERUMlinearfecode to enable nonlinear finite element reliability analysis. The J2 plasticity material is provided, and gradients can be computed by direct differentiation (DDM) or by forward finite difference. Truss and quad4 elements are available.
- **FERUMdynamicfecode** is yet another extension of FERUMlinearfecode to enable limit-state functions being defined in terms of response quantities from a dynamic finite element analysis.
- **FERUMlargedefofecode** is an add-on to enable limit-state functions being defined in terms of response quantities from a finite element code capable of large deformation analysis.
- **FERUMsystems** enables FERUM to perform system reliability analysis using the Matrix-based System Reliability (MSR) method. This part of FERUM was created by Bora Gencturk during his CEE491 term project, and is maintained by Junho Song.
- **FERUMrandomfield** is an add-on to the simple finite element codes provided with FERUM. It addresses the issue of characterizing material properties as random fields. Options for the simple 1D case was provided with the initial versions of FERUM. However, the main contributions to the current version have been made by Bruno Sudret, who has also provided a user's/theory manual for the random field part of FERUM (see the User's Guide section).
- **FERUMfedeasconnection** enables the finite element program FedeasLab developed by Professor Filip Filippou at UC Berkeley to be connected to FERUM. This provides for a quite powerful computational platform for finite element reliability analysis. This part is maintained by Paolo Franchin.
- **FERUMexamples** contains a collection of example input files for FERUM.

Recently, Dr. Jean-Marc Bourinet at the French Institute of Mechanical Engineering (IFMA) further developed FERUM (Bourinet et al. 2009). His FERUM4.0 now offers new features such as directional sampling, subset simulation, global sensitivity analysis and reliability-based design optimization. URL: http://www.ifma.fr/lang/en/Recherche/Labos/FERUM

◎ FERUM Example (Example 14.3.1.1 ADK 2005)

Limit-state function for a short column (elastic-perfect-plastic) under axial force and axial bending:

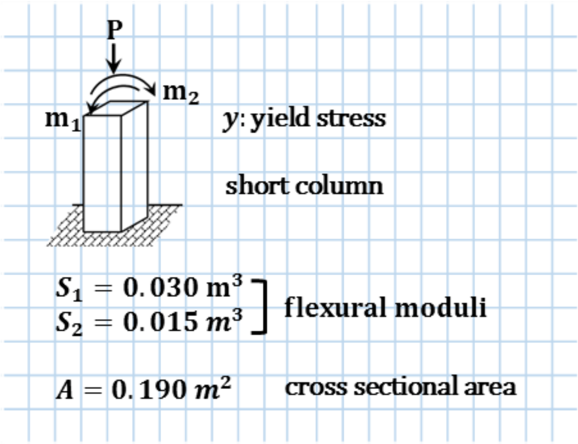
$$g(\mathbf{x}) = 1 - \frac{m_1}{s_1 y} - \frac{m_2}{s_2 y} - \left(\frac{P}{Ay}\right)^2$$

m_1 : Normal

m_2 : Normal

P : Gumbel

y : Weibull



⇒ FERUM results

$$\beta_{FORM} = 2.47$$
$$\mathbf{u}^* = \{1.21 \;\; 0.699 \;\; 0.941 \;\; -1.80\}^T$$
$$\mathbf{x}^* = \{341 \;\; 170 \;\; 3223 \;\; 31.8\}^T$$
$$\hat{\boldsymbol{\alpha}} = \{0.491 \;\; 0.283 \;\; 0.381 \;\; -0.731\}$$
$$P_f \cong \Phi(-\beta_{FORM}) = 0.00682$$

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% FERUM INPUTFILE

```
clear probdata femodel analysisopt gfundata randomfield systems results
output_filename

output_filename = 'output_Ch14_Example.txt';


probdata.marg(1,:) =  [ 1   2.5e5   2.5e5*0.3   2.5e5  0 0 0 0 0];
probdata.marg(2,:) =  [ 1  1.25e5  1.25e5*0.3  1.25e5  0 0 0 0 0];
probdata.marg(3,:) =  [15   2.5e6   2.5e6*0.2   2.5e6  0 0 0 0 0];
probdata.marg(4,:) =  [16   4.0e7   4.0e7*0.1   4.0e7  0 0 0 0 0];


probdata.correlation = [1.0 0.5 0.3 0.0;
                    0.5 1.0 0.3 0.0;
                    0.3 0.3 1.0 0.0;
                    0.0 0.0 0.0 1.0];


probdata.parameter = distribution_parameter(probdata.marg);


analysisopt.ig_max    = 100;
analysisopt.il_max    = 5;
analysisopt.e1       = 0.001;
analysisopt.e2       = 0.001;
analysisopt.step_code = 1;
analysisopt.grad_flag = 'DDM';
analysisopt.sim_point = 'dspt';
analysisopt.stdv_sim  = 1;
analysisopt.num_sim   = 100000;
analysisopt.target_cov = 0.0125;


gfundata(1).evaluator = 'basic';
gfundata(1).type = 'expression';
gfundata(1).parameter = 'no';
gfundata(1).expression = '1-x(1)/0.030/x(4)-x(2)/0.015/x(4)-
(x(3)/0.190/x(4))^2';
gfundata(1).dgdq = { '-1/0.030/x(4)' ;
                  '-1/0.015/x(4)';
                  '-2*x(3)/0.190^2/x(4)^2';
'x(1)/0.030/x(4)^2+x(2)/0.015/x(4)^2+2*x(3)^2/0.190^2/x(4)^3'};

femodel = 0;
randomfield.mesh = 0;
```

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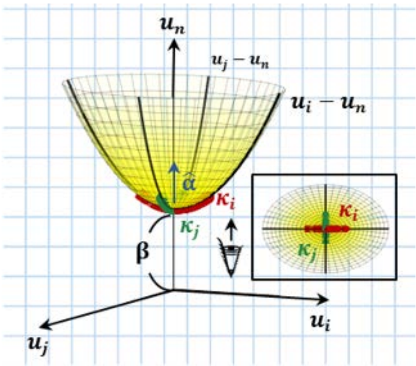
◎ Second- Order Reliability Method (read CRC ch.14)



P_f □ Prob in paraboloid in $\mathbf{u} \sim N(\mathbf{0},\mathbf{I})$

$$= p_2$$

$$= P(\beta - u_n + \frac{1}{2}\sum_{i=1}^{n-1} \kappa_i u_i^2 \le 0)$$

(κ : principal curvature in $u_i - u_n$ plane)

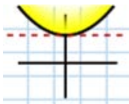
※ Formulas for p_2

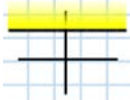
① Tvedt (exact; under the condition $\beta\kappa_i > -1$)

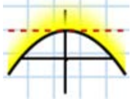
$$p_2 = \varphi(\beta)\,\mathrm{Re}\left\{ i\sqrt{\frac{2}{\pi}} \int_0^{i\infty} \frac{1}{s} \exp\left[\frac{(s+\beta)^2}{2}\right] \prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\kappa_i s}}\, ds \right\}$$

② (Karl) Breitung (simpler; derived earlier; approximate)

$$p_2 \cong \Phi(-\beta)\prod_{i=1}^{n-1}\frac{1}{\sqrt{1+\beta\kappa_i}}$$

$\kappa > 0$  p_2 p_1

$\kappa = 0$  p_2 p_1

$\kappa < 0$  p_2 p_1

③ Improved Breitung

$$p_2 \cong \Phi(-\beta)\prod_{i=1}^{n-1}\frac{1}{\sqrt{1+\psi(\beta)\kappa_i}} \quad \text{where} \quad \psi(\beta) = \frac{\varphi(\beta)}{\Phi(-\beta)}$$ (←erratum in Ch.14)

※ How to get κ_i's, $i = 1, \cdots, n-1$? (κ : principal curvature)

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① Curvature-fitting SORM (see in-class material)

⇒ Find (   ) matrix $\mathbf{H} = \left[\frac{\partial^2 G}{\partial u_i \partial u_j}\right]$ at $\mathbf{u} =$

⇒ Two rotations & eigenvalue analysis to obtain $\beta - u_n + \frac{1}{2}\sum \kappa_i u_i^2 \le 0$

⇒ Getting Hessian → Costly & Inaccurate

② Gradient-based SORM (ADK & De Stefano 1991)

⇒ Find the largest principal curvature from the trajectory of $\mathbf{u}$'s during HL-RF search to get $\mathbf{u}^*$

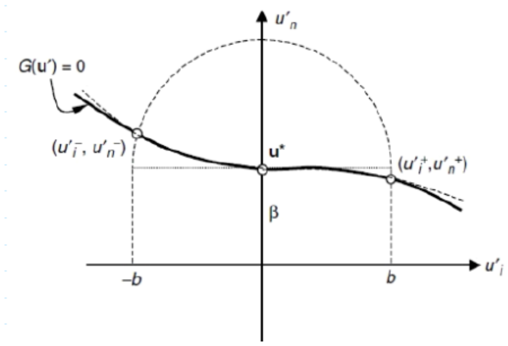
⇒ For the 2nd largest, perform HL-RF in the subspace orthogonal to u_n and u_i (that has the largest κ_i)

⇒ stop searching when $|\kappa_i| < \varepsilon$

⇒ does not need $\mathbf{H}$; can stop when $\|\kappa_i\|$ small

⇒ implementation issue?

③ Point-fitting SORM (ADK, Liu and Hwang 1987)



Fit by piecewise paraboloid surface

$$G(\mathbf{u}) \cong \beta - u_n + \frac{1}{2}\sum_{i=1}^{n-1} a_i^{\mathrm{sgn}(u_i)} \cdot u_i^2$$

where $a_i^{\mathrm{sgn}(u_i)} = \frac{2(u_n^{\mathrm{sgn}(u_i)} - \beta)}{2(u_i^{\mathrm{sgn}(u_i)})^2}$

$$b = \begin{cases} 1 & \text{if } |\beta| \le 1 \\ |\beta| & \text{if } 1 < |\beta| \le 3 \\ 3 & \text{if } |\beta| > 3 \end{cases}$$

Merit: Insensitive to the noise in calculating $g(\mathbf{x})$

Does not require derivative calculations $(\mathbf{H})$

Drawback: $2 \times (n-1)$ fitting points ⇒ solve numerically

Not invariant (rotation not unique)

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457.646 Topics in Structural Reliability

In-Class Material: Class 15

※ FERUM Example (SORM)

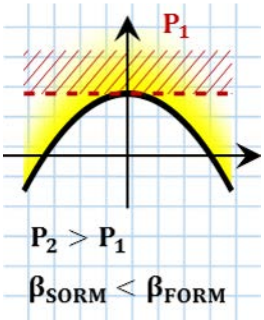
$$g(\mathbf{x}) = 1 - \frac{m_1}{s_1 y} - \frac{m_2}{s_2 y} - \left(\frac{P}{Ay}\right)^2 \le 0$$

$$\beta_{FORM} = 2.4661$$

(Curvature fitting)

$$\kappa_i \begin{cases} -1.548\times 10^{-1} \\ -3.997\times 10^{-2} \\ 8.903\times 10^{-7} \end{cases}$$

$$\beta_{SORM} = 2.3506(T),\ 2.3596(B),\ 2.341(iB)$$



(Point fitting)

$$a_i \begin{cases} \begin{array}{cc} + & - \\ -6.2969\times 10^{-2} & -4.0358\times 10^{-2} \\ -1.1986\times 10^{-2} & -9.7461\times 10^{-3} \\ -1.3778\times 10^{-1} & -1.1050\times 10^{-1} \end{array} \end{cases}$$

$$\beta_{SORM} = 2.3599(T),\ 2.3693(B),\ 2.3537(iB)$$

See supplement, “Importance and Sensitivity Vectors” (by A. Der Kiureghian)

➔ Main reference: Bjerager & Krenk (1989)

◎ FORM importance vector $\hat{\alpha}$

FORM approximation of the limit-state function

$$G(\mathbf{u}) \cong G(\mathbf{u}^*) + \nabla G(\mathbf{u}^*)(\mathbf{u} - \mathbf{u}^*)$$

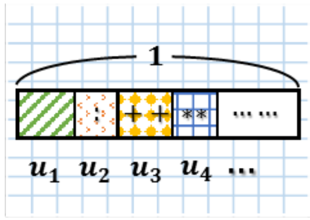
$$=$$

$$= \qquad\qquad (\beta - \hat{\alpha}\mathbf{u})$$

$$G'(\mathbf{u}) = \frac{G(\mathbf{u})^{FO}}{\left\|\nabla G(\mathbf{u}^*)\right\|} =$$

Note $\sigma_{G'}^{2} = (\qquad)\boldsymbol{\Sigma}_{\mathbf{uu}}(\qquad)$

$= \quad \hat{\boldsymbol{\alpha}} \quad \hat{\boldsymbol{\alpha}}^{T} \quad = \hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}^{T} = \boxed{\qquad\qquad} =$



Contribution (percentage) of u_i

to the total (variability)

of the limit-state function $G'(\mathbf{u})$

① Magnitude of α_i^2 $\Rightarrow$ measure of relative importance (contribution to the uncertainty) of u_i's

② Sign of α_i $\Rightarrow$ nature of u_i's e.g., $g(\mathbf{X}) = R - S$

$G'(\mathbf{u}) = \beta - \hat{\boldsymbol{\alpha}}\mathbf{u} = \beta -$

α_i positive $\Rightarrow$ u_i capacity or demand

α_i negative $\Rightarrow$ u_i capacity or demand

Question) Importance of u_i $\overset{?}{=}$ Importance of X_i

i) Independent : $u_i = \Phi^{-1}\left[F_{X_i}(x_i)\right]$ OK

ii) Dependent: e.g., Nataf NOT OK

$$\mathbf{u} = \mathbf{L}_0^{-1}\mathbf{z} = \mathbf{L}_0^{-1}\begin{Bmatrix} \Phi^{-1}\left[F_{X_1}(x_1)\right] \\ \vdots \\ \Phi^{-1}\left[F_{X_n}(x_n)\right] \end{Bmatrix}$$

$\therefore$ $\hat{\alpha}_i$ does NOT (Measure importance / Indicate the nature) of x_i's

when X_i's are .

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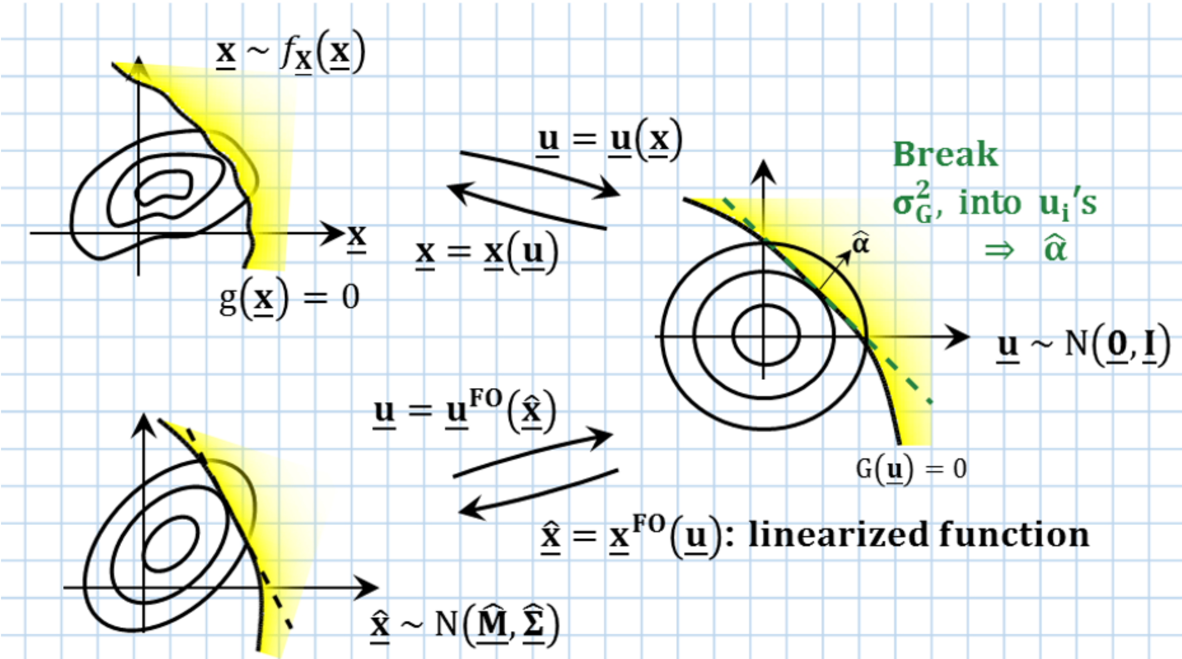
◎ Form importance vector $\hat{\gamma}$ (Question: contribution/nature of x_i? Not u_i's)



Transform to "normal equivalent" of $\mathbf{x}$

Why? Want to keep ( ) distribution

Want to recover ( )

$\mathbf{u}^{FO}(\mathbf{x})$?

$$\begin{cases} \mathbf{u} = \mathbf{u}(\mathbf{x}^*) + J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*) \\ \hat{\mathbf{x}} = \mathbf{x}^* + J_{\mathbf{u},\mathbf{x}}^{-1}(\mathbf{u} - \mathbf{u}^*) \end{cases} \qquad \textbf{(*)}$$

Note: Jacobians evaluated at $\mathbf{x} =$

$\hat{\mathbf{X}} \sim N(\hat{\mathbf{M}}, \hat{\mathbf{\Sigma}})$

$$\begin{cases} \hat{\mathbf{M}} = \\ \hat{\mathbf{\Sigma}} = \end{cases}$$

Substituting (*) into $G'(\mathbf{u}) = \beta - \hat{\boldsymbol{\alpha}}\mathbf{u}$,

$$G'(\mathbf{u}) = G''(\hat{\mathbf{x}}) = \beta - \hat{\boldsymbol{\alpha}}[\mathbf{u}^* + J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)]$$

$$= \beta - \hat{\boldsymbol{\alpha}}\mathbf{u}^* - \hat{\boldsymbol{\alpha}}J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)$$

$$= -\hat{\boldsymbol{\alpha}}J_{\mathbf{u},\mathbf{x}}(\hat{\mathbf{x}} - \mathbf{x}^*)$$

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$$\sigma_{G"}{}^2 = (-\hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}})\hat{\Sigma}(-J_{\mathbf{u},\mathbf{x}}{}^{\mathrm{T}}\hat{\boldsymbol{\alpha}}^{\mathrm{T}})$$

$$= \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}} J_{\mathbf{u},\mathbf{x}}{}^{-1} (J_{\mathbf{u},\mathbf{x}}{}^{-1})^{\mathrm{T}} J_{\mathbf{u},\mathbf{x}}{}^{\mathrm{T}} \hat{\boldsymbol{\alpha}}^{\mathrm{T}}$$

$$= \qquad = \left\| \qquad \right\|^2 \quad =$$ Contribution of each $\hat{x}_i$?

$$\hat{\boldsymbol{\Sigma}} = \hat{\mathbf{D}}\hat{\mathbf{D}} + (\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}})$$

diagonal off-diagonal

$$\sigma_{G"}^2 = \quad \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}} (\hat{\mathbf{D}}\hat{\mathbf{D}}) J_{\mathbf{u},\mathbf{x}}{}^T \hat{\boldsymbol{\alpha}}^T + \quad \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}} (\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}}) J_{\mathbf{u},\mathbf{x}}{}^T \hat{\boldsymbol{\alpha}}^T = 1$$

Contribution from variances $\sigma_{\hat{x}_i}^2$ Contribution from covariances $COV[\hat{x}_i, \hat{x}_j]$

Then, how about using $\hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}} \hat{\mathbf{D}}$ instead of $\hat{\boldsymbol{\alpha}}$?

But not normalized yet.

$$\boxed{\therefore \quad \hat{\boldsymbol{\gamma}} = \frac{\qquad}{\qquad}}$$

i) Magnitude of $\hat{\gamma}_i^2$ $\rightarrow$ contribution (importance) of $\hat{x}_i$ or x_i

ii) Sign of $\hat{\gamma}_i$ $\rightarrow$nature of $\hat{x}_i$ or x_i

Note : $G"(\hat{\mathbf{x}}) = -\hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}} (\hat{\mathbf{x}} - \mathbf{x}^*)$

$\hat{\gamma}_i$ positive $\rightarrow$ ______________ type r.v x_i

$\hat{\gamma}_i$ negative $\rightarrow$ ______________ type r.v x_i

Note : when $\mathbf{x}$ are independent, $\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\gamma}}$?

$$\hat{\boldsymbol{\Sigma}} = (J_{\mathbf{u},\mathbf{x}}{}^{-1})(J_{\mathbf{u},\mathbf{x}}{}^{-1})^T = \hat{\mathbf{D}}\hat{\mathbf{D}} + (\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{D}}\hat{\mathbf{D}})$$

$$\hat{\mathbf{D}} =$$

$$\hat{\boldsymbol{\gamma}} = \frac{\hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}} \hat{\mathbf{D}}}{\left\| \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\mathbf{x}} \hat{\mathbf{D}} \right\|} =$$

※ **FERUM Example** ($\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\gamma}}$)

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457.646 Topics in Structural Reliability

In-Class Material: Class 16

◎ FORM importance vectors; $\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\gamma}}$

◎ FORM parameter sensitivities of β **;** $\dfrac{\partial\beta}{\partial\theta}$

(Bjerager & Krenk, 1989) (See Supp)

θ — $\theta \in \boldsymbol{\theta}_g$: parameters in , $g(\mathbf{x};\boldsymbol{\theta}_g)$

e.g. $g(\mathbf{x};\boldsymbol{\theta}_g) = 1 - \dfrac{M}{M_u} - \left(\dfrac{P}{P_u}\right)^2 \le 0 \qquad \boldsymbol{\theta}_g = \{M_u \;\; P_u\}$

$\theta \in \boldsymbol{\theta}_f$: ________________ parameters in $f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta}_f)$

e.g. $\sigma, \mu, \rho, \lambda, \xi, b$

① Case $\theta \in \boldsymbol{\theta}_f$ (distribution) ※ Derivations → see Supplement

$$\boxed{\frac{d\beta}{d\theta} = \hat{\boldsymbol{\alpha}} \frac{\partial \mathbf{u}(\mathbf{x}^*, \theta)}{\partial \theta}}$$

Obtain $\hat{\boldsymbol{\alpha}}$ by FORM analysis

Derive $\dfrac{\partial \mathbf{u}(\mathbf{x},\theta)}{\partial\theta}$ from $\mathbf{u}(\mathbf{x},\theta)$ and evaluate it at $\mathbf{x} = \mathbf{x}^*$

⇒ Vector version $\nabla_{\boldsymbol{\theta}_f}\beta = \hat{\boldsymbol{\alpha}} J_{\mathbf{u},\boldsymbol{\theta}_f}(\mathbf{x}^*, \boldsymbol{\theta}_f)$

e.g. $\mathbf{x} \sim$ s.i. Normal

$$\mathbf{u} = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{X}-\mathbf{M})$$
$$= \mathbf{D}^{-1}(\mathbf{X}-\mathbf{M})$$

$u_1 =$, $u_2 =$ $\cdots$

$\dfrac{\partial u_1}{\partial \sigma_1} =$ $\qquad \therefore \dfrac{\partial u_1}{\partial \sigma_1}(\mathbf{x}^*) =$

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② Case $\theta \in \boldsymbol{\theta}_g$ (limit-state function)

$$\boxed{\frac{d\beta}{d\theta} = \frac{1}{\left\|\nabla_{\mathbf{u}} G(\mathbf{u}^*, \theta)\right\|} \frac{\partial g(\mathbf{x}^*, \theta)}{\partial \theta}}$$

↖ FORM ↖ derive from $g(\mathbf{x})$

⇒ Vector version

$$\nabla_{\boldsymbol{\theta}_g} \beta = \frac{1}{\left\|\nabla_{\mathbf{u}} G(\mathbf{u}^*, \boldsymbol{\theta})\right\|} \nabla_{\boldsymbol{\theta}_g} g(\mathbf{x}^*, \boldsymbol{\theta}_g)$$

e.g.

$$g(\mathbf{x}, \boldsymbol{\theta}_g) = 1 - \frac{M}{M_u} - \left(\frac{P}{P_u}\right)^2 \leq 0$$

θ_g

$$\frac{\partial g}{\partial \theta} = \qquad\qquad \therefore \frac{\partial g}{\partial \theta}(\mathbf{x}^*) =$$

◎ Parameter Sensitivities of failure probability P_f : $\dfrac{\partial P_f}{\partial \theta}$?

Recall $P_f = \Phi(\qquad)$

$$\frac{dP_f}{d\theta} =$$

Vector version:

$$\nabla_{\boldsymbol{\theta}} P_f = -\varphi(-\beta) \nabla_{\boldsymbol{\theta}} \beta$$

◎ Parameter sensitivities w.r.t. alternative parameters

$$\boldsymbol{\theta}_f = \boldsymbol{\theta}_f(\boldsymbol{\theta}_{f'})$$

λ, ξ $\qquad$ μ, σ

$$e.g. \begin{bmatrix} \lambda \\ \xi \end{bmatrix} = \begin{bmatrix} \ln \mu - 0.5 \ln[1 + (\frac{\sigma}{\mu})^2] \\ \sqrt{\ln[1 + (\frac{\sigma}{\mu})^2]} \end{bmatrix}$$

$\boldsymbol{\theta}_f$ $\qquad$ $\boldsymbol{\theta}_f(\boldsymbol{\theta}_{f'})$ $\qquad$ μ, σ

$$\nabla_{\boldsymbol{\theta}_{f'}} \beta = \nabla_{\boldsymbol{\theta}_f} \beta \cdot$$

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% FERUM Input File for CRC CH14 Example (with Parameter)

```
clear probdata femodel analysisopt gfundata randomfield systems results
output_filename

output_filename = 'output_Ch14_Example_param.txt';


probdata.marg(1,:) =  [ 1   2.5e5   2.5e5*0.3   2.5e5  0 0 0 0 0];
probdata.marg(2,:) =  [ 1  1.25e5  1.25e5*0.3  1.25e5  0 0 0 0 0];
probdata.marg(3,:) =  [15   2.5e6   2.5e6*0.2   2.5e6  0 0 0 0 0];
probdata.marg(4,:) =  [16  4.0e7   4.0e7*0.1   4.0e7  0 0 0 0 0];


probdata.correlation = [1.0 0.5 0.3 0.0;
                    0.5 1.0 0.3 0.0;
                    0.3 0.3 1.0 0.0;
                    0.0 0.0 0.0 1.0];

probdata.parameter = distribution_parameter(probdata.marg);

analysisopt.ig_max    = 100;
analysisopt.il_max    = 5;
analysisopt.e1       = 0.001;
analysisopt.e2       = 0.001;
analysisopt.step_code = 0;
analysisopt.grad_flag = 'DDM';
analysisopt.sim_point = 'dspt';
analysisopt.stdv_sim  = 1;
analysisopt.num_sim   = 100000;
analysisopt.target_cov = 0.05;


gfundata(1).evaluator = 'basic';
gfundata(1).type = 'expression';
gfundata(1).parameter = 'yes'; % "We have a parameter in the limit-state
function"
gfundata(1).thetag = [0.03]; % default value of S1
gfundata(1).expression = '1-x(1)/gfundata(1).thetag(1)/x(4)-
x(2)/0.015/x(4)-(x(3)/0.190/x(4))^2';
gfundata(1).dgdq = { '-1/gfundata(1).thetag(1)/x(4)' ;
                  '-1/0.015/x(4)';
                  '-2*x(3)/0.190^2/x(4)^2';

'x(1)/gfundata(1).thetag(1)/x(4)^2+x(2)/0.015/x(4)^2+2*x(3)^2/0.190^2/x(4
)^3'};
gfundata(1).dgthetag = {'x(1)/x(4)/gfundata(1).thetag(1)^2'}; %
Derivative w.r.t. S1

femodel = 0;
randomfield.mesh = 0;
```

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◎ Importance Vectors Using Parameter Sensitivities

⇒ Use $\nabla_{\mathbf{M}}\beta$ and $\nabla_{\mathbf{D}}\beta$ to quantify importance of random variables?

$\dfrac{\partial \beta}{\partial \mu_1} \gg \dfrac{\partial \beta}{\partial \mu_2} \rightarrow$ more to than

① Importance vector $\boldsymbol{\delta}$

$$\boxed{\boldsymbol{\delta} = \nabla_{\mathbf{M}}\beta \cdot \mathbf{D}}$$

$$= \left[\frac{\partial \beta}{\partial \mu_1} \cdot \quad , \frac{\partial \beta}{\partial \mu_2} \cdot \quad \cdots, \frac{\partial \beta}{\partial \mu_n} \cdot \quad \right]$$

Why?

- X_i's Can have different units & dimensions (therefore μ_i's) ⇒ make it dimensionless
- Assume variations in $\mu_i \propto$
- Change in β when μ_i change by

② Importance vector $\boldsymbol{\eta}$

$$\boxed{\boldsymbol{\eta} = \nabla_{\mathbf{D}}\beta \cdot \mathbf{D}}$$

$$= \left[\frac{\partial \beta}{\partial \sigma_1} \cdot \quad , \frac{\partial \beta}{\partial \sigma_2} \cdot \quad , \cdots, \frac{\partial \beta}{\partial \sigma_n} \quad \right]$$

Change in β when σ_i change by

③ Upgrade worth $\mathbf{I}_{\boldsymbol{\theta}}$

$$\boxed{\mathbf{I}_{\boldsymbol{\theta}} = -\nabla_{\boldsymbol{\theta}} P_f \mathbf{D}_{\boldsymbol{\theta}}}$$

$$= \left[-\frac{\partial P_f}{\partial \theta_1} \quad \cdots, -\frac{\partial P_f}{\partial \theta_n} \quad \right]$$

$$\mathbf{D}_{\boldsymbol{\theta}} = \begin{bmatrix} \ddots & & \\ & \Delta\theta_i & \\ & & \ddots \end{bmatrix}$$

Change in θ_i that can be achieved by unit _______

- Der Kiureghian, Ditlevsen & Song (2007)
- Song & Kang (2009)

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◎ Use of sensitivity / Importance Vectors

$(\nabla_{\boldsymbol{\theta}}\beta)$ $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\gamma}}, \boldsymbol{\delta}, \boldsymbol{\eta})$

① To identify important rv's

② To update β for small increment

$$\beta_{new} \cong \beta_{old} + \sum_i \frac{\partial \beta}{\partial \theta_i} \cdot \Delta\theta_i$$

③ Reliability Based Design Optimization

$\Rightarrow \frac{\partial \beta}{\partial \theta}$ needed to facilitate the use of ( )-based optimizers

④ To compute PDF of a function $y(\mathbf{x})$

$$F_Y(\theta) = P(Y(\mathbf{x}) \le \theta)$$
$$= P(Y(\mathbf{x}) - \theta \le 0)$$ here consider $Y(\mathbf{x})$-θ as the limit state function $g(\mathbf{x},\theta)$
$$\simeq \Phi(-\beta(\theta))$$

$$f_Y(\theta) = \frac{dF_y(\theta)}{d\theta} = -\varphi(-\beta(\theta))\frac{d\beta}{d\theta}$$

⑤ To help gain insight of the reliability problem

IV. System Reliability

Junho Song

Associate Professor
Department of Civil and Environmental Engineering
Seoul National University

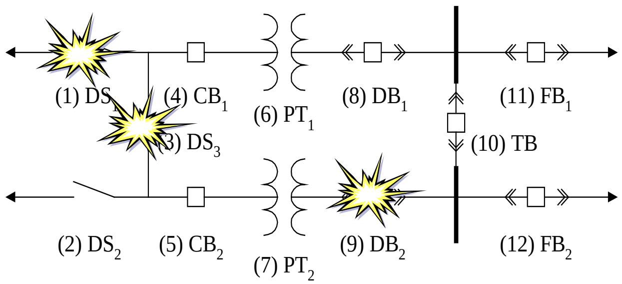
System reliability?

Failure event	E_{sys}
Abnormal flight (engine)	$E_1 \cup E_2$
Emergency	$E_1 E_2$
Landing at nearby airport	$E_1 \overline{E}_2 \cup \overline{E}_1 E_2$

$P(E_{sys})$?

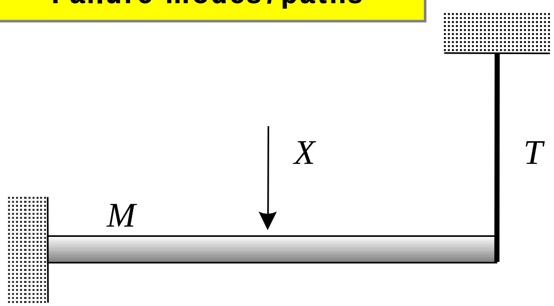
System reliability in structural engineering

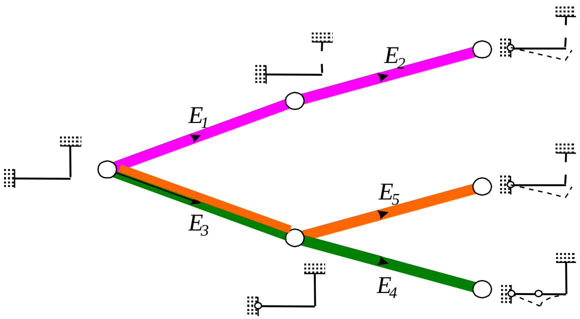
Lifeline networks



$$E_{system} = E_1E_2 \cup E_4E_5 \cup E_4E_7 \cup E_4E_9 \cup E_5E_6 \cup E_6E_7 \cup E_6E_9 \cup E_5E_8 \cup E_7E_8 \cup E_8E_9 \cup E_{11}E_{12} \cup E_1E_3E_5 \cup E_1E_3E_7 \cup \boxed{E_1E_3E_9} \cup E_2E_3E_4 \cup E_2E_3E_6 \cup E_2E_3E_8 \cup E_4E_{10}E_{12} \cup E_6E_{10}E_{12} \cup E_8E_{10}E_{12} \cup E_5E_{10}E_{11} \cup E_7E_{10}E_{11} \cup E_9E_{10}E_{11} \cup E_1E_3E_{10}E_{12} \cup E_2E_3E_{10}E_{11}$$

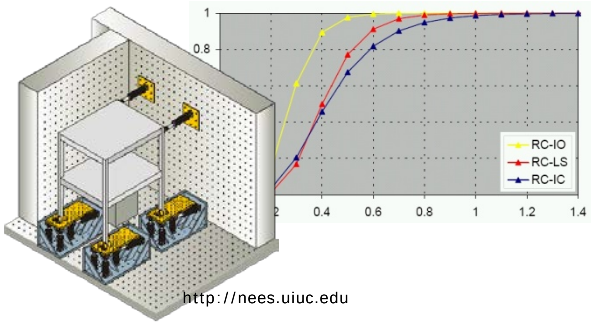
Failure modes/paths





$$E_{system} = (E_1 \cap E_2) \cup (E_3 \cap E_4) \cup (E_3 \cap E_5)$$

Damage/loss aggregation



$$L_{system} = f(\mathbf{D}, \mathbf{\Theta})$$

$$\mathrm{E}[L_{system}] \cong f(\mathrm{E}[\mathbf{D}], \mathrm{E}[\mathbf{\Theta}])$$

$$\mathrm{Var}[L_{system}] \cong \nabla f^{\mathrm{T}} \mathbf{\Sigma} \nabla f$$

$$P(L_{system} \geq c) \cong 1 - \Phi\left(\frac{c - \mathrm{E}[L_{system}]}{\sqrt{\mathrm{Var}[L_{system}]}} \right)$$

Outline

I. System reliability: definitions, existing methods and challenges

II. Bounds of system reliability by linear programming (‘LP bounds’)

III. Matrix-based system reliability (MSR) method

I. System Reliability:

- definitions, existing methods and challenges

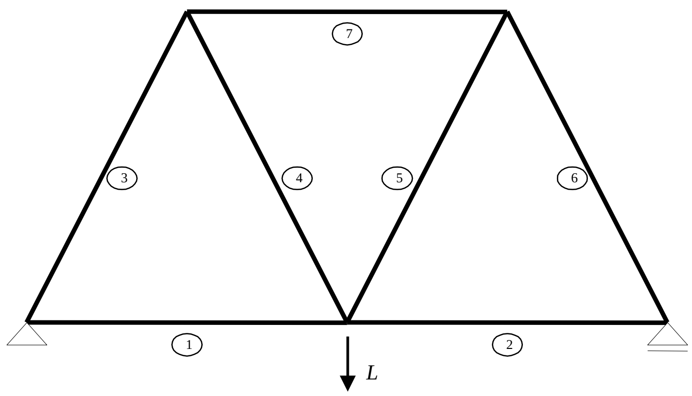
Definition of system: (1) series system

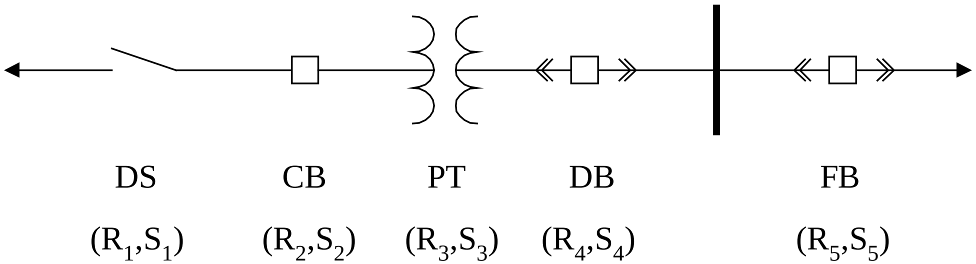
- System fails ***if any*** of its component events occur

$$E_{\text{system}} = \bigcup_{i=1}^{n} E_i$$

- Systems with no redundancy
- Examples: 1) statically determinate structure

  2) electrical substation with single-transmission-line





Song, J., and A. Der Kiureghian (2003, JEM ASCE)

Song, J., and A. Der Kiureghian (2003, ICASP9)

Definition of system: (2) parallel system

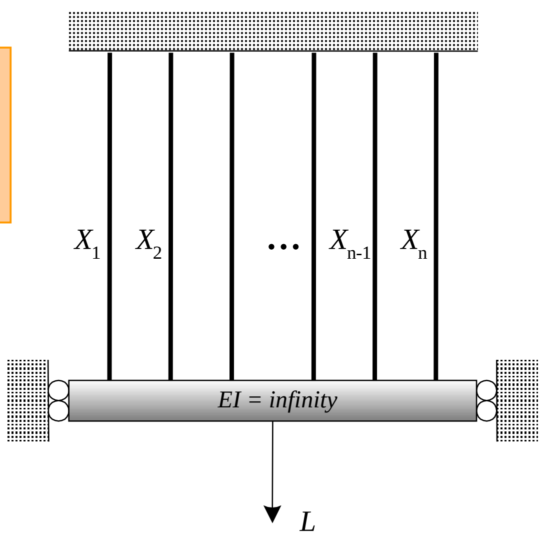
- System fails ***only if every*** component event occurs

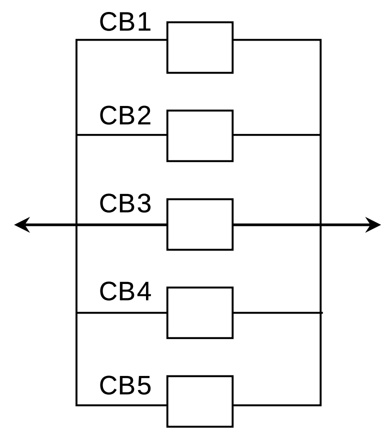
$$E_{\text{system}} = \bigcap_{i=1}^{n} E_i$$

- Systems with maximum redundancy
- Examples: 1) a bunch of wires or cables.

  2) electrical substation with equipment items in parallel.

Song, J., and A. Der Kiureghian (2003, JEM ASCE)





Definition of system: (3) general system

- System that is ***neither series or parallel*** system

1) Cut-set system:

- a series system of sub-parallel systems

$$E_{\text{system}} = \bigcup_{k=1}^{K} C_k = \bigcup_{k=1}^{K} \bigcap_{i \in C_k} E_i$$

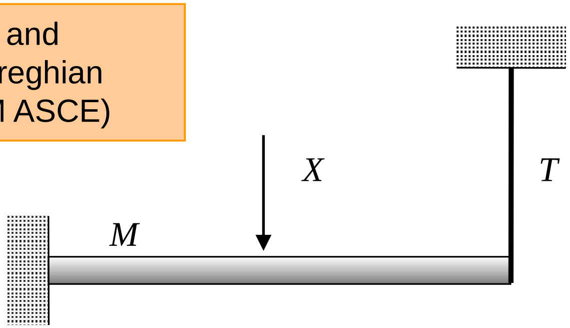
2) Link-set system:

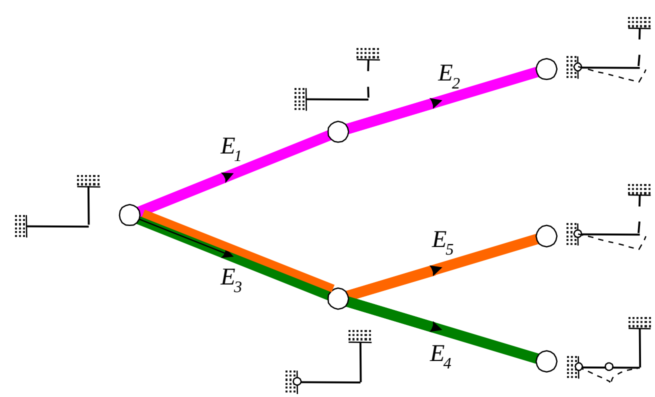
- a parallel system of sub-series systems

$$E_{\text{system}} = \bigcap_{l=1}^{L} L_l = \bigcap_{l=1}^{L} \bigcup_{i \in L_l} E_i$$

- Example: a structure with multiple failure paths (scenarios) ~ a cut-set system

Song, J., and
A. Der Kiureghian
(2003, JEM ASCE)



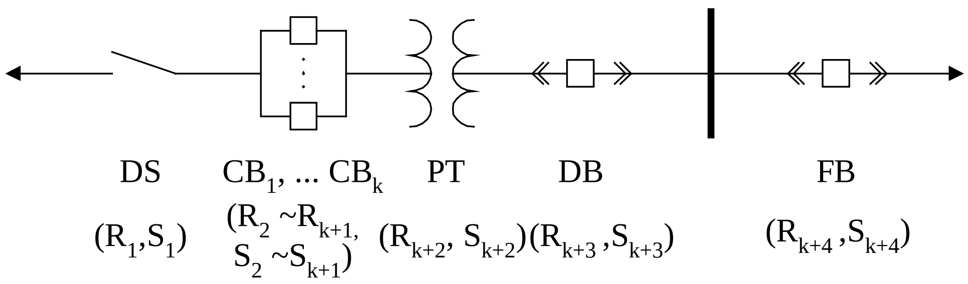


$$E_{\text{system}} = \underbrace{(E_1 \cap E_2)}_{\text{Scenario 1}} \cup \underbrace{(E_3 \cap E_4)}_{\text{Scenario 2}} \cup \underbrace{(E_3 \cap E_5)}_{\text{Scenario 3}}$$

* Component failure events and failure paths

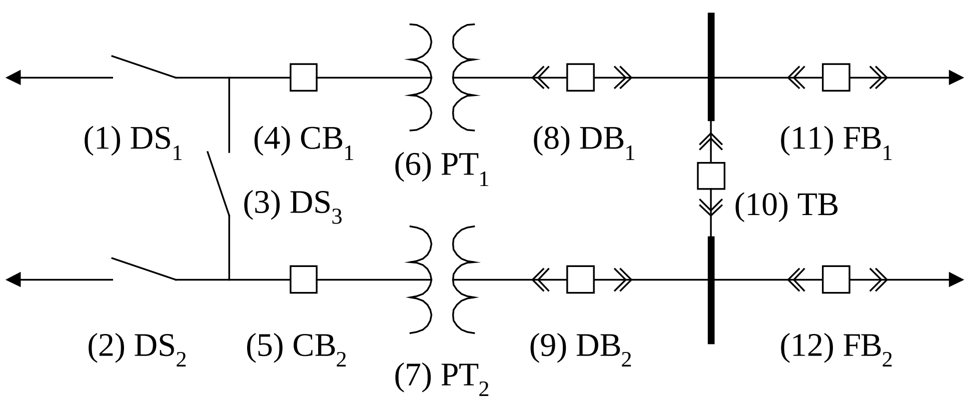
(3) General system (contd.)

- Example: electrical substations (cut-set systems)



$$P(E_{system}) =$$
$$P[E_1 \cup (E_2 E_3 \cdots E_{k+1}) \cup E_{k+2} \cup E_{k+3} \cup E_{k+4}]$$

* 5 cut sets, k+4 components



$$P(E_{system}) =$$
$$\begin{aligned}P(&E_1E_2 \cup E_4E_5 \cup E_4E_7 \cup E_4E_9 \cup E_5E_6 \cup E_6E_7 \cup \\ &E_6E_9 \cup E_5E_8 \cup E_7E_8 \cup E_8E_9 \cup E_{11}E_{12} \cup \\ &E_1E_3E_5 \cup E_1E_3E_7 \cup E_1E_3E_9 \cup E_2E_3E_4 \cup \\ &E_2E_3E_6 \cup E_2E_3E_8 \cup E_4E_{10}E_{12} \cup E_6E_{10}E_{12} \cup \\ &E_8E_{10}E_{12} \cup E_5E_{10}E_{11} \cup E_7E_{10}E_{11} \cup E_9E_{10}E_{11} \cup \\ &E_1E_3E_{10}E_{12} \cup E_2E_3E_{10}E_{11})\end{aligned}$$

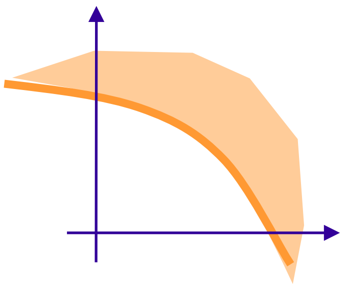
* 25 cut sets, 12 components

Song, J., and A. Der Kiureghian (2003, ICASP9)

"component" reliability vs "system" reliability

- Component reliability analysis: $P(E_i) = P(g_i(\mathbf{X}) \le 0) = \int_{g(\mathbf{X})\le 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$

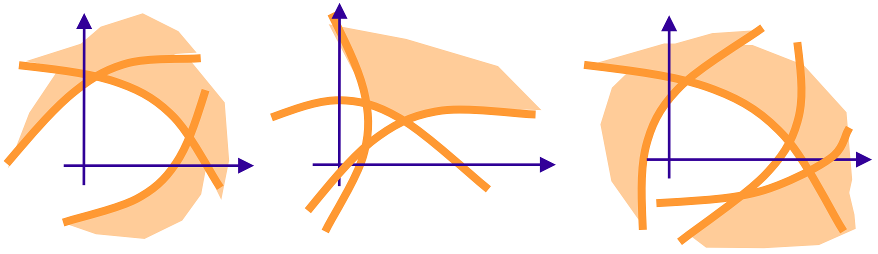
  1) FORM/SORM
  2) Response surface method
  3) Monte Carlo simulations
  4) Importance samplings



- System reliability analysis: $P(E_{\text{system}}) = P\left(\bigcup\bigcap g_i(\mathbf{x}) \le 0\right) = \int_{D_{system}} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$

  1) Complexity
  2) Dependence between component events
  3) Lack of information

  ~ synthesize components reliabilities
  or perform simulations



Series System　　Parallel System　　General System

Existing methods: (1) inclusion-exclusion formula

* Series system

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P(E_i E_j) + \cdots + (-1)^{n-1} P(E_1 E_2 \cdots E_n)$$

* Parallel system

$$P\left(\bigcap_{i=1}^{n} E_i\right) = 1 - P\left(\bigcup_{i=1}^{n} \overline{E}_i\right) = 1 - \sum_{i=1}^{n} P(\overline{E}_i) + \cdots$$

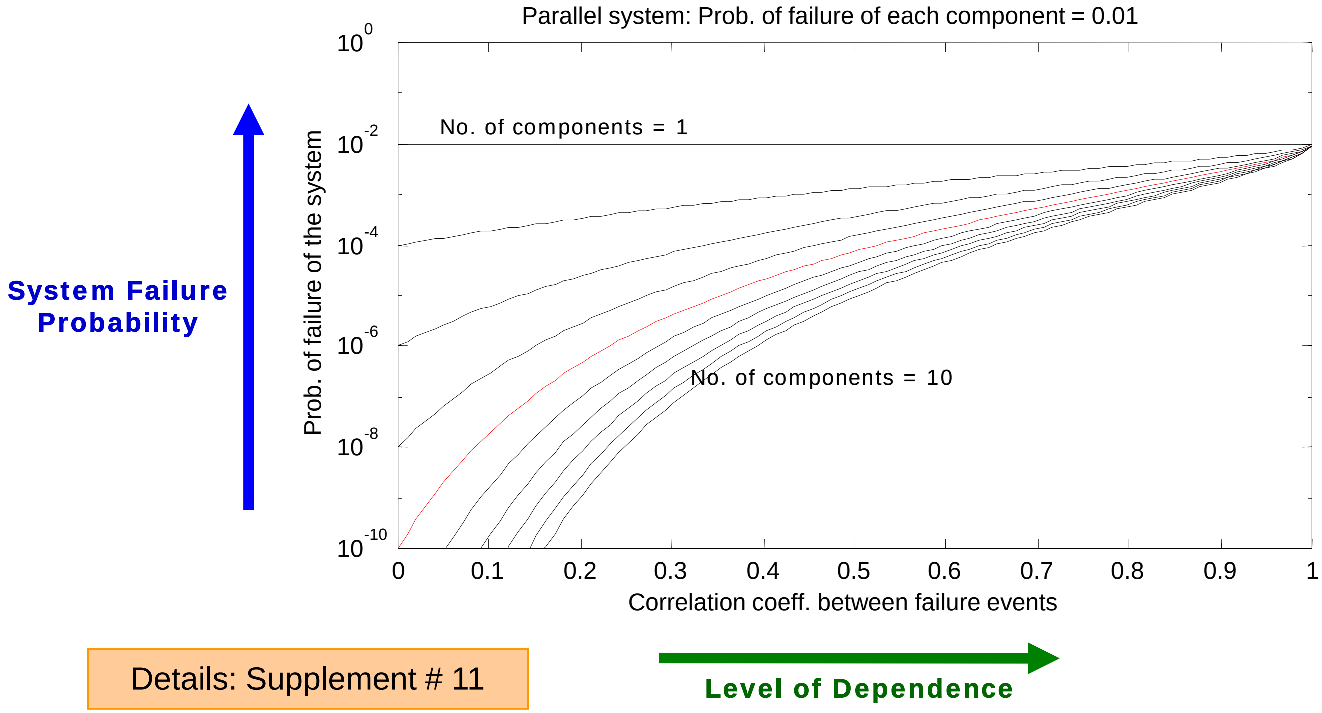
* Cut-set system

$$P\left(\bigcup_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} P(C_i) - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} P(C_i C_j) + \cdots + (-1)^{n-1} P(C_1 C_2 \cdots C_n)$$

- the number of terms increase exponentially; 2^n -1
- requires all the joint probabilities: $P(E_i)$, $P(E_iE_j)$, $P(E_iE_jE_k)$, …
- useful only if component events are statistically independent: $P(E_iE_j) = P(E_i)P(E_j)$
  ~ need marginal probabilities only

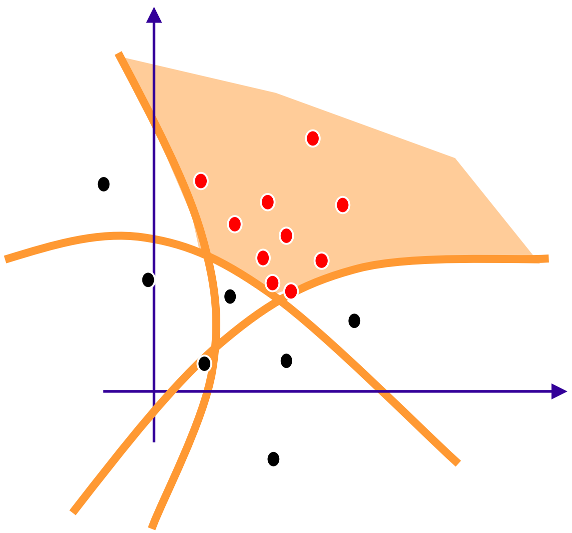
** Dependence and system reliability

- A parallel system with 1~10 components with $P(E_i) = 0.01$
  - ~ e.g. n=5: 10^{-10} (independent) ~10^{-2} (perfectly dependent)



Details: Supplement # 11

Existing methods: (2) simulations



$$P(E_{\text{system}}) = \int_D f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$$

$$\cong \frac{\#(\mathbf{x} \in D)}{\#(\mathbf{x})}$$

~ Count the number of samples in the system failure domain and estimate the ratio.

- Monte Carlo simulations, importance sampling, directional sampling, etc.
- Independent random variables: easily generated.
- Dependent random variables: need joint probability density function
  ~ not available in many cases.
- Independence assumption will lead to errors in estimating system reliability

Existing methods: (3) bounding formulas

It is desirable to derive **bounds** on system probability which involve **low-order component probabilities:**

- ✓ Uni-component probabilities: $P(E_i) = P_i$
- ✓ Bi-component probabilities: $P(E_i E_j) = P_{ij}$
- ✓ Tri-component probabilities: $P(E_i E_j E_k) = P_{ijk}$

➢ Series System

1) Uni-component bounds (Boole 1854; Fréchet 1953)

$$\max_k P_k \le P\left(\bigcup_{k=1}^{n} E_k\right) \le \min\left(1, \sum_{k=1}^{n} P_k\right)$$

2) Bi-component bounds (Kounias 1968; Hunter 1976; Ditlevsen 1979)

$$P_1 + \sum_{i=2}^{n} \max\left(P_i - \sum_{j=1}^{i-1} P_{ij}, 0\right) \le P\left(\bigcup_{k=1}^{n} E_k\right) \le P_1 + \sum_{i=2}^{n} \left(P_i - \max_{j<i} P_{ij}\right)$$

3) Tri-component bounds (Hohenbichler & Rackwitz 1983; Zhang 1993)

$$P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \max\left(0, P_i - \sum_{j=1}^{i-1} P_{ij} + \max_{k\in(1,2,\cdots,i-1)} \sum_{\substack{j=1 \\ j\neq k}}^{i-1} P_{ijk}\right) \le P\left(\bigcup_{i=1}^{n} E_i\right) \le P_1 + P_2 - P_{12} + \sum_{i=3}^{n}\left[P_i - \max_{\substack{k\in\{2,3,\ldots,i-1\} \\ j<k}} \left(P_{ik} + P_{ij} - P_{ijk}\right)\right]$$

Existing method: (3) bounding formulas (contd.)

➢ Parallel System

- Uni-component bounds (Boole 1854; Fréchet 1953)

$$\max\left(0, \sum_{k=1}^{n} P_k - (n-1)\right) \le P\left(\bigcap_{k=1}^{n} E_k\right) \le \min_k P_k$$

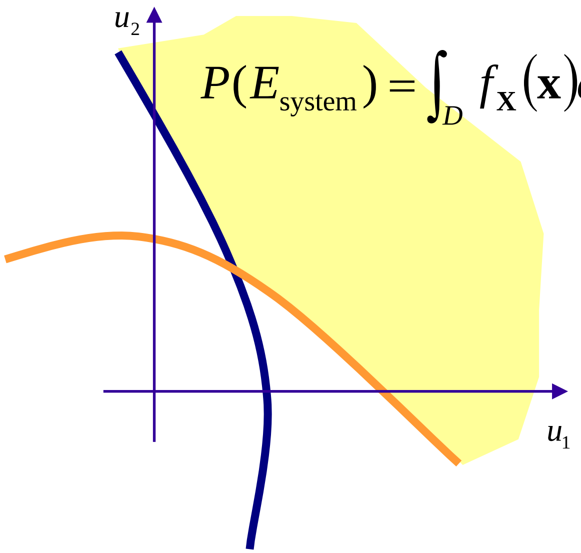
- No higher-order bounds available.

Note: De Morgan’s rule can be used to convert a parallel system to a series system, allowing use of bi- and tri-component bounding formulas for series systems.

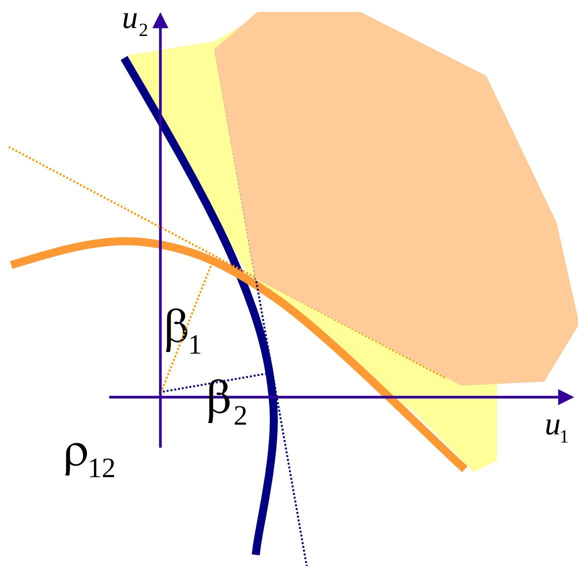
➢ General System

- No bounding formulas exist.

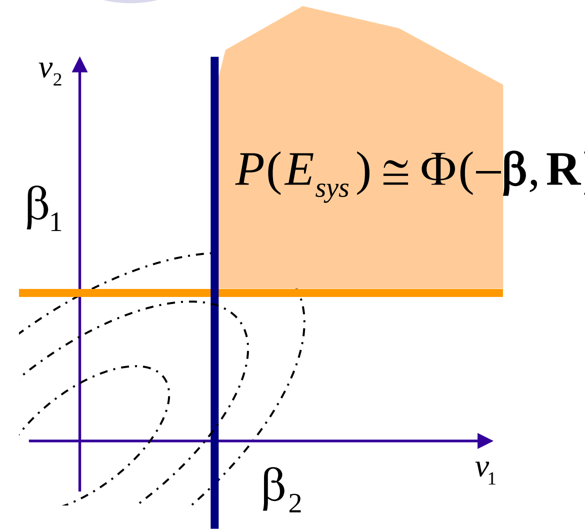
Existing methods: (4) FORM approximation



Original system reliability problem



FORM analysis for each component



Integration in standard normal space

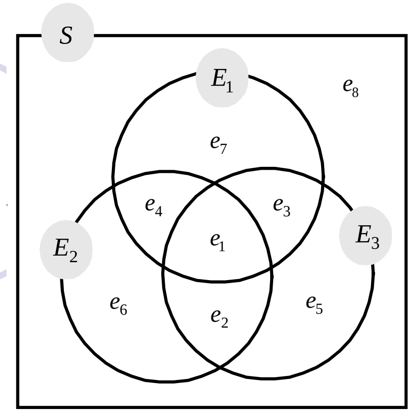
- For parallel and series system
- Find the corresponding volume in standard normal space based on FORM analyses of component events
- Errors depend on the level of nonlinearity and complexity of domain.

System reliability: challenges

- **Complexity** of system problems
  - large number of components, component states, cut sets, link sets, etc.
  - difficulty in identifying cut sets or link sets
  - computational challenges (speed and memory)
- **Dependence** between component states
  - “environmental dependence” or “common source effect”
  - members and materials by the same manufacturer or supplier
  - analysis as “independent components” is simple, but may be misleading.
- **Diversity/Lack** of available information on components
  - missing information
  - various types of information
  - should be flexible in obtaining information

II. Bounds on System Reliability by Linear Programming (‘LP Bounds’)

Bounds by linear programming (LP)



Probabilities of basic MECE events: $p_i \equiv P(e_i), \quad i = 1, 2, \ldots, 2^n$

1. The system failure probability $$P(E_{\text{system}}) = \sum_{r:e_r \subseteq E_{\text{system}}} p_r = \mathbf{c}^{\mathrm{T}}\mathbf{p}$$

2. Axioms of probability: $$\sum_{i=1}^{2^n} p_i = 1 \quad \text{and} \quad p_i \geq 0, \ \forall i$$

3. Available information on component probabilities $$P(E_i) = \sum_{r:e_r \subseteq E_i} p_r = P_i \quad (\geq P_i, \ \leq P_i)$$ $$P(E_i E_j) = \sum_{r:e_r \subseteq E_i E_j} p_r = P_{ij} \quad (\geq P_{ij}, \ \leq P_{ij}) \ \cdots$$

$$\begin{aligned} &\text{minimize (maximize)}\ \mathbf{c}^{\mathrm{T}}\mathbf{p} \\ &\text{subject to} \quad \mathbf{a}_1\mathbf{p} = \mathbf{b}_1 \\ &\qquad\qquad\quad\ \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2 \end{aligned}$$

Linear Programming Problem

* Song, J., and A. Der Kiureghian (2003). Bounds on system reliability by linear programming. *Journal of Engineering Mechanics*, ASCE, 129(**6**): 627-636.

Merits of LP approach

✓ Bounds for general systems.

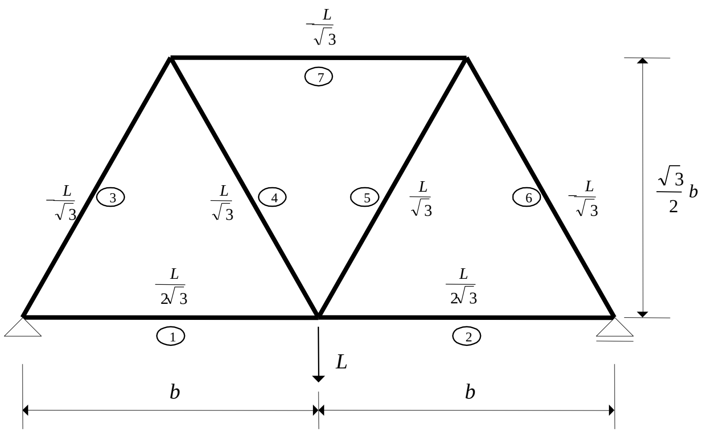
✓ Any type of information on component probabilities can be used.
- Equality: $P_{ij} = 0.02$
- Inequality: $P_{ij} \le 0.01,\ \ 0.05 \le P_i \le 0.07, P_3 \le P_2$
- Partial: $P_1 = 0.01$, $\boldsymbol{P_2 = ?}$, $P_3 = 0.03$

✓ Finds the *narrowest* possible bounds for the given information. (This is not guaranteed for existing formulas for series systems involving bi- or higher-order component probabilities.)

✓ Can be used to compute importance and sensitivity measures, and updated system reliability.

Application to structural system reliability

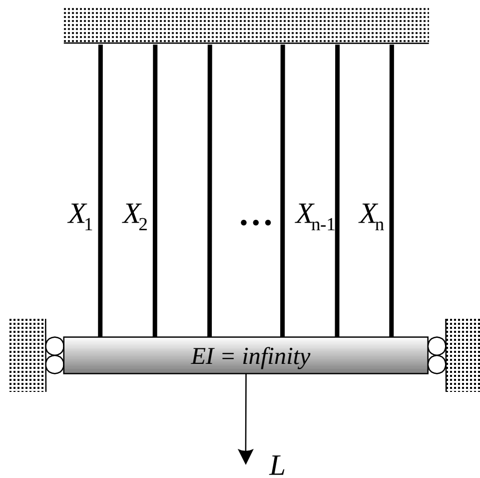
Statically determinate truss (series system)



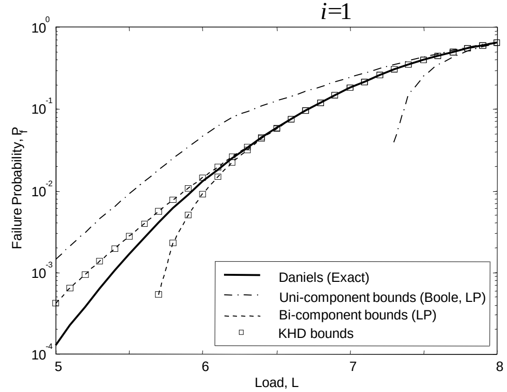
$$E_{\text{system}} = \bigcup_{i=1}^{n} E_i$$

1. Narrowest bounds
2. Incomplete set of probabilities
3. Inequality-type information

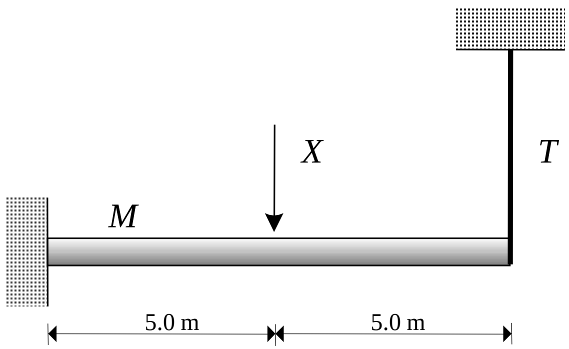
Daniels' parallel system



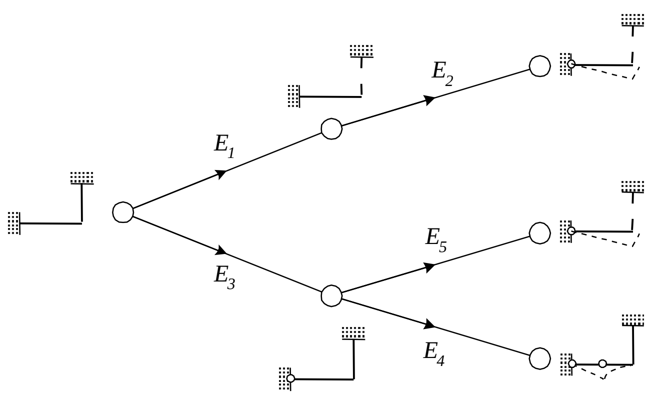
$$E_{\text{system}} = \bigcap_{i=1}^{n} E_i$$



Cantilever beam – bar (general system)

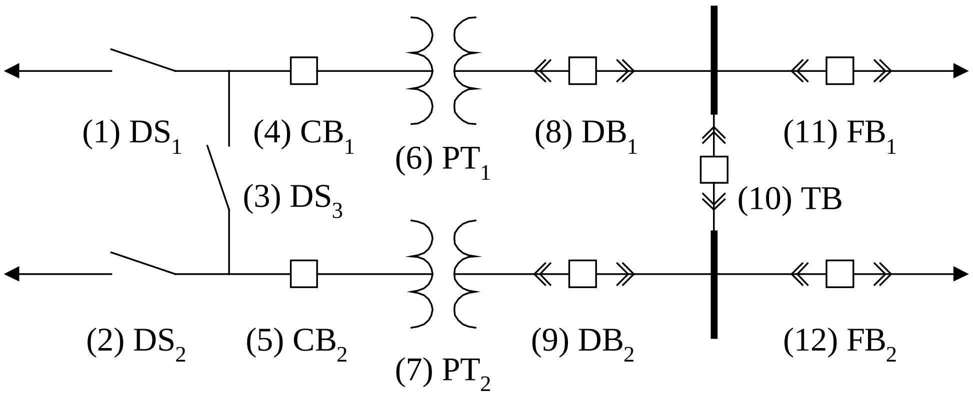


$$E_{\text{system}} = E_1E_2 \cup E_3E_4 \cup E_3E_5$$



* Song, J., and A. Der Kiureghian (2003). Bounds on system reliability by linear programming. *Journal of Engineering Mechanics*, ASCE, 129(**6**): 627-636.

Application to electrical substation systems



Two-transmission-line substations

- Component failure event, E_i

$$E_i = \{\ln R_i - \ln A - \ln S_i \le 0\},\ i = 1, \ldots, n$$

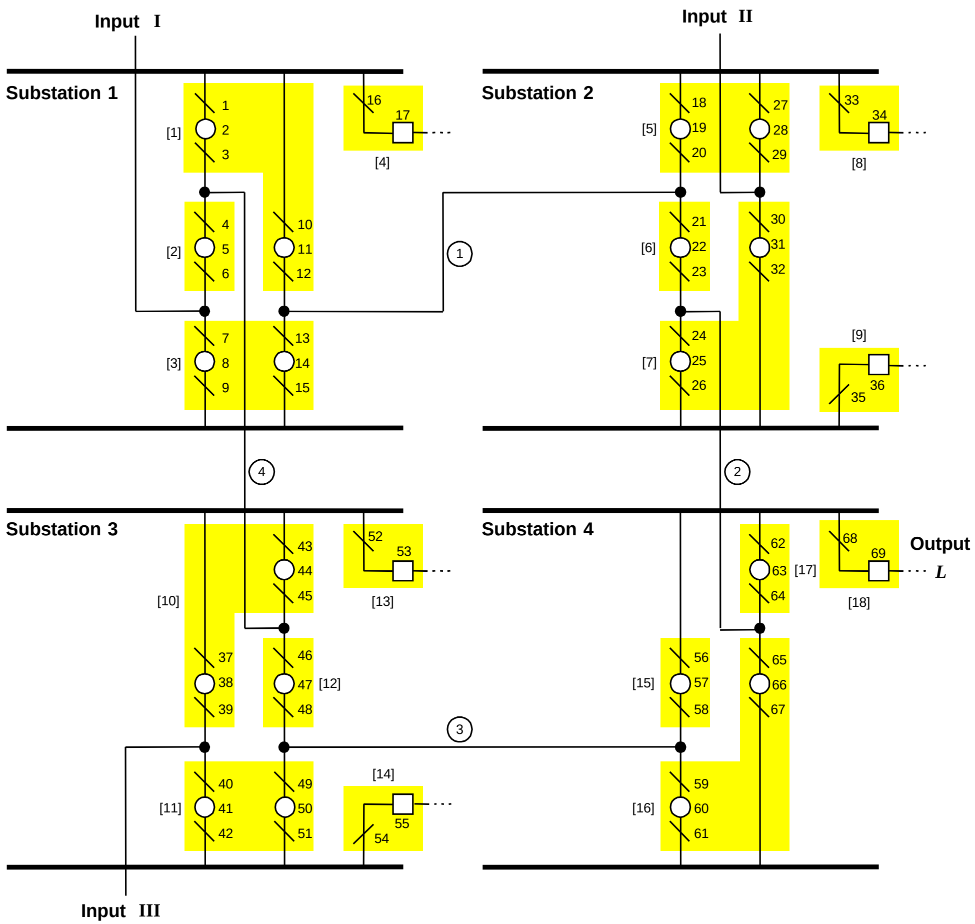
A = LN(mean=0.15, c.o.v.=0.5) PGA
S_i = LN(mean=1, c.o.v.=0.2) local site effect
R_i = LN(mean,c.o.v.,corr.) equipment capacity

DS: Disconnect Switch (0.4, 0.3, 0.3)
CB: Circuit Breaker (0.3, 0.3, 0.3)
PT: Power Transformer (0.5, 0.5, 0.5)
DB: Drawout Breaker (0.4, 0.3, 0.3)
TB: Tie Breaker (1.0, 0.3, 0.3)
FB: Feeder Breaker (1.0, 0.3, 0.3)

Case	Uni-comp.	Bi-comp.	Tri-comp.	M.C. δ=0.01
As shown in figure	1.13×10^{-12}~0.202	0.0436~0.146	0.0616~0.0942	0.0752
No information available on TB (E_{10})	1.82×10^{-11}~0.202	0.0436~0.146	0.0615~0.0943	N/A
No information available on CB_1 (E_4)	1.26×10^{-9}~0.202	0.0267~0.147	0.0395~0.1360	N/A
Upper bound available on CB_1, $P_4 \le 0.01$	5.19×10^{-9}~0.120	0.0267~0.0995	0.0395~0.0701	N/A

* Song, J., and A. Der Kiureghian (2003). Bounds on system reliability by linear programming and applications to electrical substations. *Proc. of ICASP9*, San Francisco, USA, July 6-9.

Multi-scale system reliability analysis



System of four electrical substations

(n = 59: $\boxed{5.76\times10^{17}\text{ design variables}}$)

- System decomposition
  - consider a subset of the components of a system as "super-components"
  - bounds on marginal and joint probabilities of the super-components are computed by LP approach
  - the computed bounds are used as constraints in solving the LP problem for the entire system
  - reduced to 35 LP problems, the largest of which has $\boxed{2^{15} = 32{,}768\text{ variables}}$
- multi-scale system modeling
  - helps the analyst see the "big picture," while not disregarding system details
  - particularly effective when many similar subsystems exist
  - allows different teams of analysts to work on different subsystems (parallel computing)

System reliability updating

- In the analysis of system reliability, it is often of interest to compute the **conditional probability** of a system or subsystem event, given that another system or subsystem event is known or presumed to have occurred.
  - Examples: $P(E_i \mid E_{system}),\ P(E_i \mid \overline{E}_{system})$, etc.

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{\sum_{r \in AB} p_r}{\sum_{r \in A} p_r}$$ ~ **Nonlinear** function of **p**'s

- The bounds on the conditional probabilities can be obtained after a few iterations of a parameterized LP problem (Dinkelbach 1967).

$$\begin{aligned} &\text{minimize/maximize} && (\mathbf{c}_{AB}^{\mathrm{T}} - \lambda \mathbf{c}_{A}^{\mathrm{T}})\mathbf{p} \\ &\text{subject to} && \mathbf{a}_1\mathbf{p} = \mathbf{b}_1 \\ & && \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2 \end{aligned}$$

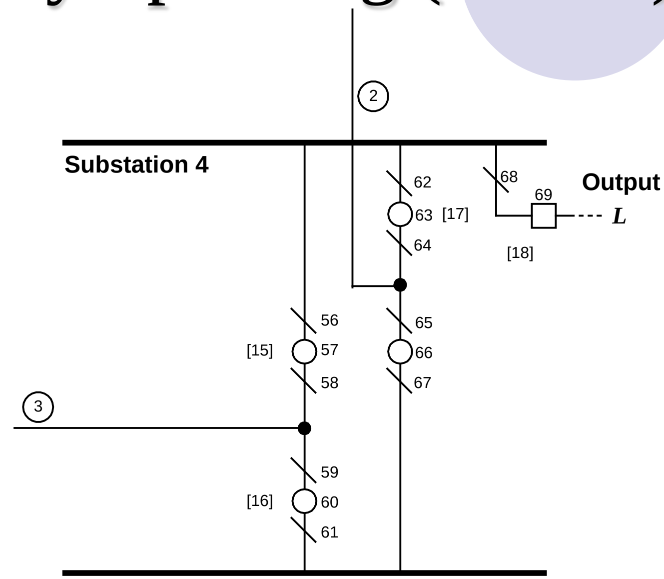
optimal solution, $\hat{\mathbf{p}}$

parameter, λ

$$\lambda = \frac{\mathbf{c}_{AB}^{\mathrm{T}}\hat{\mathbf{p}}}{\mathbf{c}_{A}^{\mathrm{T}}\hat{\mathbf{p}}}$$

* Der Kiureghian, A. and J. Song (2008). Multi-scale reliability analysis and updating of complex systems by use of linear programming. *Journal of Reliability Engineering & System Safety*, 93(2): 288-297.

System reliability updating (contd.)



Updated failure probabilities of equipment items in Substation 4

Type	Equipment No.	$P(E_i)$	$P(E_i \mid E_{sys})$	$P(E_i \mid \overline{E}_{sys})$
DS	56, 58, 62, 64	0.00371	0.243 ~ 0.375	0.000431 ~ 0.00125
	59, 61, 65, 67	0.00371	0.175 ~ 0.372	0.000431 ~ 0.00182
	68	0.00371	0.331 ~ 0.468	0
CB	57, 63	0.00953	0.506 ~ 0.660	0.00345 ~ 0.00458
	60, 66	0.00953	0.338 ~ 0.623	0.00357 ~ 0.00613
PT	69	0.00232	0.206 ~ 0.292	0

Identification of critical components and cut sets

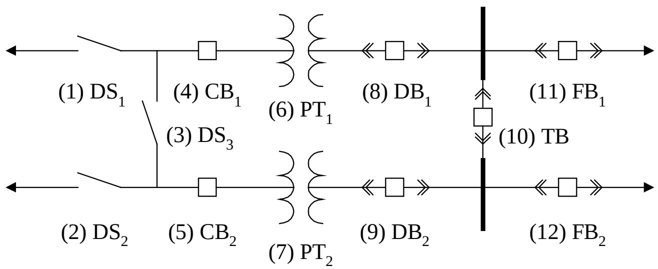
- LP approach can identify components and cut sets which make **significant contributions** to the system failure probability by iteratively solving parameterized LP's.

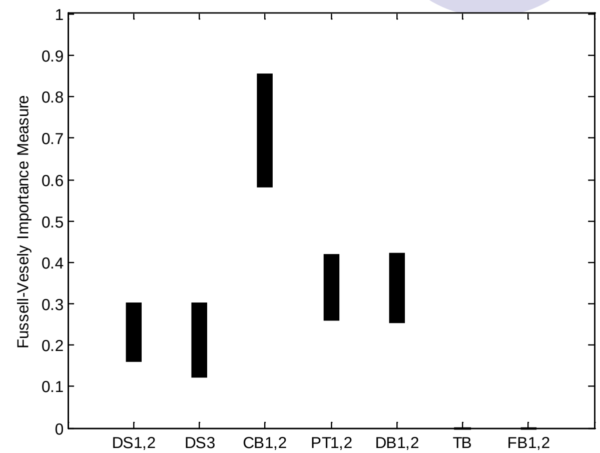
- **Importance Measures (IM)**

  quantifies **participation** in system failure probability

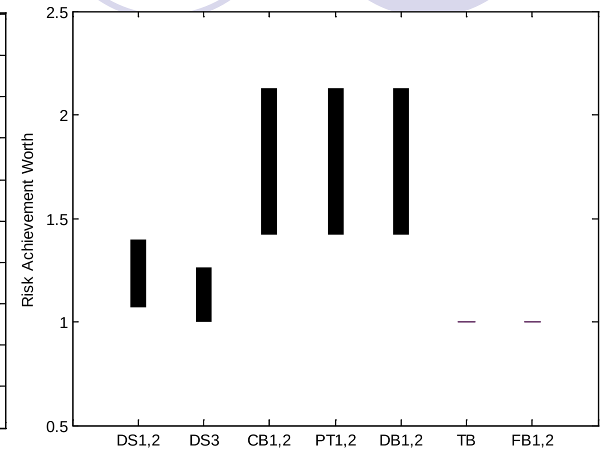
  - Fussell-Vesely: $FV_i = P(\bigcup_{k:E_i \subseteq C_k} C_k) / P(E_{system})$
  - Risk Achievement Worth: $RAW_i = P(E_{system}^{(i)}) / P(E_{system})$
  - Risk Reduction Worth: $RRW_i = P(E_{system}) / P(E_{system}^{\overline{(i)}})$
  - Boundary Probability: $BP_i = P(E_{system}^{(i)}) - P(E_{system}^{\overline{(i)}})$
  - Fussell-Vesely Cutset IM: $FVC_k = P(C_k) / P(E_{\text{system}})$

Identification of critical components and cut sets (contd.)

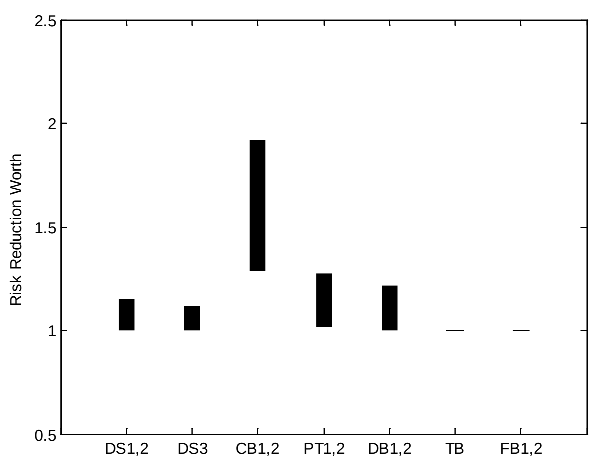




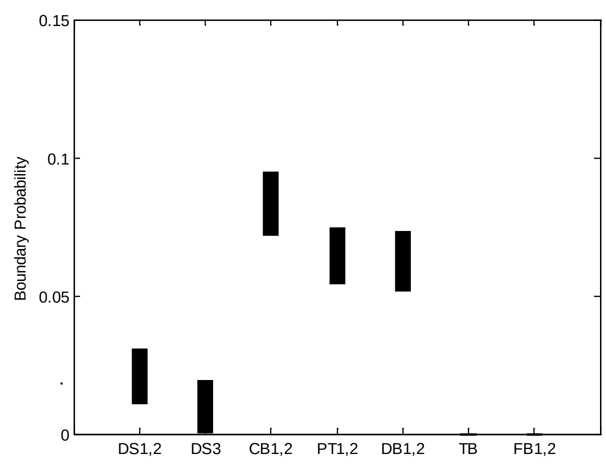
FV IM



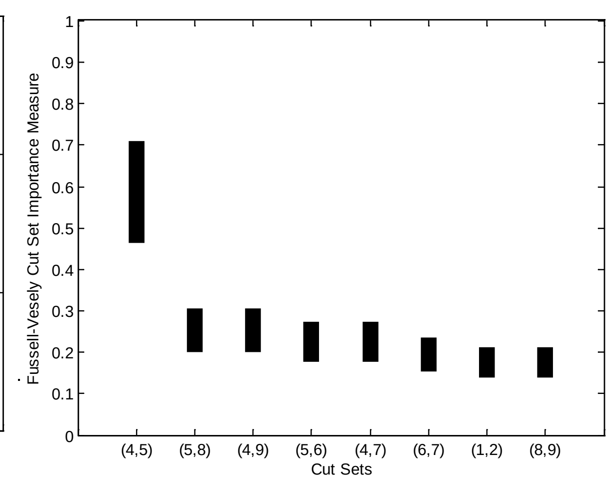
RAW



RRW



BP



FVC IM

* Song, J. and A. Der Kiureghian. Component importance measures by linear programming bounds on system reliability. *Proc. of ICOSSAR9*, Rome, Italy, June 19-23.

Sensitivity and optimal upgrade

- General-purpose LP algorithms provide the sensitivity of an optimal solution with respect to the values in the right-hand side vector, **b**.

$$\begin{aligned} &\text{minimize (maximize)}\ \mathbf{c}^{\mathrm{T}}\mathbf{p} \\ &\text{subject to} \quad \mathbf{a}_1\mathbf{p} = \mathbf{b}_1 \\ &\qquad\qquad\quad\ \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2 \end{aligned}$$

LP problem for system reliability

LP algorithm $\Longrightarrow$

$$\frac{\partial P_{sys}}{\partial b_i}$$

Sensitivity w.r.t. failure probabilities

$\frac{\partial b_i}{\partial \theta_j}$

Chain rule $\Longrightarrow$

$$\frac{\partial P_{sys}}{\partial \theta_j}$$

Sensitivity w.r.t. design parameters

- **Optimal upgrade** of system reliability within the limit of upgrade cost (*in progress*)

$$\begin{aligned} &\min_{\mathbf{x}} \max_{\mathbf{p}} \quad \mathbf{c}^{\mathrm{T}}\mathbf{p}(\mathbf{x}) \\ &\text{subject to} \ \ \mathbf{a}_1\mathbf{p} = \mathbf{b}_1(\mathbf{x}), \quad \mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2(\mathbf{x}) \\ &\qquad\qquad\quad \mathbf{Q}\mathbf{x} \leq \mathbf{q}, \ \ \mathbf{m}^{\mathrm{T}}\mathbf{x} \leq m_c \\ &\qquad\qquad\quad \mathbf{x} : \text{binary integers} \end{aligned}$$

~ minimize the upper bound of P_{sys}

~ component failure probabilities: f(actions)

~ constraints on the actions (workability, cost)

~ indicators for upgrade actions (1: yes, 0: no)

LP Bounds approach and decision-making

$$\text{minimize (maximize) } \mathbf{c}^{\mathrm{T}}\mathbf{p}$$
$$\text{subject to} \quad \mathbf{a}_1\mathbf{p} = \mathbf{b}_1$$
$$\mathbf{a}_2\mathbf{p} \geq \mathbf{b}_2$$

LP Bounds Approach

System Reliability

consequence-based engrg.
Life-cycle cost analysis

Identification of Critical Components and Cut sets

Priority in upgrade project
(cost limit not considered)

System Reliability Updating

Strategy for post-hazard
inspection/ recovery

Sensitivity of System Reliability

Plan for optimal upgrade
(cost limit considered)

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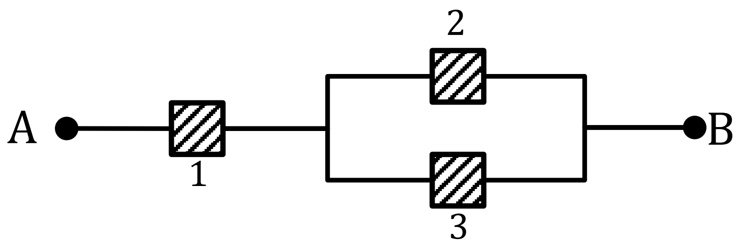
457.646 Topics in Structural Reliability

In-Class Material: Class 17

◎ General system by cut set formulation



E_{sys} : cannot travel from A to B

① Cut set: a subset of components whose joint _________ constitutes the _______ of the system

$C = \{ \qquad\qquad\qquad\qquad\qquad \}$

$E_{sys} =$

② "Minimum" cut sets ~ cut sets with no r_________ components

$C = \{ \qquad\qquad\qquad\qquad\qquad \}$

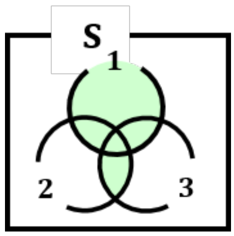
$E_{sys} =$

$\Rightarrow$cut sets which cease to be a cut set if any of the components is ________

③ "Disjoint" cut sets $P(E_{sys}) = P(\cup C_k) = \Sigma P(C_k)$

$C_{disj} = \{ \qquad\qquad\qquad\qquad\qquad \}$

$= \{ \qquad\qquad\qquad\qquad\qquad \}$



$E_{sys} = E_1 \cup \bar{E}_1 E_2 E_3$

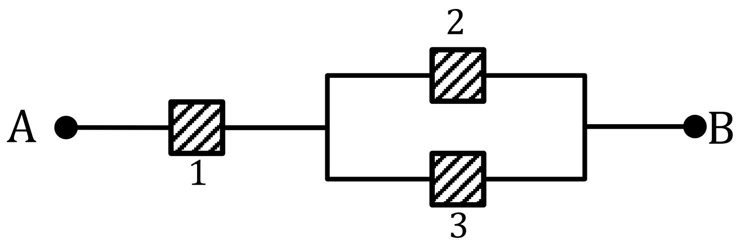
★

$$E_{sys} = \bigcup_{k=1}^{Ncut} C_k$$

$$= \bigcup_{k=1}^{Ncut} \bigcap_{i \in C_k} E_i$$

	1	2	3	
	O	O	O	$\overline{E_1} \cdot \overline{E_2} \cdot \overline{E_3}$
	O	O	X	$\overline{E_1} \cdot \overline{E_2} \cdot E_3$
	O	X	O	$\overline{E_1} \cdot E_2 \cdot \overline{E_3}$
	X	O	O	$E_1 \cdot \overline{E_2} \cdot \overline{E_3}$
		$\vdots$		$\vdots$
		$\vdots$		$\vdots$

◎ General system by link set formulation



① Link set: a subset of components whose joint (   ) assures (   ) of the system

$$L = \{ \qquad\qquad\qquad\qquad \}$$

② "Minimum" link sets ~ link sets with no r__________ component

$$L_{\min} = \{ \qquad\qquad\qquad \}$$

③ "Disjoint" Link set

$$L_{disj} = \{ \qquad\qquad\qquad \}$$

$$\bigstar\ \overline{E}_{sys} = \bigcup_{k=1}^{Nlink} L_k \qquad = \bigcup_{k=1}^{Nlink} \bigcap_{i=L_k} \overline{E}_i$$

De morgan's law

$$\therefore E_{sys} = \bigcap_{k=1}^{Nlink} \left(\bigcup_{i=L_k} \qquad \right)$$

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457.646 Topics in Structural Reliability
In-Class Material: Class 18

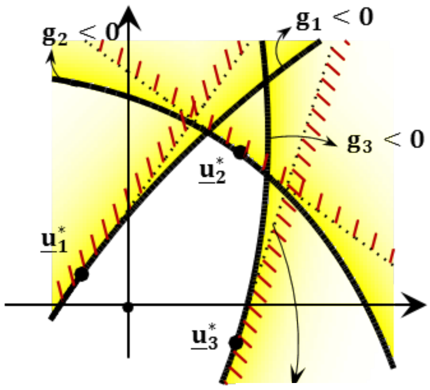
◎ FORM approximation (Hohenbichler & Rackwitz 1983)

① Series system

$$P(E_{sys}) = P(\bigcup_{i=1}^{m} E_i)$$

$$= P(\bigcup_{i=1}^{m} g_i(\mathbf{x}) \le 0)$$

$$\overset{FORM}{\cong} P(\bigcup_{i=1}^{m} \qquad \le 0)$$



$n \rightarrow$ # rv's

$m \rightarrow$ # comp's

Let $Z_i = \hat{\boldsymbol{\alpha}}_i \mathbf{u}$, $i = 1, \cdots, m$

$$G_i(\mathbf{u}) \approx G_i(\mathbf{u}_i^*) + \nabla G_i(\mathbf{u}_i^*)(\mathbf{u} - \mathbf{u}_i^*)$$
$$= \nabla G_i(\mathbf{u}_i^*)(\mathbf{u} - \mathbf{u}_i^*) \le 0$$

$E[Z_i] =$

$\Leftrightarrow \qquad\qquad \le 0$

$Var[Z_i] = \| \quad \|^2 =$

Therefore, $Z_i \sim \quad (\ , \ )$

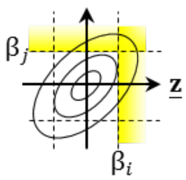
$$\rho_{Z_i Z_j} = \frac{[\quad , \quad]}{} = E[\qquad] - E[\quad] \cdot E[\quad]$$

$$= E[\quad \cdot \quad ^{\mathrm{T}}] = E[\qquad] = \qquad E[\mathbf{uu}^T] \qquad =$$

$\therefore \mathbf{Z} \sim \quad (\ , \ )$, $\rho_{Z_i Z_j} =$

$$\therefore P(E_{sys}) \overset{FORM}{\cong} P(\bigcup_{i=1}^{m} \qquad \le 0)$$

$$= 1 - P(\bigcap_{i=1}^{m} \qquad \le \qquad)$$

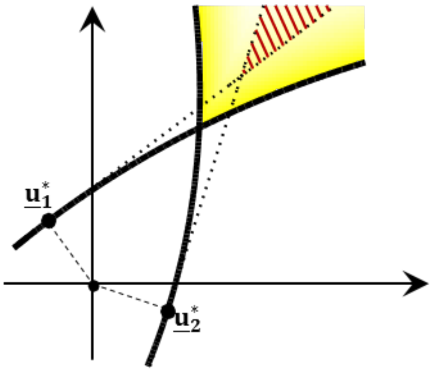


$$= 1 - \Phi_m(\quad , \cdots , \quad ; \mathbf{R})$$

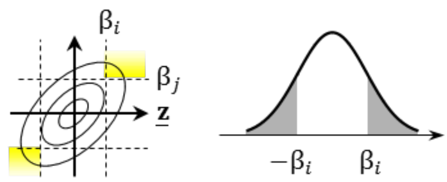
Joint normal CDF of $\mathbf{Z} \sim N(\mathbf{0}; \mathbf{R})$

Where $\Phi_m(\boldsymbol{\beta}; \mathbf{R}) = \int_{-\infty}^{\beta_1} \cdots \int_{-\infty}^{\beta_m} \varphi_m(\mathbf{Z}; \mathbf{R}) d\mathbf{z}$

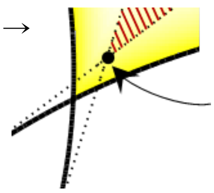
② **Parallel system**



$$
\begin{aligned}
P(E_{sys}) &= P\Big(\bigcap_{i=1}^{m} E_i\Big) \\
&= P\Big(\bigcap_{i=1}^{m} g_i(\mathbf{x}) \le 0\Big) \\
&\overset{FORM}{\cong} P\Big(\bigcap_{i=1}^{m} \qquad\qquad \le 0\Big) \\
&= P\Big(\bigcap_{i=1}^{m} \qquad \ge \qquad\Big) \\
&\overset{sym}{=} P\Big(\bigcap_{i=1}^{m} \qquad \le \qquad\Big) \\
&= \Phi_m(\qquad, \cdots, \qquad; \mathbf{R})
\end{aligned}
$$



→ may have huge errors due to curvatures

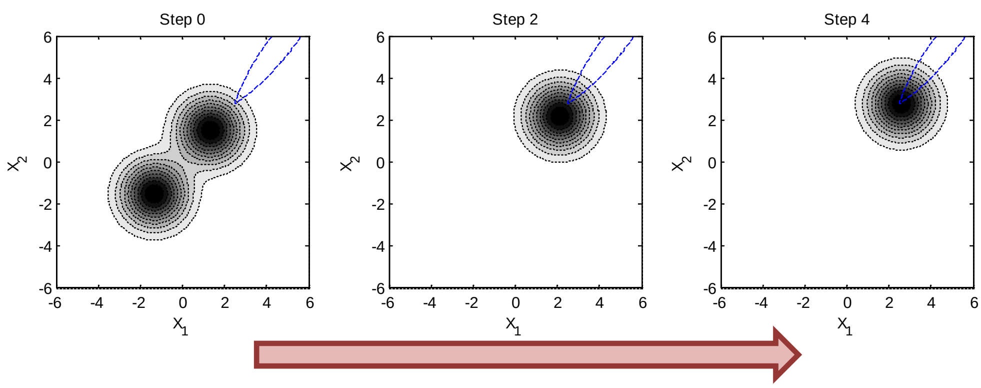
→  better linearization point?

"joint design point"

Hard to find or may not exist

Note: One could find such important domain using an adaptive sampling technique

Kurtz, N., and J. Song (2013). Cross-entropy-based adaptive importance sampling using Gaussian mixture. *Structural Safety*. Vol. 42, 35-44.



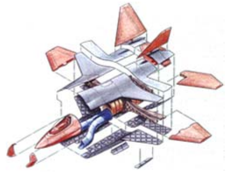
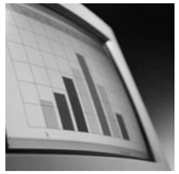
③ **General system?**

⇒ No direct FORM approximation

Risk-quantification of Complex Systems by Matrix-based System Reliability Method



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Matrix-based Formulation

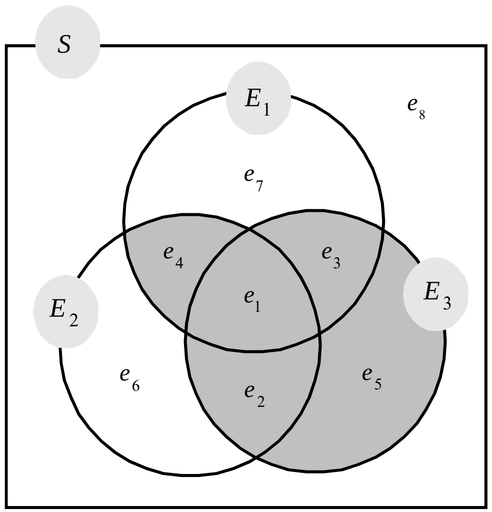
- Matrix-based formulation of system failure:

$$\boxed{P(E_{sys}) = \mathbf{c}^{\mathrm{T}}\mathbf{p}}$$

* Example: $P(E_1 E_2 \cup E_3) = p_1 + p_2 + p_3 + p_4 + p_5$

$$= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0] \cdot$$

$$[p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \quad p_8]^{\mathrm{T}}$$



- **c**: "**event**" vector
  ~ describes the system event of interest

- **p**: "**probability**" vector
  ~ likelihood of component joint failures

Identification of event vector, c

- Matrix-based event operations:

$$\mathbf{c}^{\bar{E}} = \mathbf{1} - \mathbf{c}^{E}$$

$$\mathbf{c}^{E_1 \cdots E_n} = \mathbf{c}^{E_1} .* \mathbf{c}^{E_2} .* \cdots .* \mathbf{c}^{E_n}$$

$$\mathbf{c}^{E_1 \cup \cdots \cup E_n} = \mathbf{1} - (\mathbf{1} - \mathbf{c}^{E_1}) .* (\mathbf{1} - \mathbf{c}^{E_2}) .* \cdots .* (\mathbf{1} - \mathbf{c}^{E_n})$$

  - Efficient and easy to implement by matrix-based computing languages, e.g. Matlab®, Octave
  - Can construct directly from event vectors of components and other system events
  - Can develop/use problem-specific algorithms to identify event vectors

Identification of event vector, c

- Event vectors for component events:

$$\mathbf{C}_{[1]} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{C}_{[i]} = \begin{bmatrix} \mathbf{C}_{[i-1]} & \mathbf{1} \\ \mathbf{C}_{[i-1]} & \mathbf{0} \end{bmatrix} \quad \text{for } i = 2,\ldots,n$$

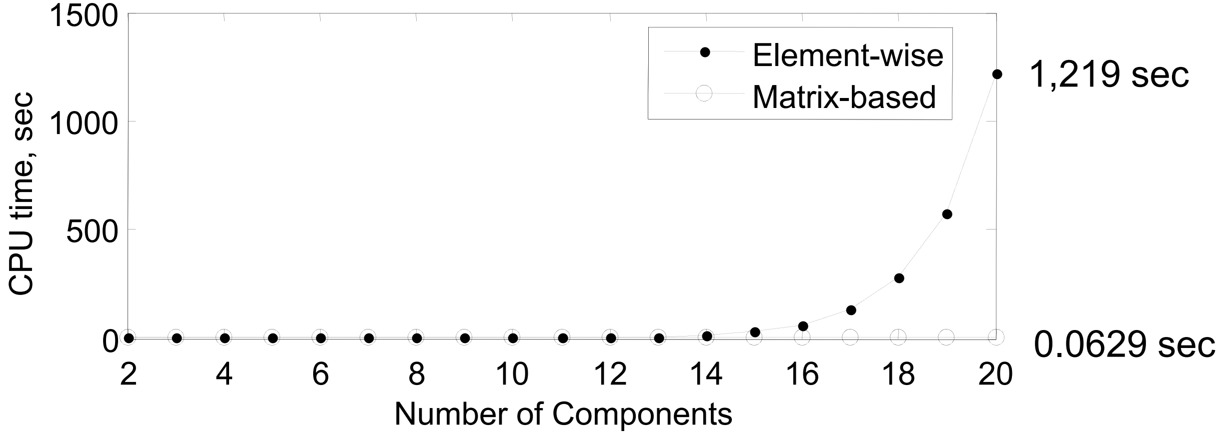
  - **0** and **1** denote the column vectors of $2^{(i\text{-}1)}$ zeros and ones
  - After $\mathbf{C}_{[n]}$ is constructed, the i-th column of the matrix is the event vector of the i-th component event.

Computation of probability vector, p

- Iterative matrix-based procedure for statistically independent (s.i.) components

$$\mathbf{p}_{[1]} = \begin{bmatrix} P_1 & 1 - P_1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{p}_{[i]} = \begin{bmatrix} \mathbf{p}_{[i-1]} \cdot P_i \\ \mathbf{p}_{[i-1]} \cdot (1 - P_i) \end{bmatrix} \quad \text{for } i = 2, \ldots, n$$



Statistical dependence b/w components

- By total probability theorem,

$$\boxed{\begin{aligned} P(E_{sys}) &= \int_{\mathbf{s}} P(E_{sys} \mid \mathbf{s})\, f_{\mathbf{S}}(\mathbf{s})\, d\mathbf{s} \\ &= \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}} \mathbf{p}(\mathbf{s})\, f_{\mathbf{S}}(\mathbf{s})\, d\mathbf{s} \\ &= \mathbf{c}^{\mathrm{T}} \widetilde{\mathbf{p}} \end{aligned}}$$

  - Utilize **conditional s.i.** of components given an outcome of random variables **S** causing component dependence e.g. Earthquake magnitude for a bridge system
  - Event vector **c** is independent of this consideration ~ no need to construct the probability vector for new system events

"What if not explicitly identified?"

- Example: approximation by Dunnett-Sobel (DS) correlation matrix (1955)

$$\boxed{\begin{aligned} &Z_i \sim N(\mathbf{0}, \mathbf{R}),\ \rho_{ij} = r_i \cdot r_j \\ &Z_i = \sqrt{1 - r_i^2} \cdot U_i + r_i S, \end{aligned}}$$

  - Z_i, $i=1,\ldots,n$ are conditional s.i. given $S=s$
  - Fit the given correlation matrix with a DS correlation matrix with the least square error
  - Generalized DS model (Song and Kang, Structural Safety)

$$\boxed{\begin{aligned} &Z_i \sim N(\mathbf{0}, \mathbf{R}), \quad \rho_{ij} = \Sigma_{k=1}^{m}(r_{ik} r_{jk}) \\ &Z_i = \sqrt{1 - \Sigma_{k=1}^{m} r_{ik}^2} \cdot U_i + \Sigma_{k=1}^{m}(r_{ik} S_k) \end{aligned}}$$

Conditional prob./importance measure

- Conditional probability Importance Measure (CIM)

$$\boxed{CIM_i = P(E_i \mid E_{sys}) = \frac{P(E_i E_{sys})}{P(E_{sys})}}$$

- Fussell-Vesely IM

$$\boxed{FV_i = \frac{P(\bigcup_{k:C_k \supseteq E_i} C_k)}{P(E_{sys})}}$$

  - $P(E_{sys}')/P(E_{sys}) = (\mathbf{c}'^{T}\mathbf{p}) / (\mathbf{c}^{T}\mathbf{p})$
  - Once the system reliability is done, only additional task is to find the event vector for a new system event

Parameter sensitivity of system reliability

- Statistically independent components

$$P(E_{sys}) = \mathbf{c}^{\mathrm{T}}\mathbf{p}$$

$$\boxed{\frac{\partial P_{sys}}{\partial \theta} = \mathbf{c}^{\mathrm{T}} \frac{\partial \mathbf{p}}{\partial \theta}}$$

- Statistically dependent components

$$P(E_{sys}) = \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}}\mathbf{p}(\mathbf{s})\, f_{\mathbf{S}}(\mathbf{s})ds$$

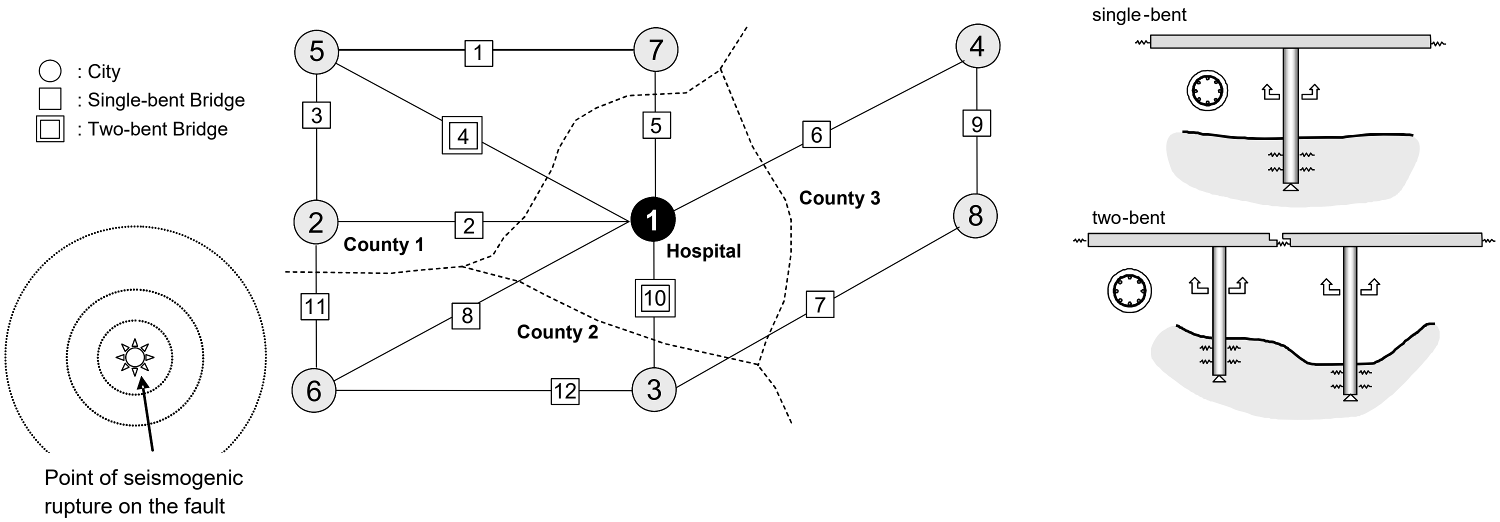
Zero unless θ is a parameter related to common source.

$$\boxed{\frac{\partial P_{sys}}{\partial \theta} = \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}} \left[\frac{\partial \mathbf{p}(\mathbf{s})}{\partial \theta} f_{\mathbf{S}}(\mathbf{s}) + \mathbf{p}(\mathbf{s}) \frac{\partial f_{\mathbf{S}}(\mathbf{s})}{\partial \theta} \right] ds}$$

* Song, J. and W.-H. Kang "System Reliability and Sensitivity under Statistical Dependence by Matrix-based System Reliability Method," *Structural Safety*, Vol. 31(**2**), 148-156.

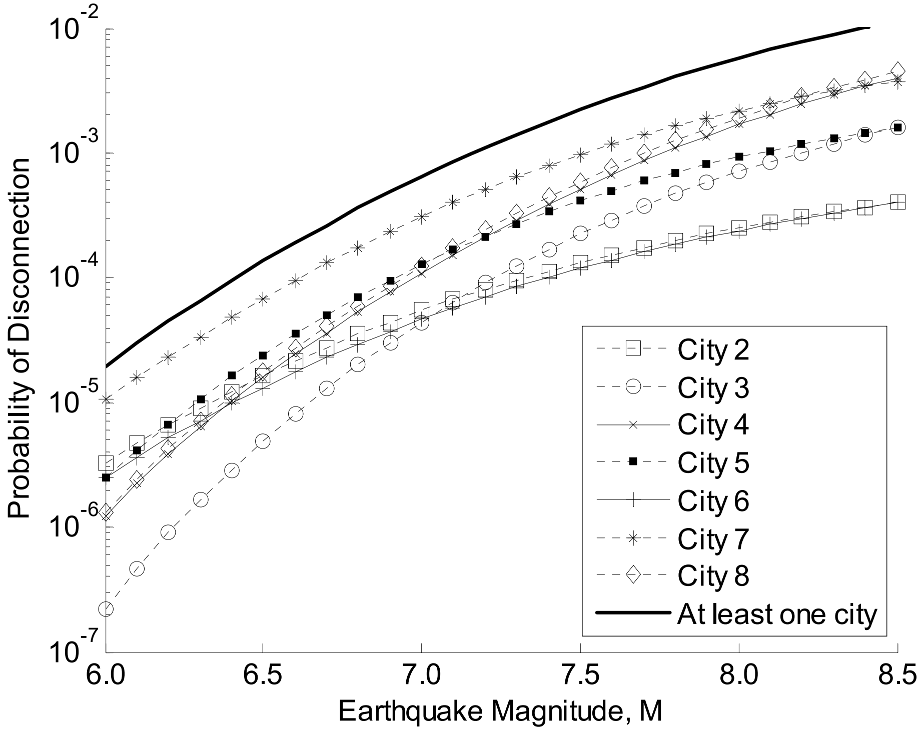
Appl. I: Connectivity of a transportation network

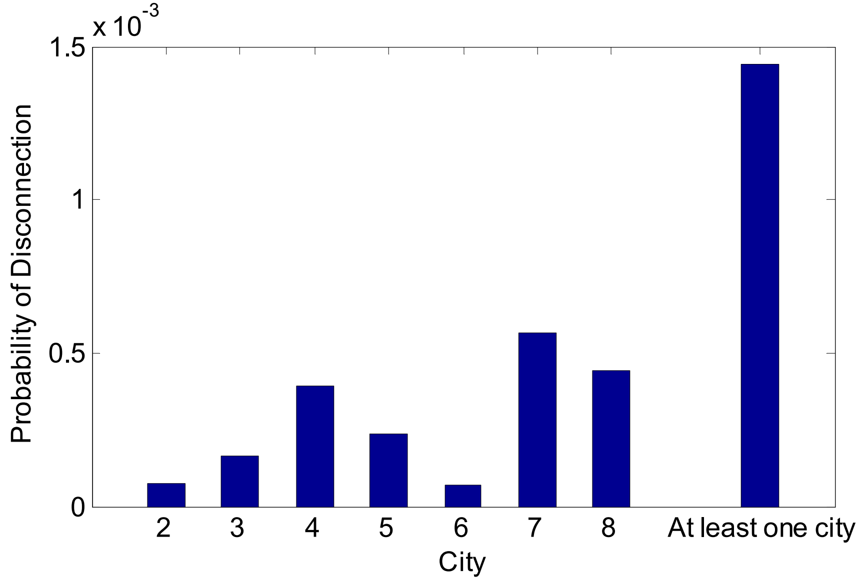
* Kang, W.-H., J. Song, and P. Gardoni (2008) "Matrix-based system reliability method and applications to bridge networks," *Reliability Engineering & System Safety*, Vol. 93, 1584-1593.



- Post-earthquake disconnection from the critical facility
- Fragilities for bridges (Gardoni et al. 2003)
- Deterministic attenuation relationship used
- For given magnitude, the bridge component failures are conditional s.i.

Connectivity of a transportation network





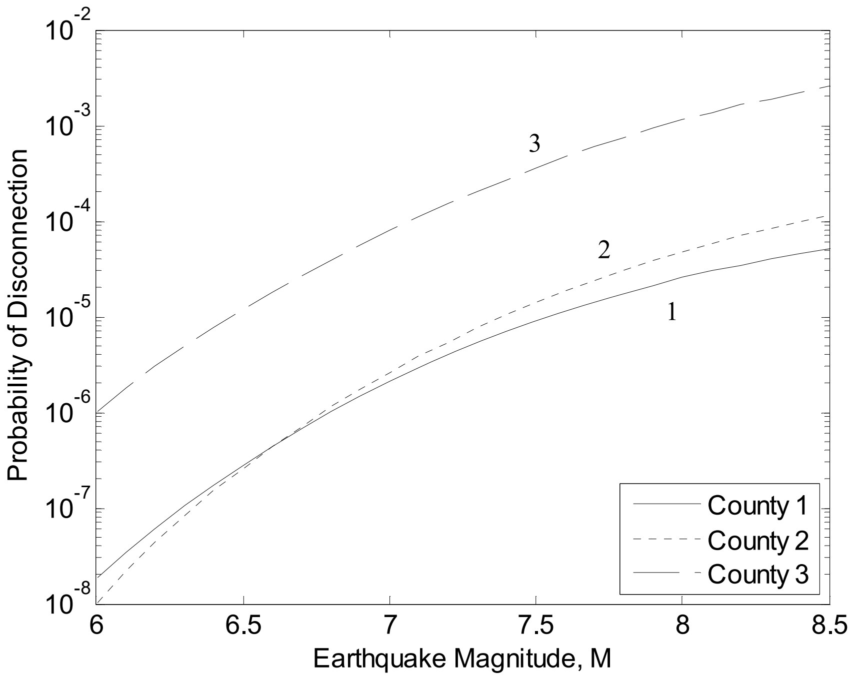
$$P(E_{sys} \mid M = m) = \mathbf{c}^{\mathrm{T}} \mathbf{p}(m)$$

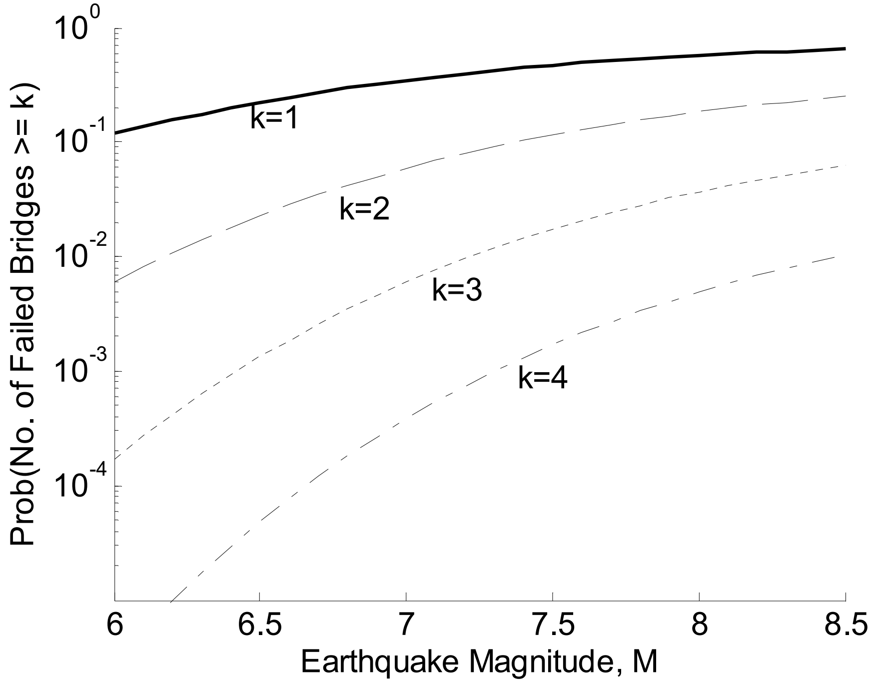
$$P(E_{sys}) = \int_{m_0}^{m_c} \mathbf{c}^{\mathrm{T}} \mathbf{p}(m)\, f_M(m) dm = \mathbf{c}^{\mathrm{T}} \tilde{\mathbf{p}}$$

Conditional probability of disconnection of cities

Probability of disconnection of cities

Connectivity of a transportation network





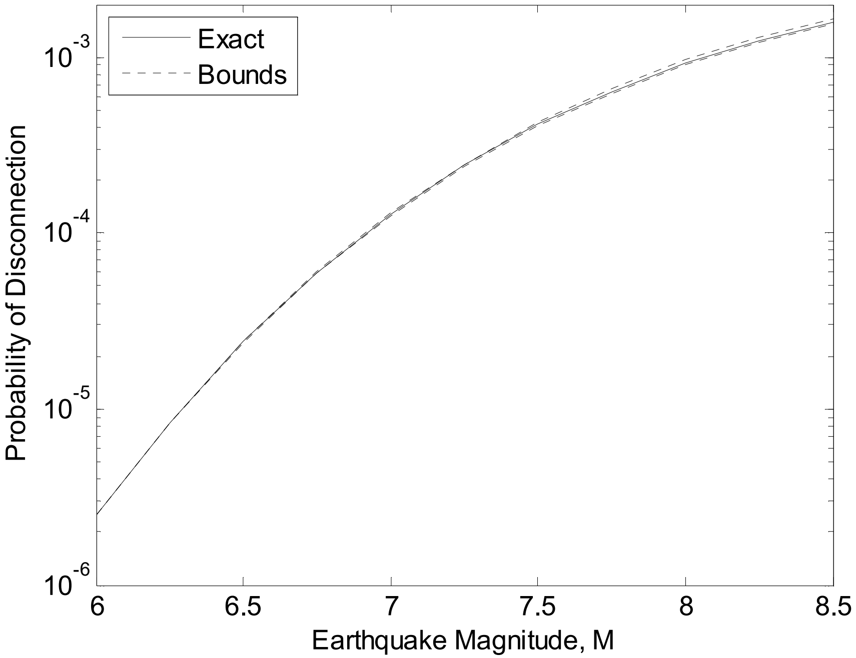
$$P(E'_{sys}) = \mathbf{c}'^{\mathrm{T}}\mathbf{p}(m)$$

$$P(E'_{sys}) = \mathbf{c}''^{\mathrm{T}}\mathbf{p}(m)$$

Conditional probability of disconnection of counties

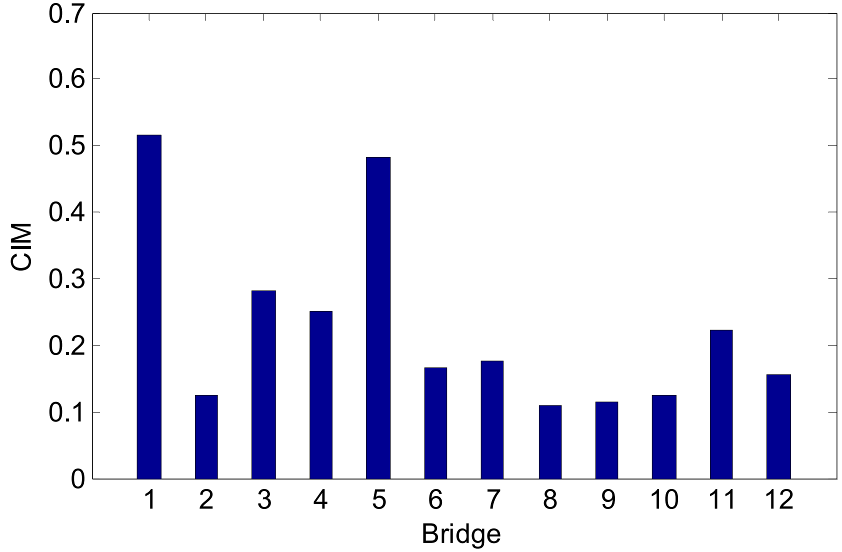
Prob (No. of failed bridges ≥ k)

Connectivity of a transportation network



$$\min(\max) \quad \mathbf{c}^{\mathrm{T}}\mathbf{p}\,(m)$$

Bounds on P(City 5 disconnected)
(No information on Bridge 12)

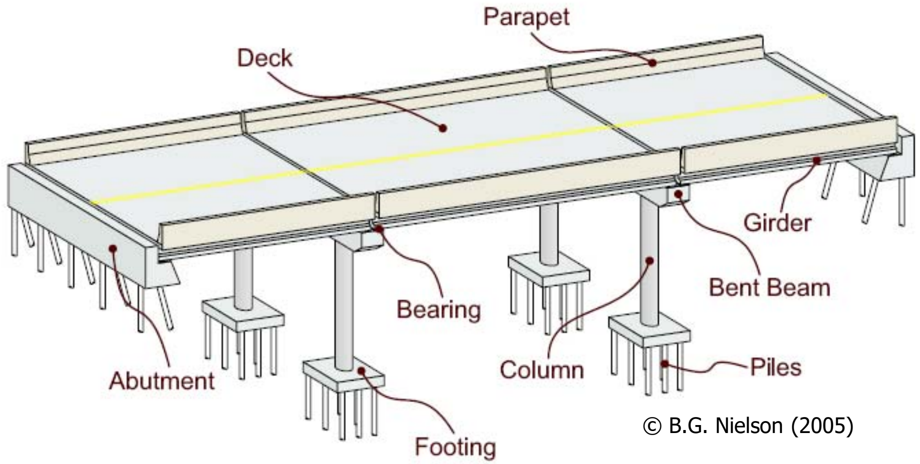


$$P(E_i \mid E_{sys}) = \frac{P(E_i E_{sys})}{P(E_{sys})} = \frac{\mathbf{c}'^{\mathrm{T}}\tilde{\mathbf{p}}}{\mathbf{c}^{\mathrm{T}}\tilde{\mathbf{p}}}$$

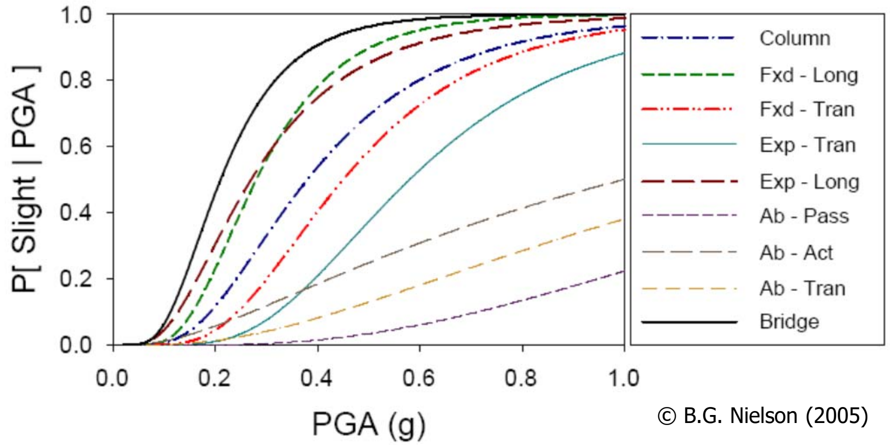
Importance measure of components
w.r.t. the likelihood of at least a disconnection

Appl. II: Damage of a bridge structural system

* Song, J. and W.-H. Kang "System Reliability and Sensitivity under Statistical Dependence by Matrix-based System Reliability Method," *Structural Safety*, Vol. 31(**2**), 148-156.



© B.G. Nielson (2005)



© B.G. Nielson (2005)

- Nielson (2005) developed analytical fragilities of bridge components such as bearings, abutments and columns
- Identified the statistical dependence between demands
- Probability that at least one component fails (series system)
- Performed MCS to account for component dependence

Damage of a bridge structural system

* Safety Factor $F_i = \ln C_i - \ln D_i$

* Fragility

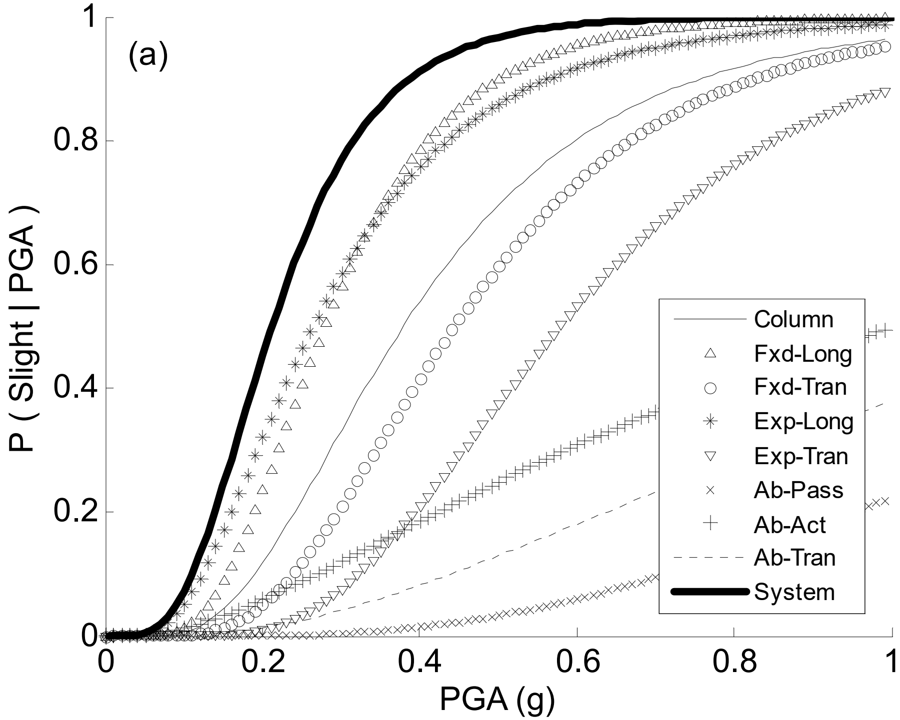
$$P(LS_i \mid IM) = P(F_i \le 0 \mid IM)$$
$$= P\left(Z_i \le -\frac{\mu_{F_i}}{\sigma_{F_i}} \mid IM \right)$$
$$= \Phi\left[-\frac{\mu_{F_i}(IM)}{\sigma_{F_i}(IM)} \right]$$

* Correlation

$$\rho_{Z_i Z_j} = \rho_{F_i,F_j} = \frac{(\zeta_{D_i} \cdot \zeta_{D_j})}{(\zeta_{C_i}^2 + \zeta_{D_i}^2)^{1/2} (\zeta_{C_j}^2 + \zeta_{D_j}^2)^{1/2}} \cdot \underline{\rho_{\ln D_i, \ln D_j}}$$

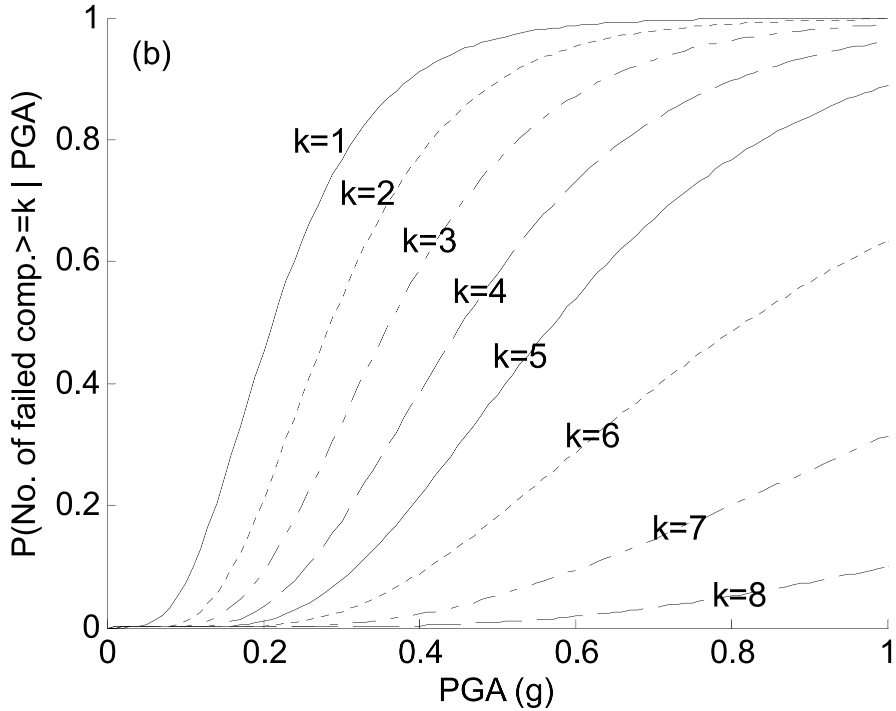
* Fitting by DS-class corr. matrix: average of percentage error ~ 3%

Damage of a bridge structural system



$$P(E_{sys} \mid PGA = pga) = \mathbf{c}^{\mathrm{T}}\mathbf{p}(pga)$$
$$= \int \mathbf{c}^{\mathrm{T}}\mathbf{p}(pga, x)\varphi(x)dx$$

System fragility (at least one)

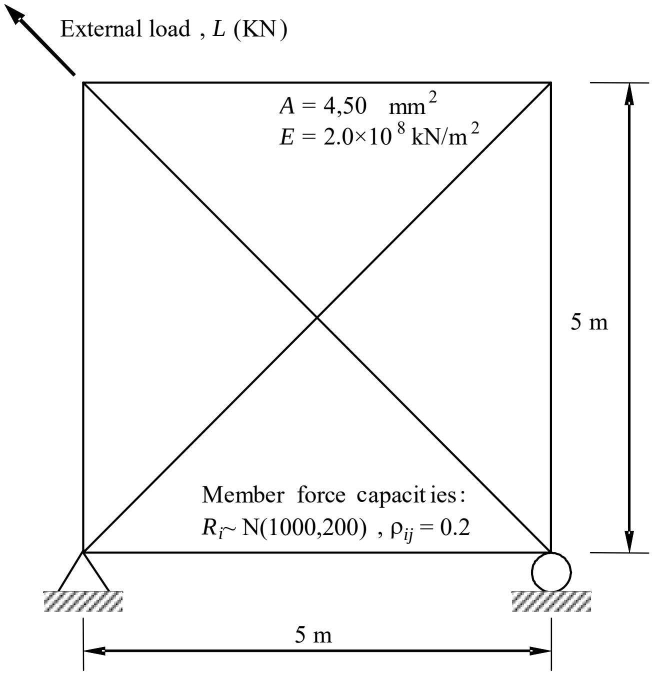


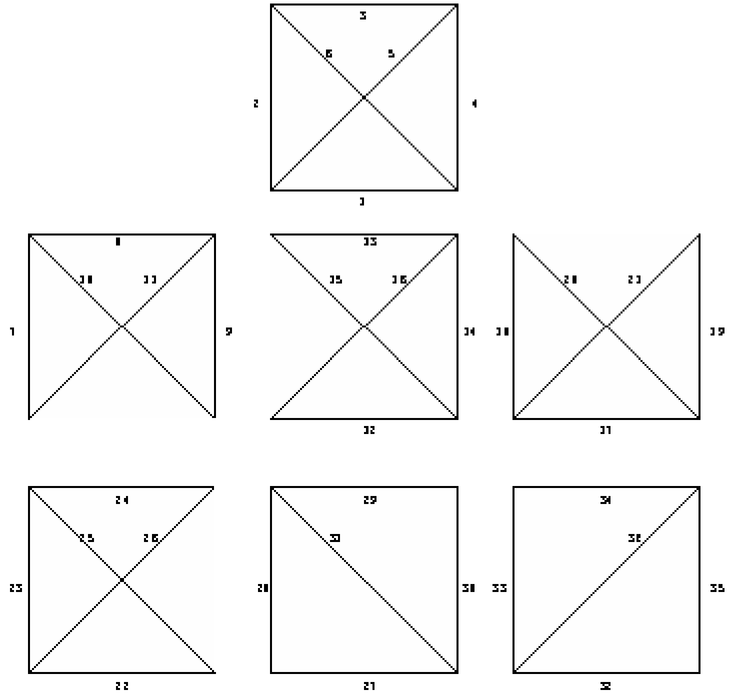
$$P(E_{sys} \mid PGA = pga) = \mathbf{c}'^{\mathrm{T}}\mathbf{p}(pga)$$

P(No. of failed components ≥ k)

Appl. III: Progressive failure of a truss structure

* Song, J. and W.-H. Kang "System Reliability and Sensitivity under Statistical Dependence by Matrix-based System Reliability Method," *Structural Safety*, Vol. 31(**2**), 148-156.





$$P(\overline{E}_{sys}) = P[\overline{E}_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5\overline{E}_6 \cup (E_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5\overline{E}_6)(\overline{E}_7\overline{E}_8\overline{E}_9\overline{E}_{10}\overline{E}_{11})$$

$$\cup (\overline{E}_1 E_2\overline{E}_3\overline{E}_4\overline{E}_5\overline{E}_6)(\overline{E}_{12}\overline{E}_{13}\overline{E}_{14}\overline{E}_{15}\overline{E}_{16}) \cup \cdots$$

$$\cup (\overline{E}_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5 E_6)(\overline{E}_{32}\overline{E}_{33}\overline{E}_{34}\overline{E}_{35}\overline{E}_{36})]$$

Progressive failure of a truss structure

$$
\begin{aligned}
P(\overline{E}_{sys}) = P[&\overline{E}_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5\overline{E}_6 \bigcup (E_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5\overline{E}_6)(\overline{E}_7\overline{E}_8\overline{E}_9\overline{E}_{10}\overline{E}_{11}) \\
&\bigcup (\overline{E}_1 E_2\overline{E}_3\overline{E}_4\overline{E}_5\overline{E}_6)(\overline{E}_{12}\overline{E}_{13}\overline{E}_{14}\overline{E}_{15}\overline{E}_{16}) \bigcup \cdots \\
&\bigcup (\overline{E}_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5 E_6)(\overline{E}_{32}\overline{E}_{33}\overline{E}_{34}\overline{E}_{35}\overline{E}_{36})]
\end{aligned}
$$

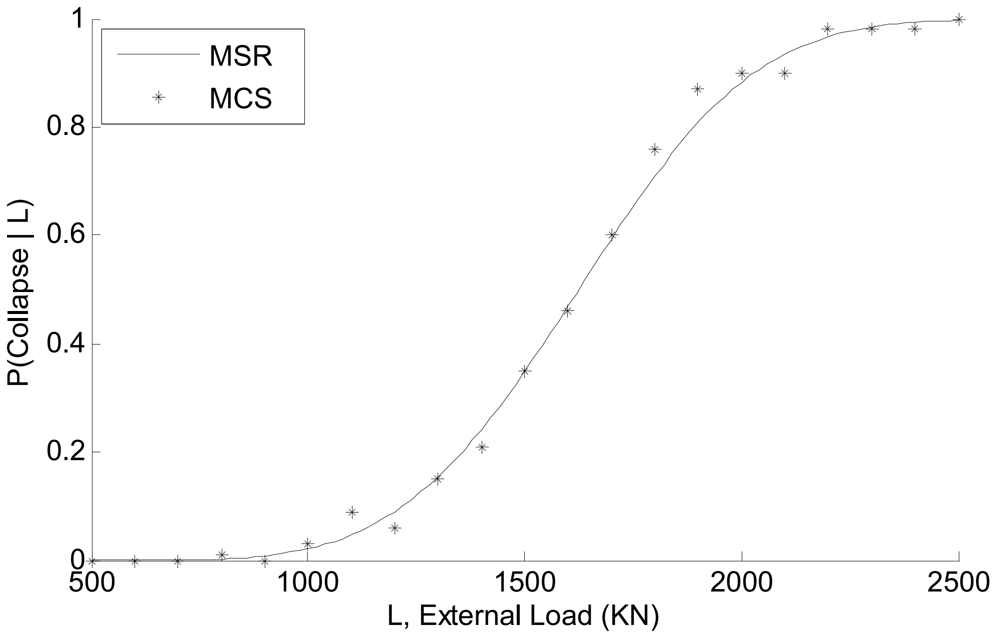
Disjoint link sets (36→11)

$$
\begin{aligned}
P(\overline{E}_{sys}) = P(\overline{E}_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5\overline{E}_6) + P(E_1\boxed{\overline{E}_2}\overline{E}_3\overline{E}_4\overline{E}_5\overline{E}_6\boxed{\overline{E}_7}\overline{E}_8\overline{E}_9\overline{E}_{10}\overline{E}_{11}) \\
\cdots + P(\overline{E}_1\overline{E}_2\overline{E}_3\overline{E}_4\overline{E}_5 E_6\overline{E}_{32}\overline{E}_{33}\overline{E}_{34}\overline{E}_{35}\overline{E}_{36})
\end{aligned}
$$

Perfect correlation

7 systems with 6 components

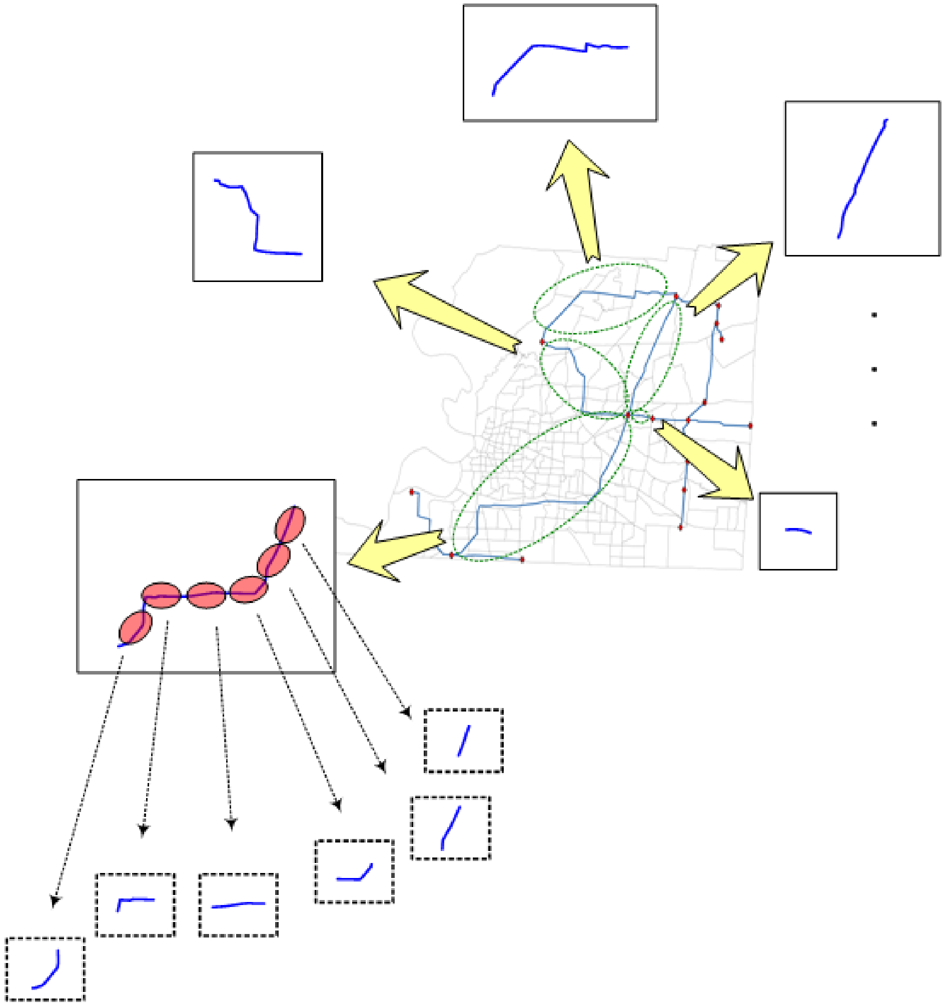
Progressive failure of a truss structure



- System collapse fragility curve given abnormal load
- Verified through MCS
- Importance of members (components)
- Sensitivity of fragility w.r.t. design parameters

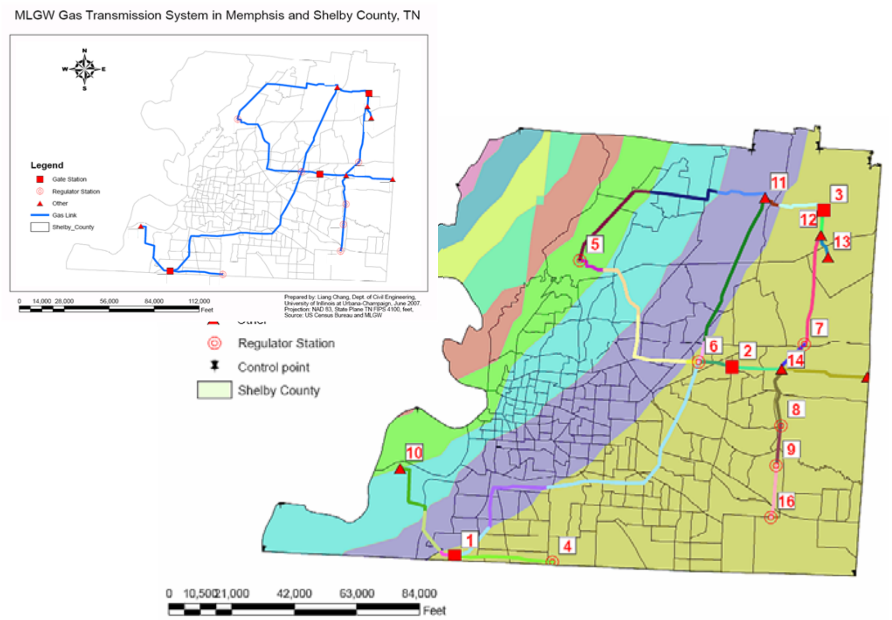
Appl. IV: Multi-scale SRA of lifeline networks

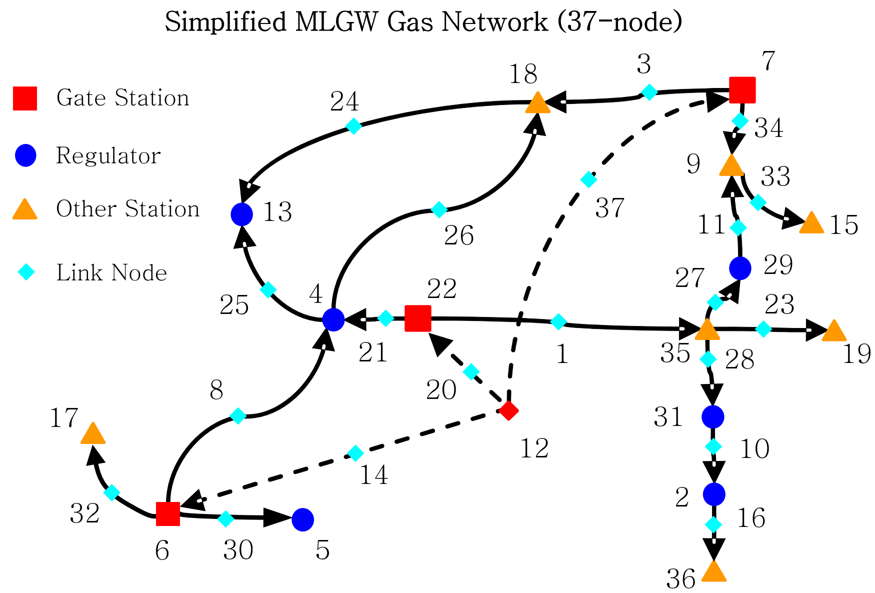
* Song, J., and S.-Y. Ok (2010). Multi-scale system reliability analysis of lifeline networks under earthquake hazards. *Earthquake Engineering and Structural Dynamics*, Vol. 39(**3**), 259-279.



- **"Divide and Conquer" approach**
  - Lower-scale system reliability analyses are performed for "supercomponents" and followed by higher-scale system reliability analyses
  - Proposed to facilitate the use of **LP bounds method** (Song and Der Kiureghian, 2003) for large-size systems
  - **MSR method** is a good tool for SRA at multiple scales
- **Advantages**
  - **Multi-scale modeling** of a system – seeing big picture without disregarding the details
  - Helps identify **important** components and parameters at **multiple scales**
  - **Collaborative** risk management
  - Facilitates parallel computing

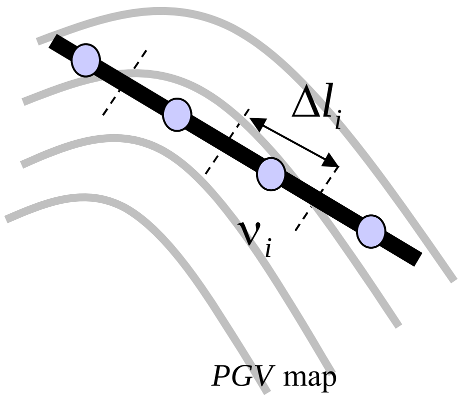
Example: MLGW gas network





- Gas pipeline network of Memphis Light, Gas, and Water (MLGW), Shelby County, TN
- A simplified network in Chang et al. (1996) was modified based on comments from R. Bowker (MLGW)
- 37-node and 40-arc network: nodes representing pipelines and stations
- Earthquake hazard scenarios: Epicenter at N35.54°-W90.43° at Blytheville, AR
- Fragilities of pipelines and stations – *HAZUS-MH*
- PGV and PGA maps from *MAEviz*

Failure prob. of pipeline segments



- Failure probability of the i-th segment of a pipeline

$$P_i = 1 - \exp(-\nu_i \cdot \Delta l_i)$$

- Failure occurrence rate of a pipeline (HAZUS-MH: FEMA 2003)

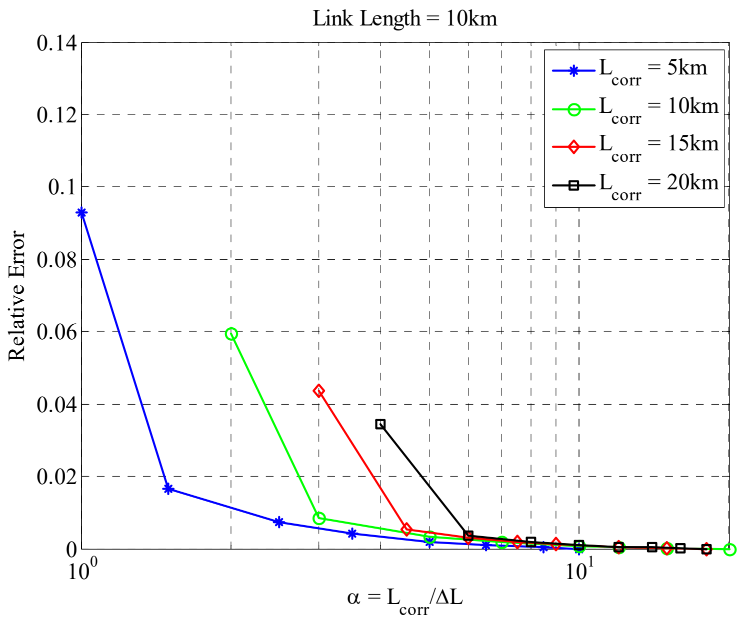
$$\nu_i = k \cdot (PGV_i)^{\gamma}$$

- Uncertainty in PGV (Adachi & Ellingwood, 2007)

$$PGV_i = \overline{PGV_i} \times \varepsilon_i$$

Lognormal r.v. (median = 1, c.o.v. = 0.6)

Attenuated PGV (Fernandez and Rix 2006)



- Spatial Correlation (Wang & Takada, 2005)

$$\rho_{\ln PGV_i, \ln PGV_j} = \rho_{\ln \varepsilon_i, \ln \varepsilon_j} = \exp(-\|\mathbf{x}_i - \mathbf{x}_j\| / L_{corr})$$

- Generalized Dunnett-Sobel (Song and Kang, 2008)

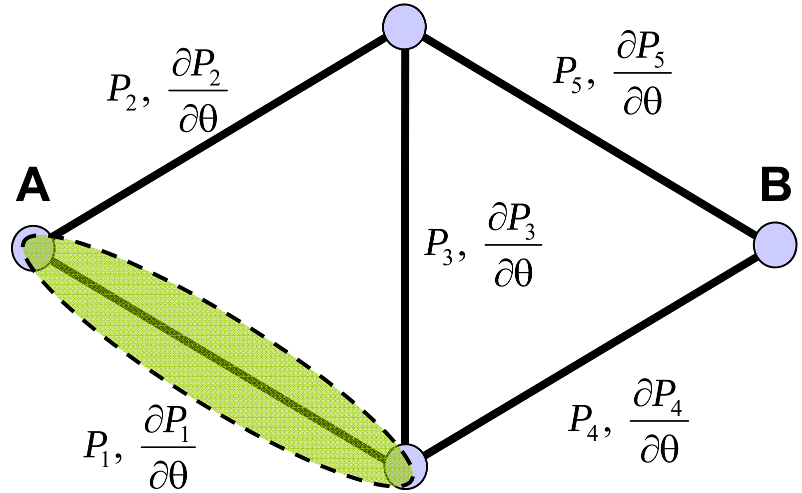
$$Z_i = \ln \varepsilon_i / \zeta_i \sim N(\mathbf{0}, \mathbf{R})$$ → Find gDS that fits best

- (←) Discretization error

  choose number of segments considering corr. length

Multi-scale SRA using MSR Method

Higher-scale

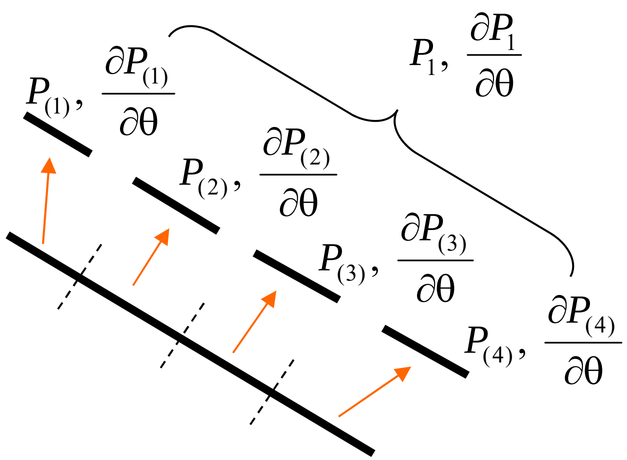


$$P(E_{sys}) = \mathbf{c}^{\mathrm{T}}\mathbf{p}$$

$$\frac{\partial P(E_{sys})}{\partial \theta} = \mathbf{c}^{\mathrm{T}}\frac{\partial \mathbf{p}}{\partial \theta} = \mathbf{c}^{\mathrm{T}}\hat{\mathbf{P}}\frac{\partial \mathbf{P}}{\partial \theta}$$

→ MSR analysis using failure probability and sensitivity of links $P_i, \frac{\partial P_i}{\partial \theta} \quad i = 1, \ldots, n_{link}$

Lower-scale

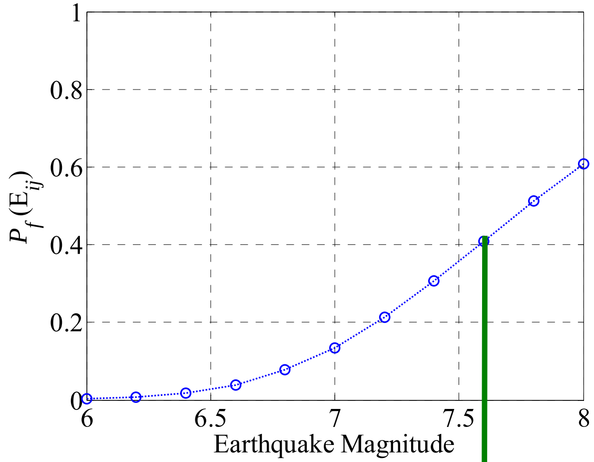


$$P_1 = \mathbf{c}_1^{\mathrm{T}}\mathbf{p}_1$$

$$\frac{\partial P_1}{\partial \theta} = \mathbf{c}_1^{\mathrm{T}}\frac{\partial \mathbf{p}_1}{\partial \theta} = \mathbf{c}_1^{\mathrm{T}}\hat{\mathbf{P}}_1\frac{\partial \mathbf{P}_1}{\partial \theta}$$

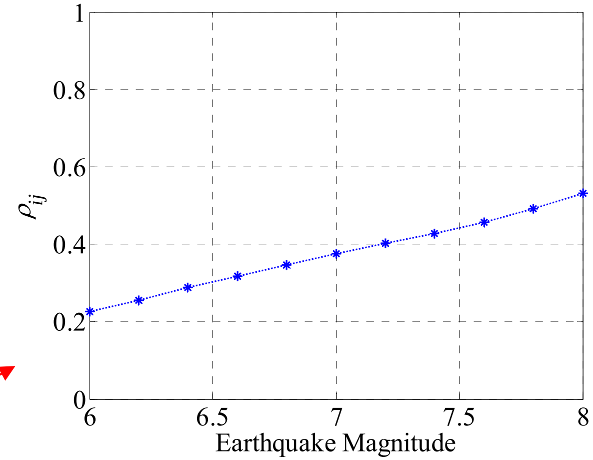
→ MSR analysis using failure probability and sensitivity of segments $P_{(i)}, \frac{\partial P_{(i)}}{\partial \theta} \quad i = 1, \ldots, n_{seg}$

Correlation between pipelines



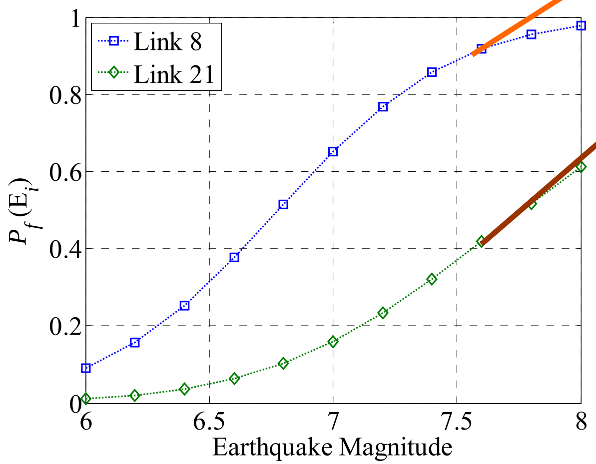
MSR for joint failure of pipelines

$$(E_{i,1} \cup \ldots \cup E_{i,N_i}) \cap (E_{j,1} \cup \ldots \cup E_{j,N_j})$$



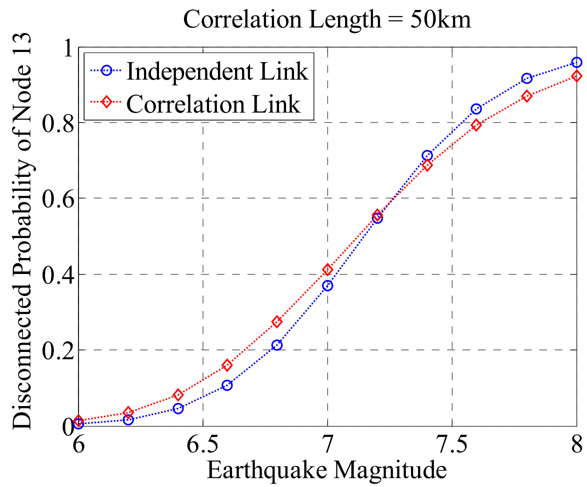
$$\Phi_2(-\beta_1, -\beta_2, \rho) = \Phi(-\beta_1)\Phi(-\beta_2) + \int_0^{\rho} \varphi(-\beta_1, -\beta_2, \rho')\, d\rho'$$

MSR w/ correlation

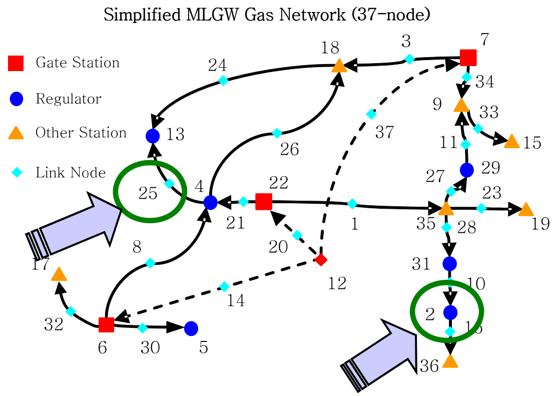


Lower-scale MSR

$$(E_{i,1} \cup \ldots \cup E_{i,N_i})$$
$$(E_{j,1} \cup \ldots \cup E_{j,N_j})$$

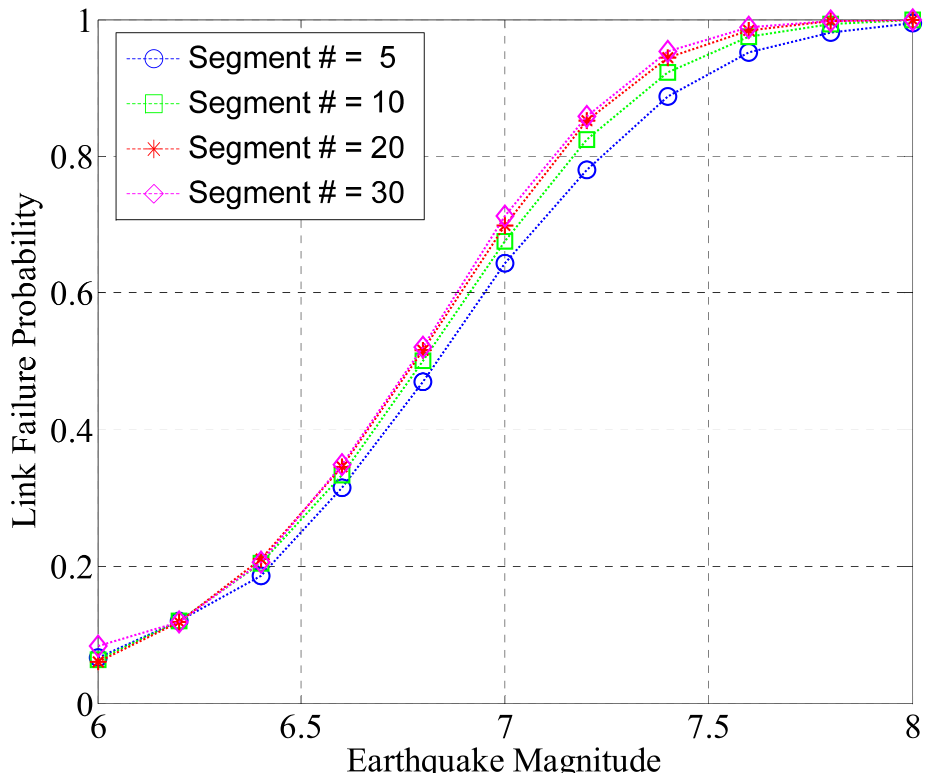


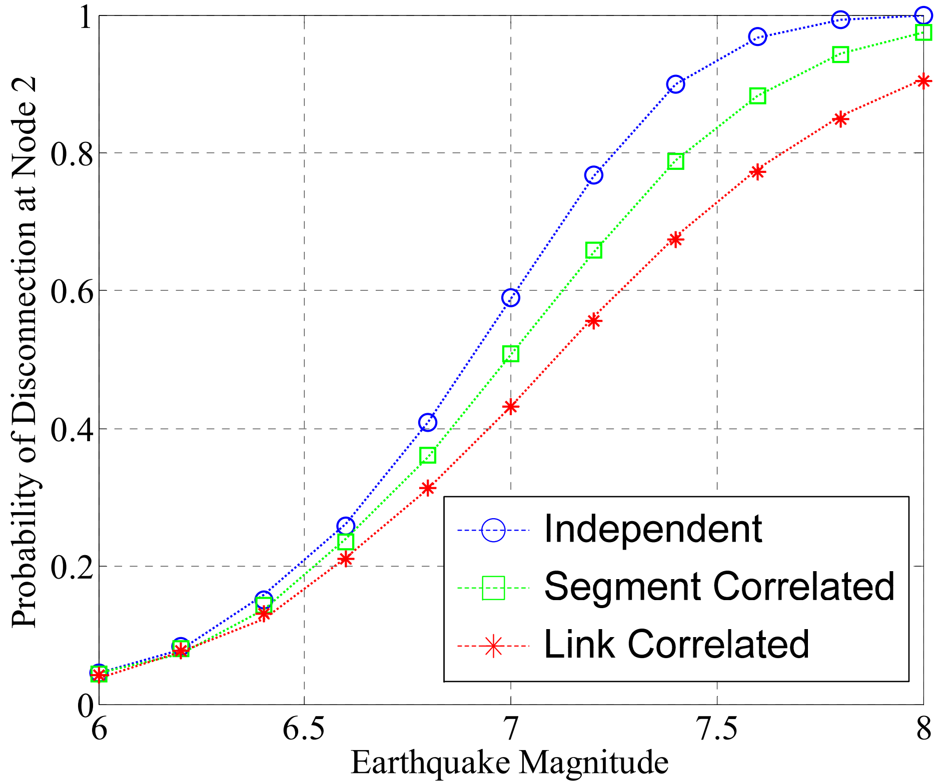
Risk at multiple scales



Lower-scale: pipelines

Failure probability of **Link** 25





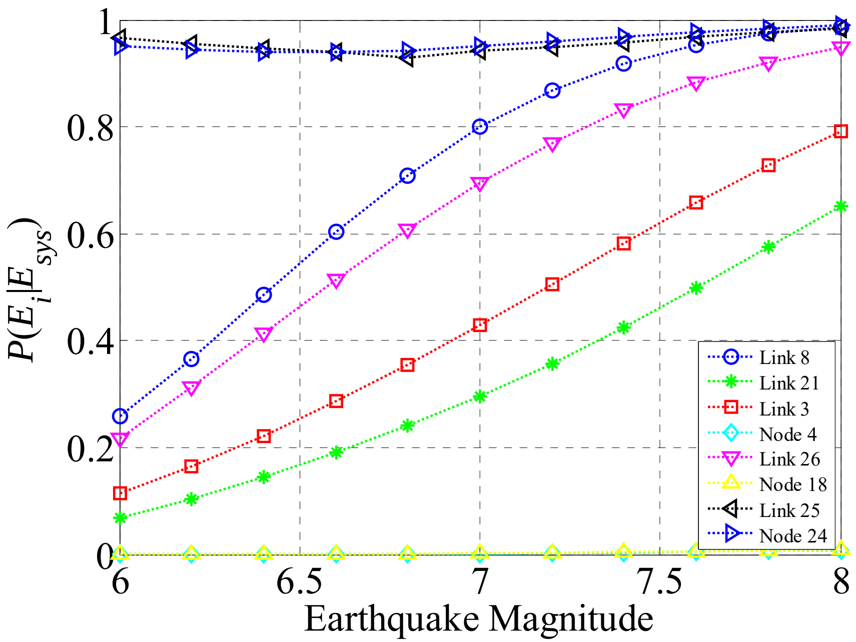
Higher-scale: service nodes

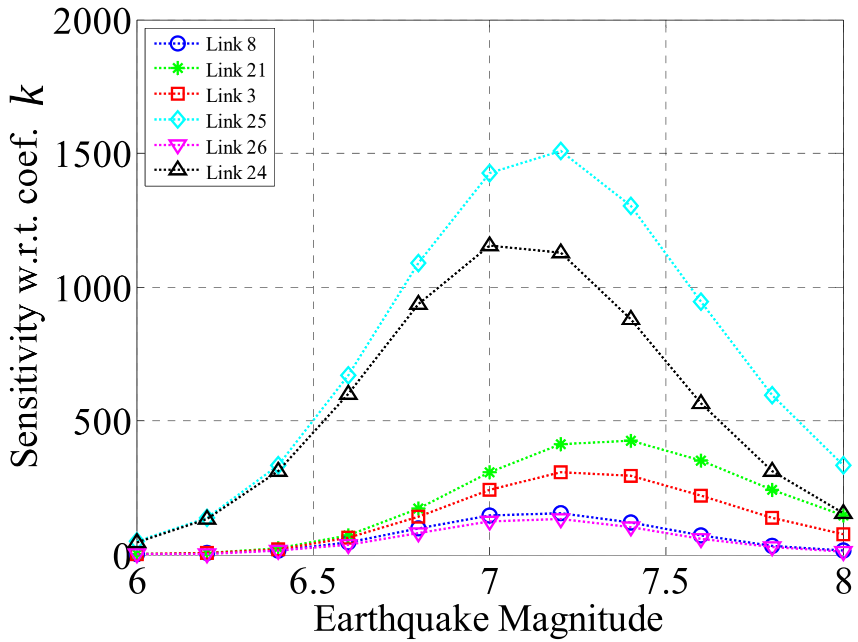
Prob. of Disconnection at **Node** 2

Probabilistic inference and sensitivity

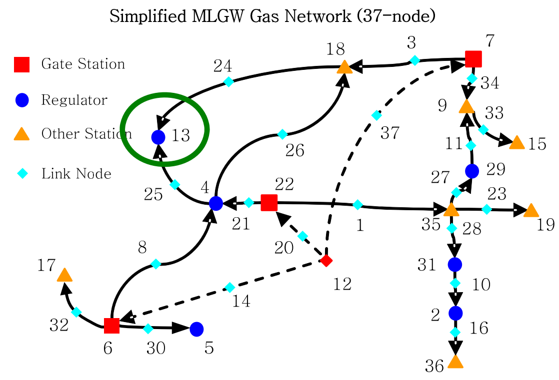
Conditional Probabilities

Parameter Sensitivity

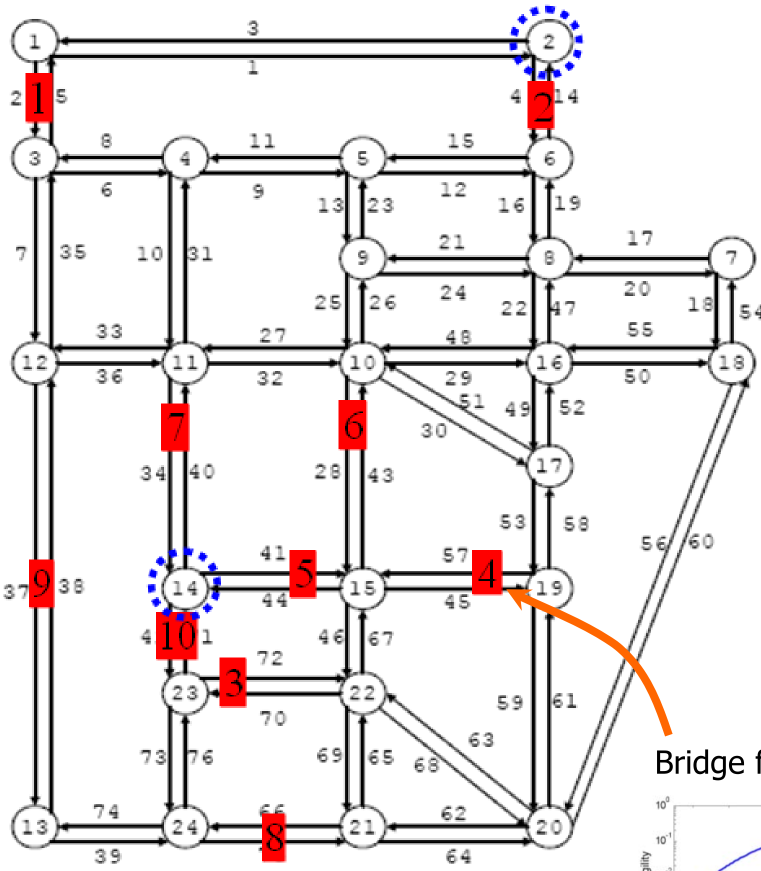




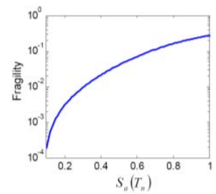
- Conditional probability of link failure probability given observed system event (e.g. disconnection)
- Sensitivity of system failure probability with respect to parameters in PGV-based model for failure occurrence rate: $\nu_i = k \cdot (PGV_i)^{\gamma}$



Appl. V: Post-hazard flow capacity of a network



Bridge fragility



Example: Modified Sioux-Falls network

Red: bridges; **Circles**: Starting & Ending points

- Traffic flow **capacity** between two points in a network → determined by combinations of bridge damage

$\mathbf{q}$: a vector of network flow capacity for bridge failure combinations (obtained by maximum flow capacity analysis)

$$\mu_Q = \mathbf{q}^{\mathrm{T}}\mathbf{p}$$

: average post-hazard flow capacity

$$\sigma_Q^2 = (\mathbf{q}.*\mathbf{q})^{\mathrm{T}}\mathbf{p} - (\mathbf{q}^{\mathrm{T}}\mathbf{p})^2$$

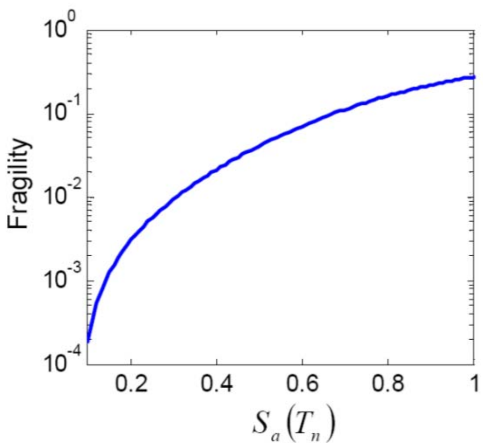
: variance of post-hazard flow capacity

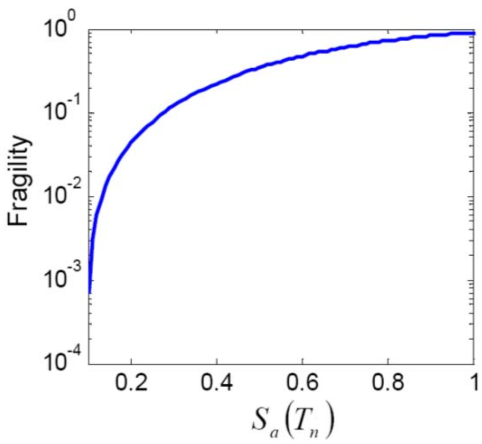
$$P(Q < a) = \sum_{\forall i: q_i < a} p_i$$

: probability that flow capacity is lower than a

Multi-state Fragility

- Fragility curves (Gardoni *et al*. 2002, 2003)





⇒ **Only two states, "connected" or "disconnected"**

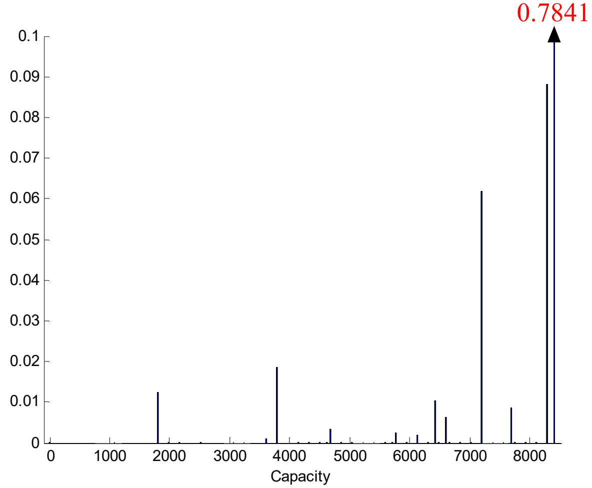
P(*Complete failure*) = $0.3 \times P_f$
P(*Heavy damage*) = $0.45 \times P_f$
P(*Moderate damage*) = $0.25 \times P_f$
P(*No damage*) = $1\text{-}P_f$

F(*Complete failure*) = 0
F(*Heavy damage*) = 0.3×Full capacity
F(*Moderate damage*) = 0.7×Full capacity
F(*No damage*) = 1.0×Full capacity

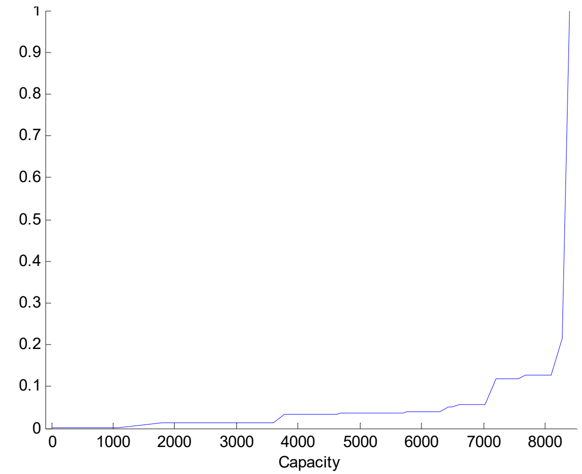
Uncertainty quantification of flow capacity

- Capacity distribution for a given seismic intensity (M=7.0)

PMF



CDF

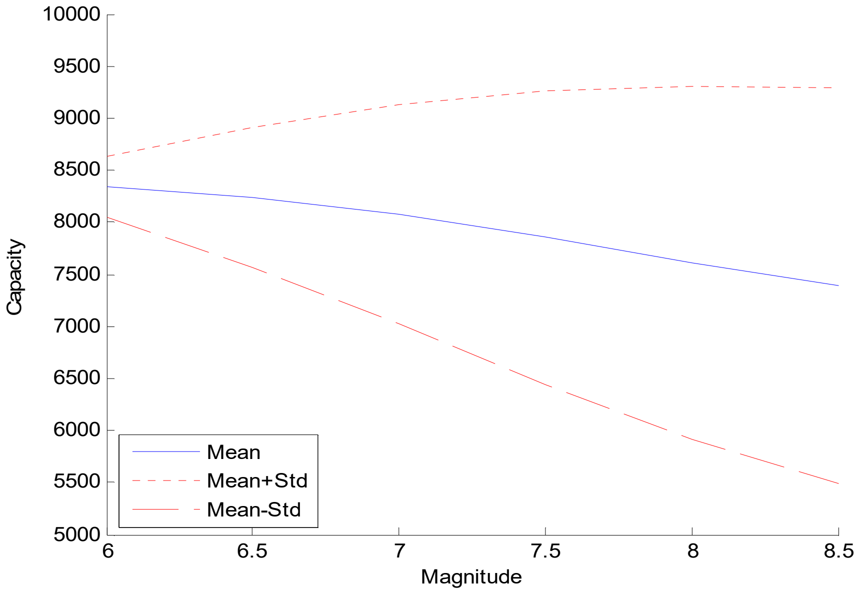


- Statistical parameters of flow capacity (M=6.0~8.5)

$$\mu_Q = \mathbf{p}^{\mathrm{T}}\mathbf{f}$$

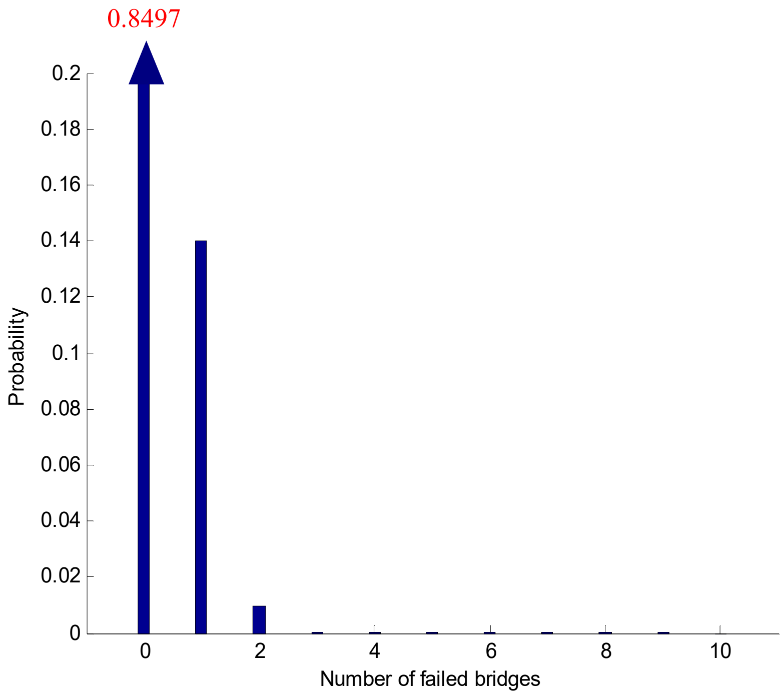
$$\sigma_Q = (\mathbf{p}^{\mathrm{T}}(\mathbf{f}.*\mathbf{f}) - \mu_Q{}^2)^{1/2}$$

$$\delta_Q = \sigma_Q / \mu_Q$$

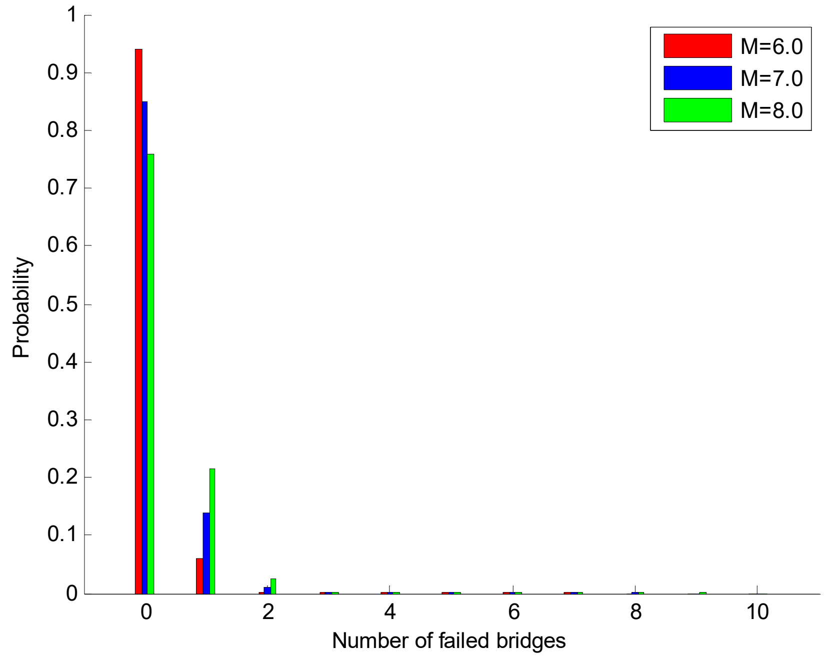


Analysis Results

- Probability with number of failed bridges



M=7.0

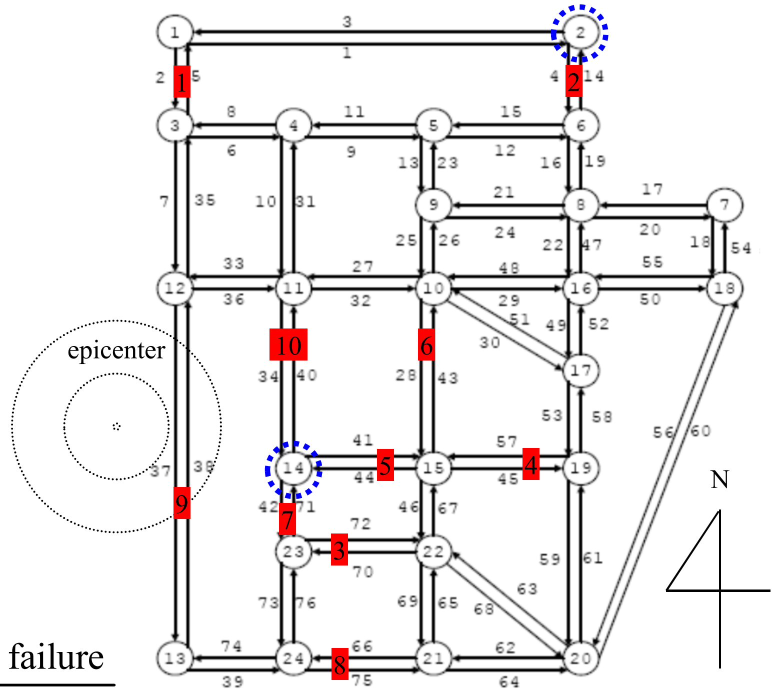


M=6.0~8.0

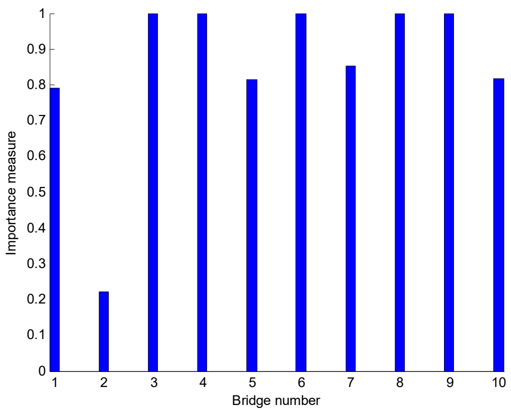
Analysis Results

- Conditional flow capacity (For 10th bridge, M=7.0)

Parameter		Value
Mean	$\mu_{Q\|10th}$	6591.9 **(8076.3)**
Standard deviation	$\sigma_{Q\|10th}$	1268.9 **(1056.6)**
C.O.V.	$\delta_{Q\|10th}$	0.1925 **(0.1308)**



- Importance measure for all bridges (M=7.0)

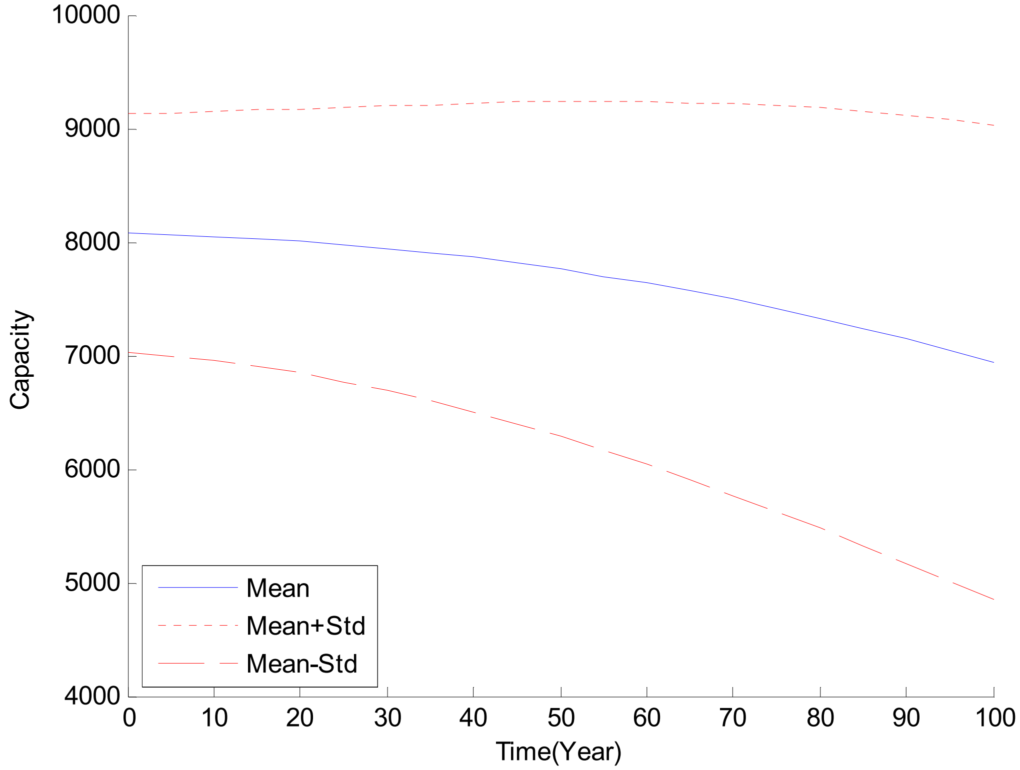


$$RF = 1 - \frac{\mu_{Q|\text{bridge failure}}}{\mu_Q}$$

1st, 2nd, 5th, 7th, and 10th bridges are most important

Analysis Results

- Flow capacity with deterioration



- Assumptions

P(T, *Complete failure*)
= P(*Complete failure*) × (1.0+0.0005 × T^2)

P(T, *Heavy damage*)
= P(*Heavy damage*) × (1.0+0.015 × T)

P(T, *Moderate damage*)
= P(*Moderate damage*) × (1.0-0.015 × T)

P(T, *No damage*) = 1 - P(T, *Complete failure*)
- P(T, *Heavy damage*)
- P(T, *Moderate damage*)

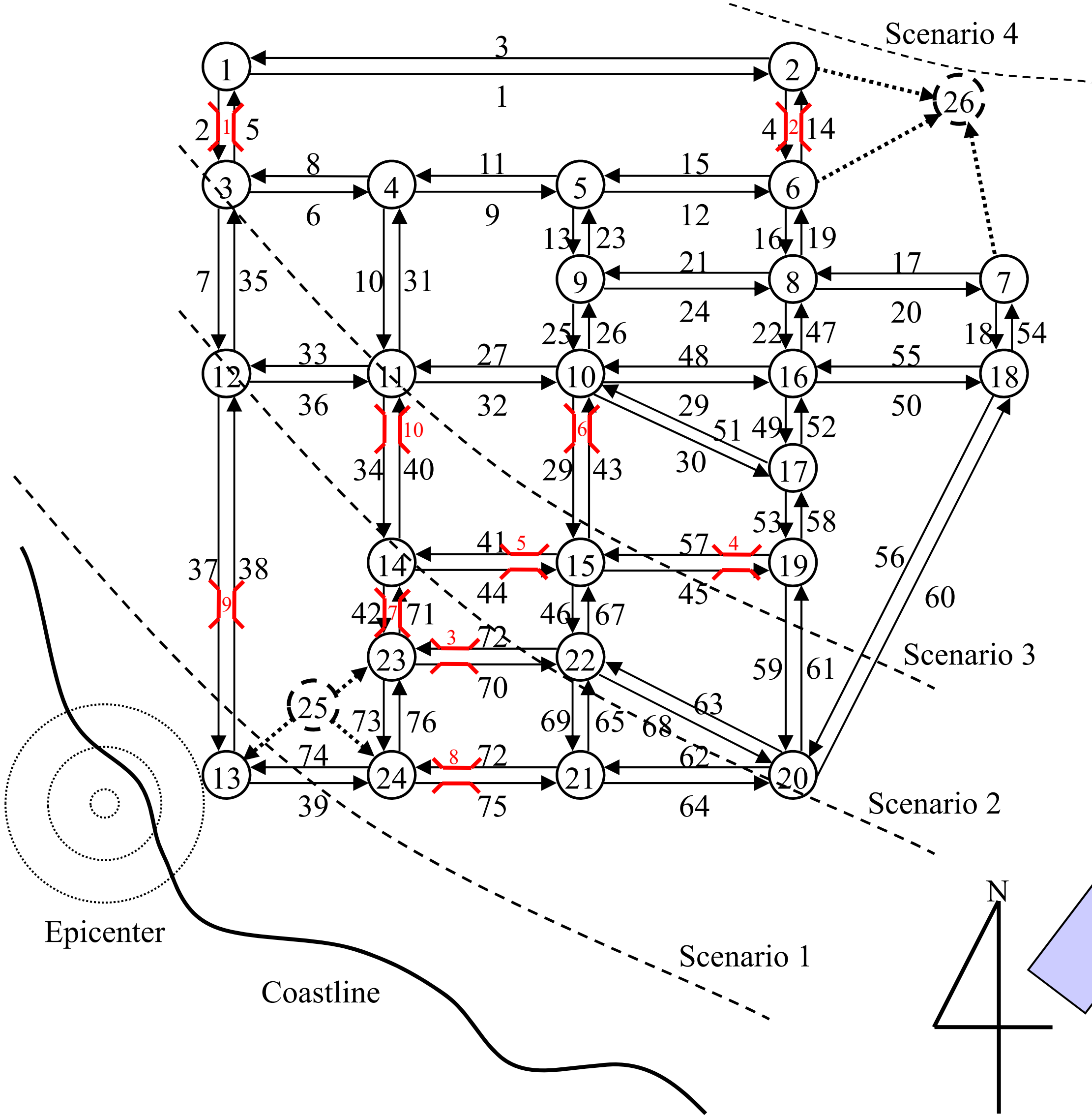
, where T:[Years]

$$\mu_Q(t) = \mathbf{q}^{\mathrm{T}}\mathbf{p}(t)$$

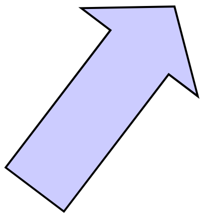
$$\sigma_Q(t) = \sqrt{(\mathbf{q}.*\mathbf{q})^{\mathrm{T}}\mathbf{p}(t) - \mu_Q^2(t)}$$

Extension to multi-hazard environment

* Lee, Y.-J., J. Song, P. Gardoni, and H.-W. Lim. (2010). Post-hazard flow capacity of bridge transportation network considering structural deterioration of bridges, *Structure and Infrastructure Engineering*, Accepted for Publication.

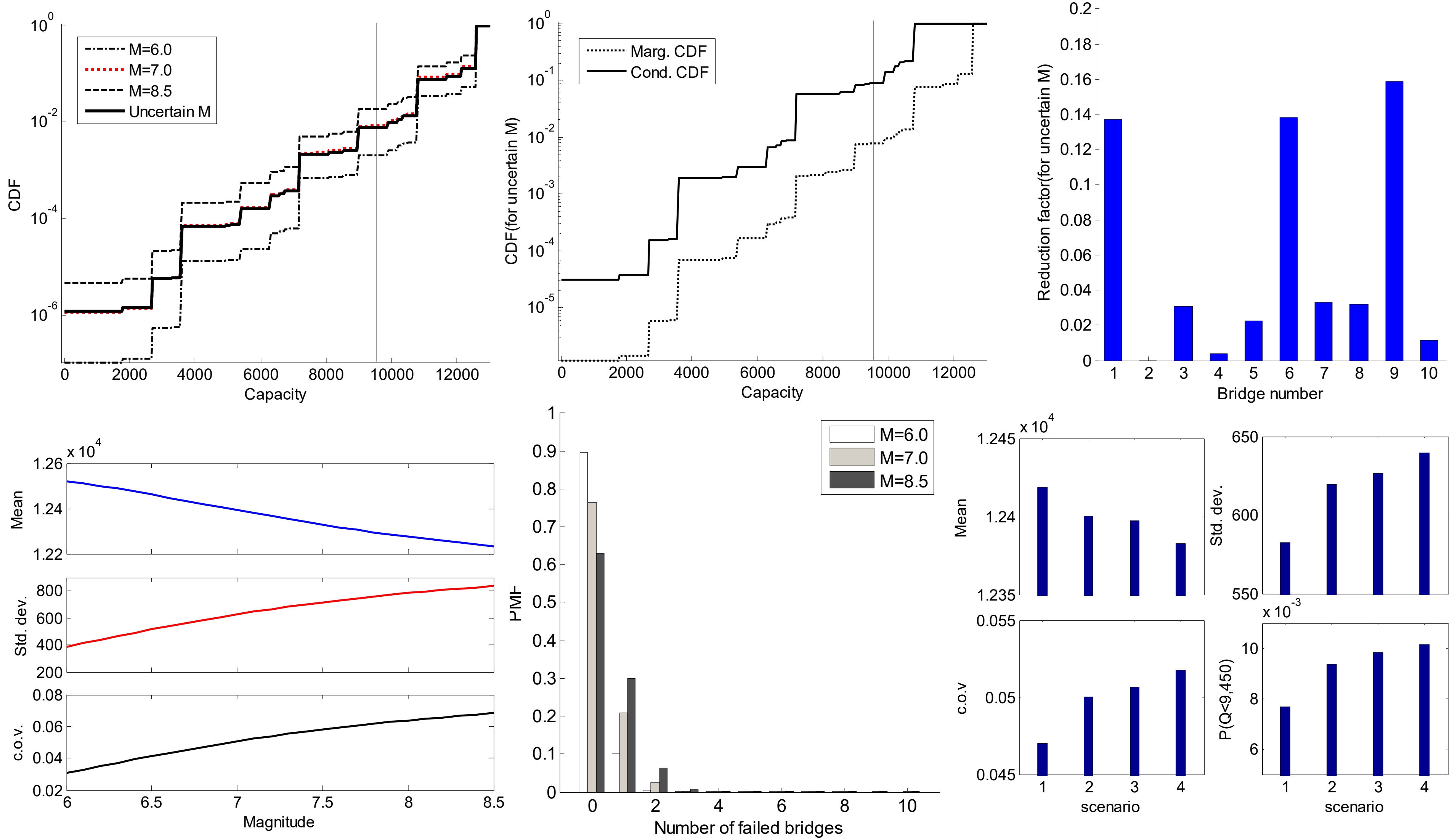


- More realistic assumptions
  - Multi-state fragility estimates w.r.t. drift capacity levels
  - Attenuation relationship (PSA & PGV)
  - Deterioration fragility estimates (Choe *et al.* 2007)
  - Multi-state flow capacity level proportional to number of open lanes
  - Deterioration scenarios
- Area-to-area flow capacity
- Further analysis for uncertain earthquake magnitude



Progress of Structural Deterioration (Corrosion) by Sea Air

Analysis Results

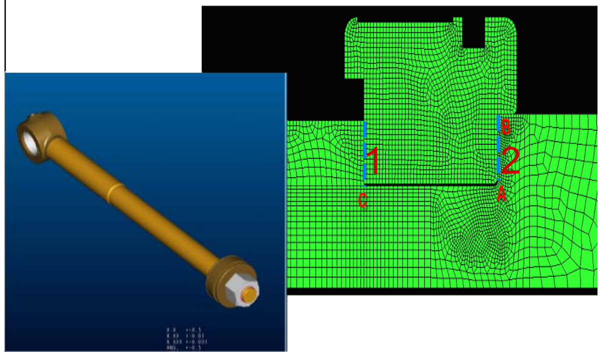


Application VI: FE system reliability analysis

* Lee, Y.-J., J. Song, and E.J. Tuegel (2008). Finite element system reliability analysis of a wing torque box. *Proc. 10th AIAA NDA*, April 7-10, Schaumburg, IL.

- FE reliability analysis: component vs. system
  - System-level risk is a logical function of multiple component events characterized by **failure modes**, **locations** and **load cases**
  - Using MSR methods, the **system-level risk and parameter sensitivities** are estimated based on the results of FE "component" reliability analysis.

1. Mechanical structures (single-nut piston)



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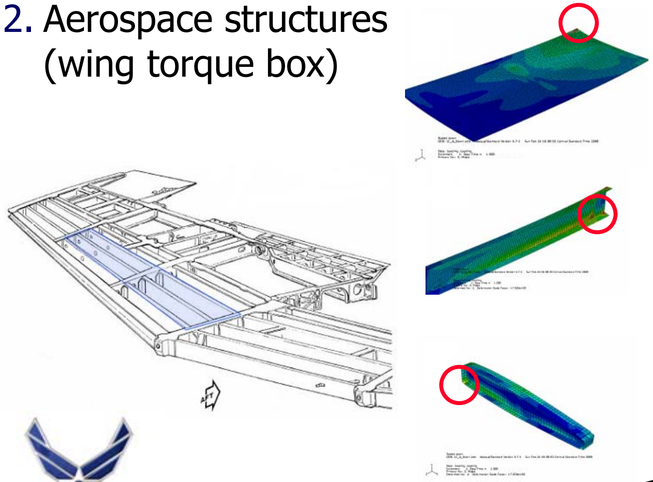
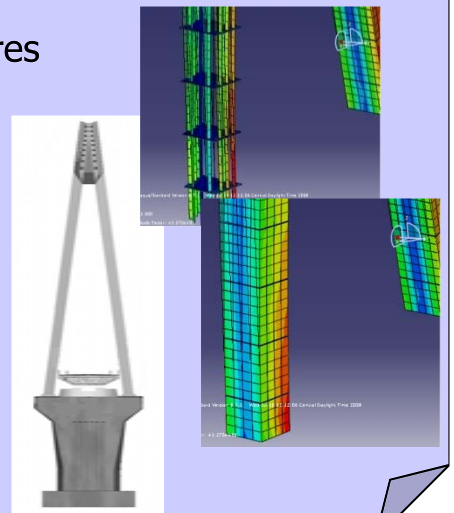
2. Aerospace structures (wing torque box)

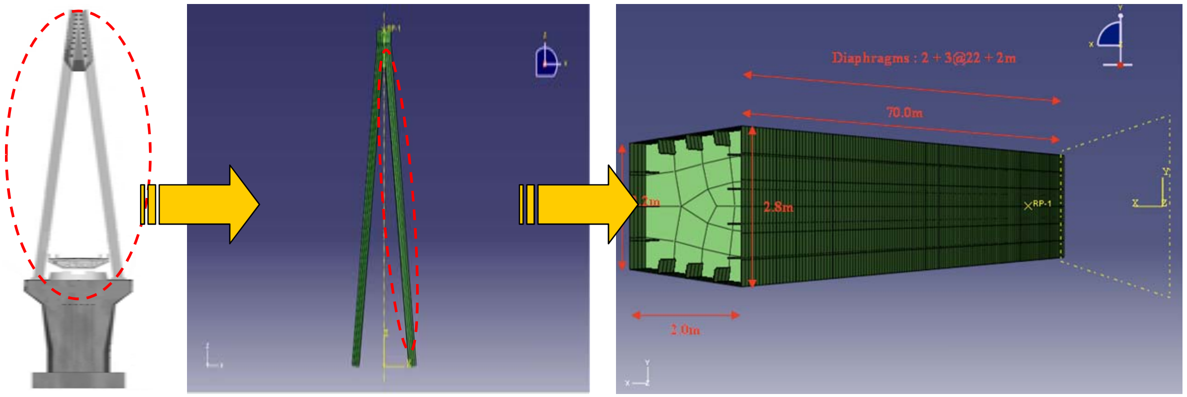


U.S. AIR FORCE

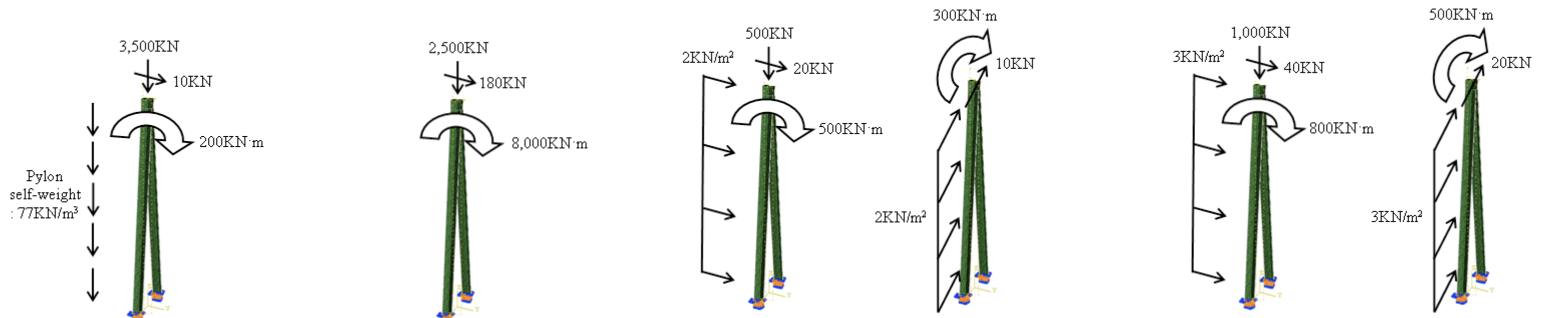
3. Civil structures (Bridge pylon)



Example: FE-SRA of bridge pylon system



- Bridge pylon system
  - Consists of 2 arms – each has 13 stiffeners and 23 diaphragms
  - Yielding failure considered in this example
  - Uncertainties in <u>Young's modulus</u>, <u>yield strength</u> and <u>scale factors of load cases</u> (dead, live, in-service wind and out-of-service wind loads) considered
  - Two load combinations considered: LC1 = D+L+Wi, LC2 = D+Wo



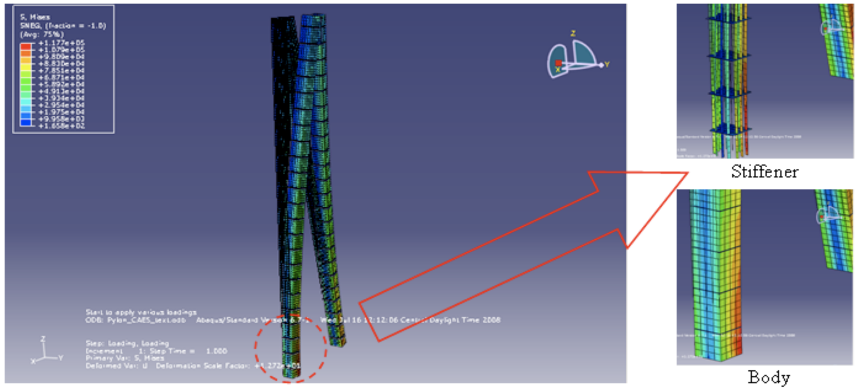
(i) Dead load(D)

(ii) Live load(L)

(iii) In-service wind load (Wi, In-plane & Out-of-plane)

(iv) Out-of-service wind load (Wo, In-plane & Out-of-plane)

FE component reliability analysis



- Identification of significant components
  - Deterministic FE analysis using the mean values of random variables → identify "hot spots" for each load combination
  - FE reliability analysis for identified "hot spots" by FORM → neglect if (1) Pf is too low or (2) highly correlated with other (more likely) component events

Components identified (E_1–E_8)

Truncated due to high correlation (E_9–E_{12})

Component event	Failure probability ($\times 10^{-4}$)
E_1 (LC1; 1st spot on right body)	1.295
E_2 (LC1; 1st spot on left body)	1.295
E_3 (LC1; 1st spot on right stiffener)	0.606
E_4 (LC1; 1st spot on left stiffener)	0.606
E_5 (LC2; 1st spot on right body)	6.996
E_6 (LC2; 1st spot on left body)	6.996
E_7 (LC2; 1st spot on right stiffener)	2.445
E_8 (LC2; 1st spot on left stiffener)	2.445
E_9 (LC1; 2nd spot on right body)	0.430
E_{10} (LC1; 2nd spot on left body)	0.430
E_{11} (LC2; 2nd spot on right body)	4.044
E_{12} (LC2; 2nd spot on left body)	4.044

- Correlation between components
  - Correlation b/w components are computed by

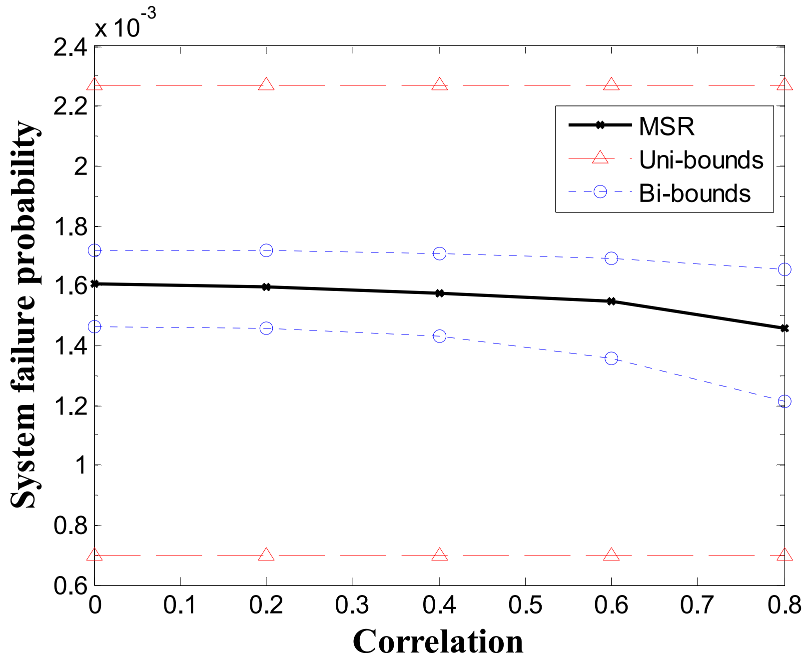
$$\rho_{ij} = \hat{\alpha}_i^{T} \hat{\alpha}_j$$

Correlation	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
E_1	1	0.814	0.708	0.744	0.646	0.502	0.448	0.476
E_2		1	0.744	0.708	0.502	0.646	0.476	0.448
E_3			1	0.683	0.423	0.451	0.680	0.429
E_4				1	0.451	0.423	0.429	0.680
E_5					1	0.887	0.820	0.842
E_6						1	0.842	0.820
E_7			Symmetric				1	0.801
E_8								1

FE system reliability analysis by MSR

- FE-SRA by MSR
  - Probability of most dominant component: 6.996×10^{-4} vs. system failure probability 1.550×10^{-3} → component reliability analysis may underestimate the risk significantly
  - Using component failure probability and sensitivity, the MSR method computes the system level parameter sensitivity
  - Can analyze other system events just by replacing event vector **c**

$$P(E_{sys}) = P\left(\bigcup_{i=1}^{8} E_i\right) \cong P\left[\bigcup_{i=1}^{8} \beta_i - Z_i \le 0\right]$$
$$= \int_{\Omega} \varphi_N(\mathbf{z};\mathbf{R})\,dz$$
$$= \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}}\mathbf{p}(\mathbf{s})\, f_{\mathbf{S}}(\mathbf{s})\,d\mathbf{s}$$



Random variables		$\delta_i = \frac{\partial P_1}{\partial \mu_i}\sigma_i$	$\eta_i = \frac{\partial P_1}{\partial \sigma_i}\sigma_i$
Young's modulus	Diaphragm (Left)	−0.0004	0
	Diaphragm (Right)	−0.0003	0
	Body (Left)	−0.6480	1.8018
	Body (Right)	−0.6624	1.8159
	Stiffener (Left)	0.3463	1.3114
	Stiffener (Right)	0.3558	1.3198
Load scale factor	Dead load	0.5130	0.0171
	Live load	2.1175	1.8348
	In-service wind load (In-plane)	2.9923	14.873
	In-service wind load (Out-of-plane)	0.4900	1.9121
	Out-of-service wind load (In-plane)	13.989	66.648
	Out-of-service wind load (Out-of-plane)	2.3301	8.599
Yield strength	Body (Left)	−8.0319	8.8381
	Stiffener (Left)	−2.5299	2.925
	Body (Right)	−8.0583	8.8729
	Stiffener (Right)	−2.5132	2.9001

App. VII: Reliability-Based Design Optimization

>> Deterministic Optimization

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} \quad f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X})$$

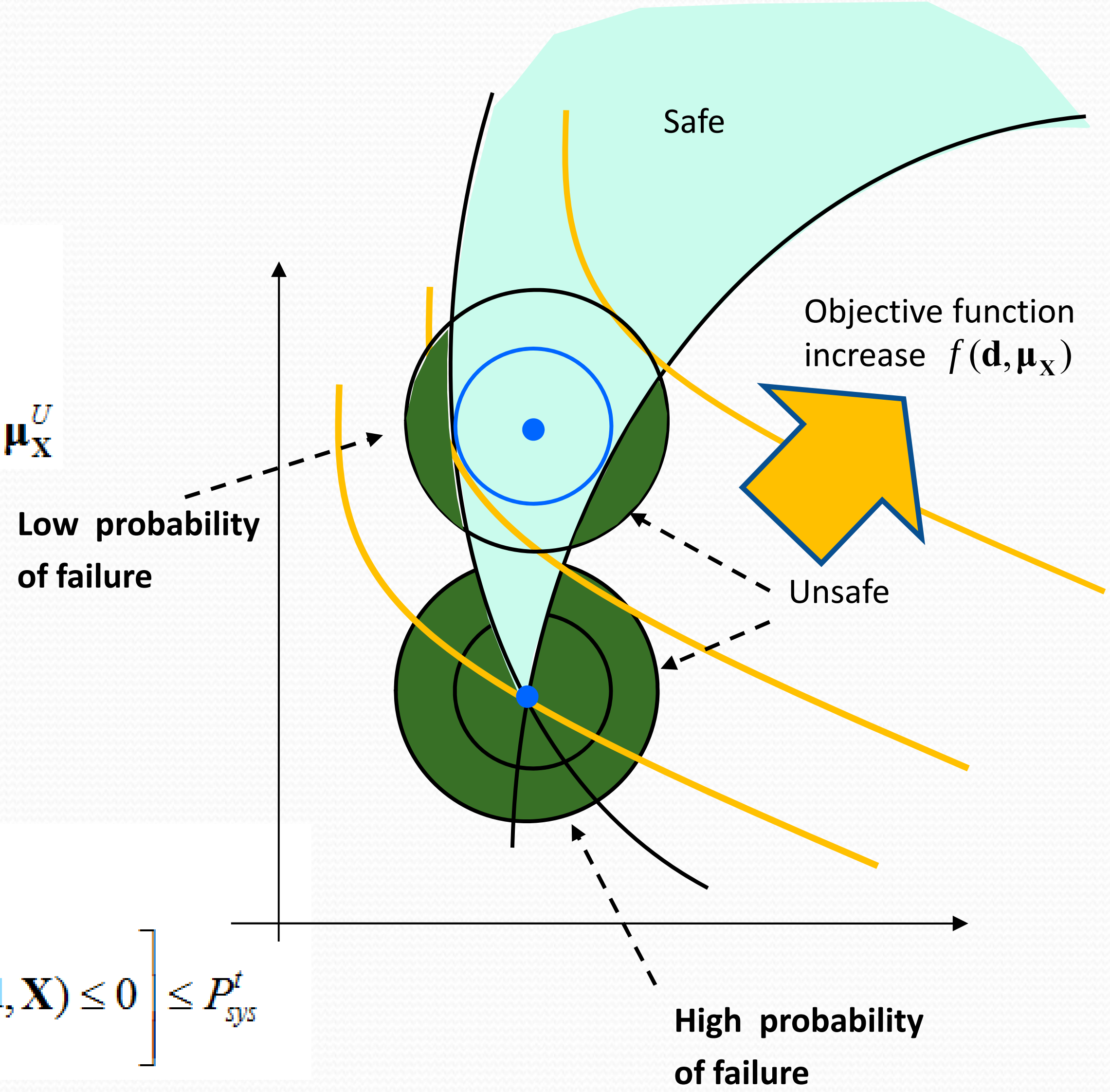
$$s.t. \quad g_i(\mathbf{d},\mathbf{X}) > 0$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \qquad \boldsymbol{\mu}_\mathbf{X}^L \le \boldsymbol{\mu}_\mathbf{X} \le \boldsymbol{\mu}_\mathbf{X}^U$$

>> Reliability-Based Design Optimization (RBDO)

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} \quad f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X})$$

$$s.t. \quad P_{sys} = P(E_{sys}) = P\left[\bigcup_k \bigcap_{i\in C_k} g_i(\mathbf{d},\mathbf{X}) \le 0\right] \le P_{sys}^t$$

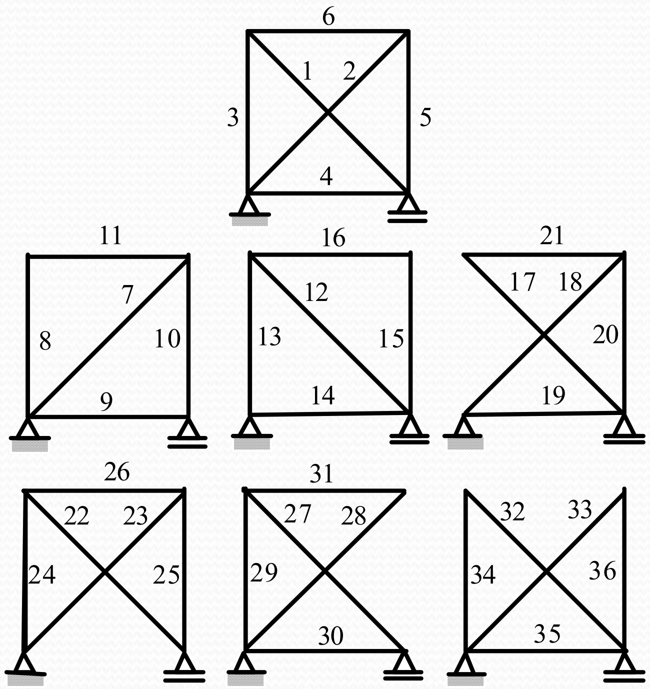


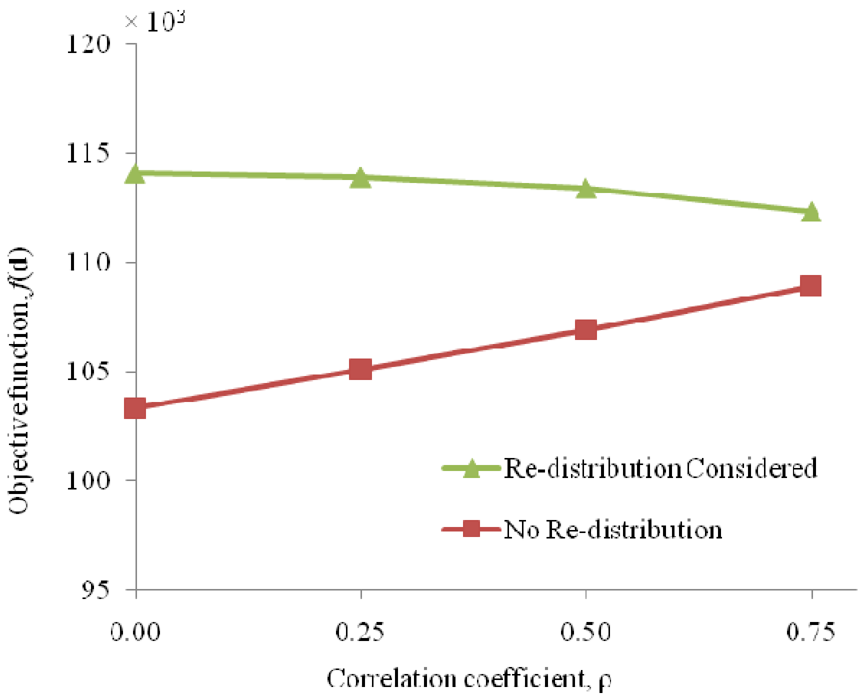
System RBDO by MSR method

RBDO of Truss system: Minimize the cross section areas under target failure probability of system collapse

Using MSR method, we can consider

- Effects of **load re-distributions** (sequential failures)
- Effects of **correlation between components**

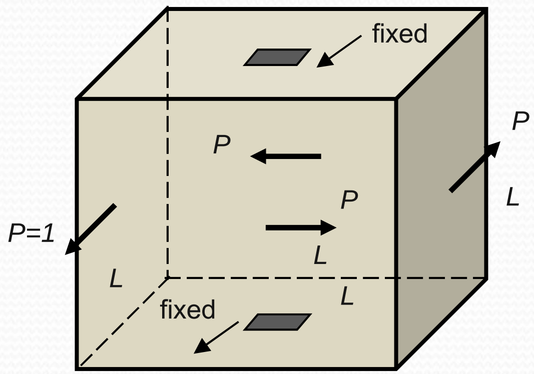


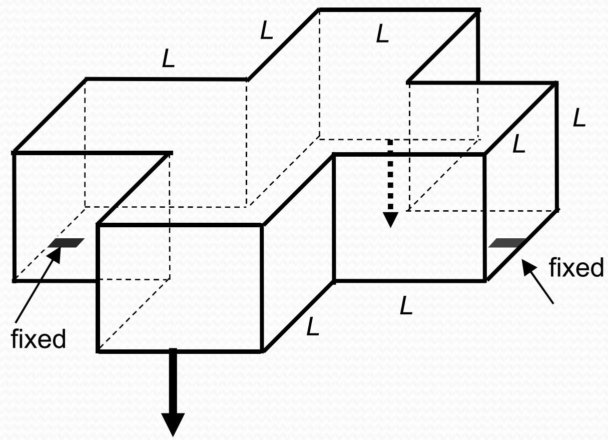


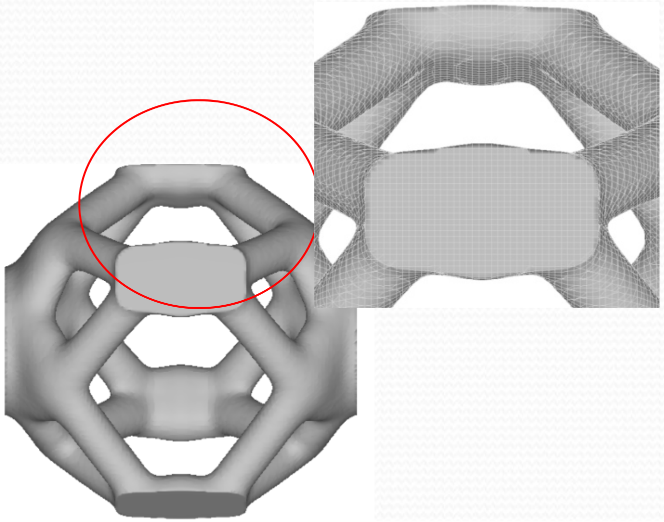
Nguyen, T.H., J. Song, and G.H. Paulino (2010). "Single-loop system reliability-based design optimization using matrix-based system reliability method: theory and applications," *J. of Mechanical Design*, ASME, Vol. 132, 011005-1~11.

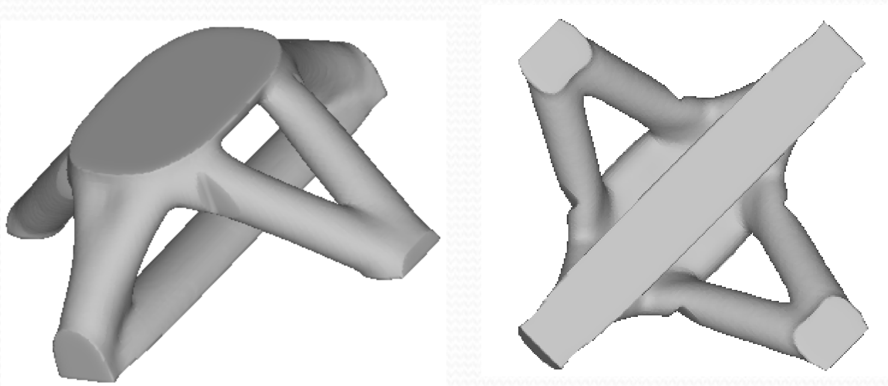
System RBTO by MSR method

RBTO of 2D or 3D continuum: Minimize the volume or compliance under target failure probability of *system* failure





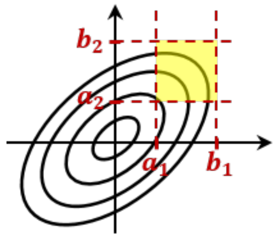




Nguyen, T.H., Paulino, G.H., and Song, J., and Le, C.H., “A Computational Paradigm for Multiresolution Topology Optimization (MTOP),” *Structural and Multidisciplinary Optimization,* vol. 41(**4**), 525-539.

457.646 Topics in Structural Reliability
In-Class Material: Class 19

◎ Multivariate normal integrals



$\mathbf{Z} \sim N(\mathbf{0};\mathbf{R})$

$$F(\mathbf{a},\mathbf{b};\mathbf{R}) \;=\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} \qquad\qquad d\mathbf{z}$$

If $a_i = -\infty$, $i = 1,\cdots,m$, it becomes Joint of

$\mathbf{Z} \sim N(\mathbf{0};\mathbf{R})$

$$\Phi_m(b_1,\cdots,b_m;\mathbf{R}) \;=\int_{-\infty}^{b_1}\cdots\int_{-\infty}^{b_m} \qquad\qquad d\mathbf{z}$$

Ⅰ) **Ditlevsen & Madsen (1996)**

$$m = 2:\;\; \Phi_2(b_1,b_2;\rho_{12}) = \qquad\qquad\qquad\qquad + \int_0^{\rho_{12}} \varphi_2(b_1,b_2;\quad)d\rho$$

________ assumption   error by ________ assumption

Note: double-fold integral involving $(-\infty, b_i)$ $\Rightarrow$ single-fold integral in $(0,\rho_{12})$

Note: $\rho_{12} > 0$: s.i assumption under/overestimate

$\rho_{12} < 0$: s.i assumption under/overestimate

※ $m = 3$ Song & ADK (2005) double-fold integral

Ⅱ) **Sequentially Conditioned Importance Sampling (SCIS)**

(Ambartzumian et al. 1998)

~sequentially sampling based on <u>conditional</u> PDF

given sampled value

~"scis.m" (developed by Prof. Young Joo Lee at UNIST
available at http://systemreliability.wordpress.com/software/

III) Product of Conditional Marginals (Pandey & Sarkar 2002)

$$\Phi_m(\mathbf{b};\mathbf{R}) \cong \prod_{k=1}^{m} \Phi\left(\frac{b_k - \mu_{k|k-1}}{\sigma_{k|k-1}}\right)$$

→ reasonable accuracy & very efficient

→ parallel or series

→ error↑ as m↑

→ Improved PCM (Yuan & Pandey 2006)

IV) Sequential Compounding Method (Kang & Song 2010)

$$\underbrace{\underbrace{\{(Z_1 < -\beta_1) \bigcup (Z_2 < -\beta_2)\}}_{Z_A < -\beta_A,\ \rho_{A3}} \bigcap (Z_3 < -\beta_3)}_{Z_B < -\beta_B}$$

→ applicable to <u>general</u> system

→ efficient and accurate

→ handle large m

→ when the same component event appears multiple times → difficult

→ parameter sensitivity of system reliability using SCM (Chun, Song, and Paulino, 2015, *Structural Safety*)

V) Matrix-based System Reliability (MSR) Method (Kang & Song 2008) (Kang et al. 2012)

VI) Method by Genz (1992) http://www.math.wsu.edu/faculty/genz/homepage

Transformations to uniform hypercube

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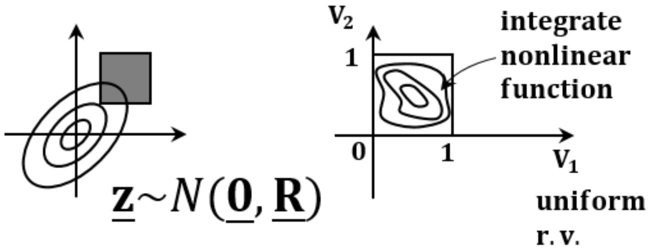


→ Parallel system

→ Very accurate & efficient even for large-size system

→ Integration by qusai-MCS

→ `mvncdf.m` in Matlab

Genz, A., and Bretz, F. (2009) *Computation of Multivariate Normal and t Probabilities, Lecture Notes in Statistics*, Springer-Verlag, NY.

457.646 Topics in Structural Reliability

In-Class Material: Class 20

“　　　　　　　　” (cf.　　　　　　　)

Ⅴ. Structural Reliability under Model & Stastical Uncertainties

(Ref.: “Analysis of Structural Reliability under Model and Statistical Uncertainties: A Bayesian Approach” ~ eTL)

◎ Formulation of Reliability Problems under Epistemic Uncertainties

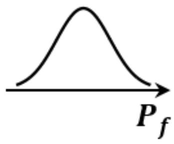
① Reliability Problem with Aleatoric uncertainties (only)

$P_f = \int f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$　　　　$\mathbf{x}$: r.v’s representing aleatoric uncertainties in the problem

→ Use component and/or system reliability method

② Reliability Problem under Aleatoric & Epistemic certainties

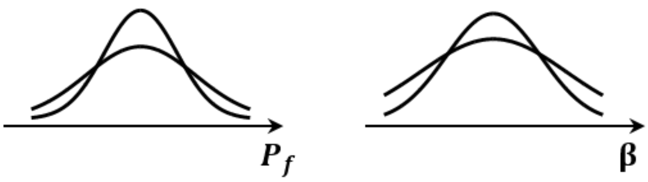
$P_f(\boldsymbol{\theta}) = \int_{\cup\cap g(\mathbf{x};\quad)\le 0} f_{\mathbf{x}}(\mathbf{x};\quad)d\mathbf{x} \Rightarrow \beta(\boldsymbol{\theta}) = \Phi^{-1}[\qquad\quad]$



$\boldsymbol{\theta} = [\boldsymbol{\theta}_f \quad \boldsymbol{\theta}_g]$

uncertain parameters

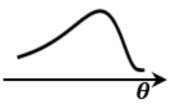
$\Rightarrow P_f$ & β become __________ due to uncertainty in $\boldsymbol{\theta}_f$ and/or $\boldsymbol{\theta}_g$



cf. $\beta(\boldsymbol{\theta}),\ P_f(\boldsymbol{\theta}) \Rightarrow$ _________ reliability index given value of uncertain parameters

◎ Three approaches for estimating reliability under epistemic uncertainties

Suppose $f_{|\boldsymbol{\theta}|}(\boldsymbol{\theta})$ is available,



① Point estimate of Reliability: $P_f(\boldsymbol{\theta})$ at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$

$\hat{\boldsymbol{\theta}}$: point estimate (representative) of $\boldsymbol{\theta}$

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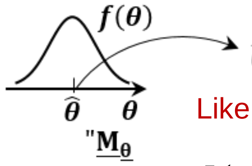
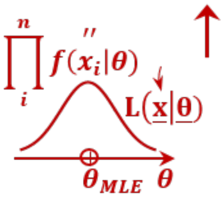
e.g. $\begin{cases} \hat{\boldsymbol{\theta}} = \mathbf{M}_{\boldsymbol{\theta}} = \int \boldsymbol{\theta} f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\, d\boldsymbol{\theta} & \text{Bayesian} \\ \hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_{MLE} = \arg\max_{\boldsymbol{\theta}} L(\mathbf{x};\boldsymbol{\theta}) & \text{Non-Bayesian} \end{cases}$



Likelihood function

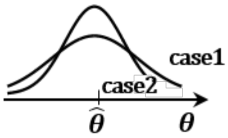
$L(\mathbf{x};\boldsymbol{\theta}) \propto P(x,\cdots,x_n|\boldsymbol{\theta})$



$\Rightarrow$ $P_f(\hat{\boldsymbol{\theta}})$, $\beta(\hat{\boldsymbol{\theta}})$: Perform reliability analysis with $\boldsymbol{\theta} =$ fixed

Note i) $P_f(\mathbf{M}_{\boldsymbol{\theta}}) = \mathbf{M}^{FO}_{P_f(\boldsymbol{\theta})}$

ii) Variability in $\boldsymbol{\theta}$ not considered



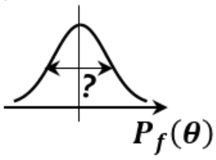
② "Predictive" Reliability

$$\tilde{P}_f = E_{\boldsymbol{\theta}}\left[P_f(\boldsymbol{\theta})\right]$$
$$= \int P_f(\boldsymbol{\theta}) \cdot f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\, d\boldsymbol{\theta}$$

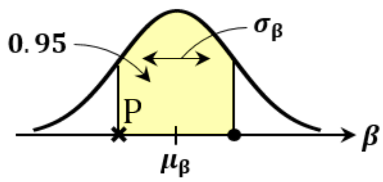
$$\tilde{\beta} = \Phi^{-1}[\qquad\qquad]$$

$\rightarrow$ incorporates variability in $\boldsymbol{\theta}$

$\rightarrow$ but still point estimate, i.e. does not measure variability in $P_f(\boldsymbol{\theta})$ caused by that in $\boldsymbol{\theta}$



③ Bounds on Reliability (Confidence Intervals)



$100 \times p(\%)$ confident that β is b/w x and o

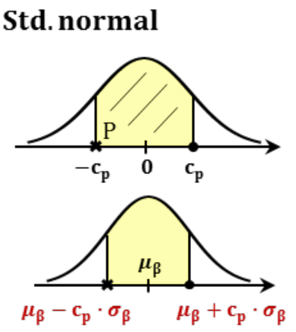
First, find mean and variance of $\beta(\boldsymbol{\theta})$

reliability analysis $\qquad f_{\boldsymbol{\theta}}(\boldsymbol{\theta})$

$$\begin{cases} \mu_\beta \cong \mu_\beta^{FO} = \beta(\mathbf{M}_{\boldsymbol{\theta}}) \\ \sigma_\beta^{\,2} \cong (\sigma_\beta^{\,2})^{FO} = \nabla_{\boldsymbol{\theta}}\beta(\mathbf{M}_{\boldsymbol{\theta}})\boldsymbol{\Sigma}_{\boldsymbol{\theta\theta}}\nabla_{\boldsymbol{\theta}}\beta(\mathbf{M}_{\boldsymbol{\theta}})^T \end{cases}$$

Parameter sensitivity (e.g. FORM)

Second, assume $\beta \sim N(\mu_\beta, \sigma_\beta)$



$$u_\beta = \frac{\beta - \mu_\beta}{\sigma_\beta}$$

P	c_p
0.70	1.04
0.80	1.28
0.90	1.64
0.95	1.96
0.99	2.58

$$\langle \beta \rangle_{100\times p(\%)} = \mu_\beta \pm c_p \sigma_\beta$$

(if $\tilde{\beta}$ available, $\tilde{\beta} \pm c_p \sigma_\beta$)

$$\langle P_f \rangle_{100\times p(\%)} = \Phi\left[-\left(\tilde{\beta} \pm c_p \sigma_\beta\right)\right] \qquad P_f = \Phi(-\beta)$$

Then, $f_{\boldsymbol{\theta}_f}(\boldsymbol{\theta}_f)$, $f_{\boldsymbol{\theta}_g}(\boldsymbol{\theta}_g)$??

(Review) Rel. Analysis under Epistemic Uncertainties (Model or Statistical)

① Point Estimate $\quad P_f(\hat{\theta}),\ \beta(\hat{\theta})$

② Predictive Reliability $\quad \tilde{P}_f = E_\theta\left[P_f(\theta)\right]$

③ Bounds $\quad \langle \beta \rangle_{100\times p(\%)} = \mu_\beta \pm c_p \sigma_\beta$

$f_{\theta_f}(\theta_f)$? $f_{\theta_g}(\theta_g)$?

◎ Bayesian Parameter Estimation

cf. Bayes rule

$$P(A|B) = \frac{1}{P(B)} \cdot P(B|A) \cdot P(A)$$

f c L p

$$f(\boldsymbol{\theta}) = c \cdot L(\boldsymbol{\theta}) \cdot p(\boldsymbol{\theta})$$

① $P(\boldsymbol{\theta})$: ( ) distribution

- represents state of our knowledge ( ) making observations (objective information)
- may incorporate ( ) info. such as "engineering judgment"

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② $L(\boldsymbol{\theta})$: (   ) function

- represents (   ) information gained from the observation
- function (   ) to conditional prob. of the observation given $\boldsymbol{\theta}$

$$L(\boldsymbol{\theta}) \propto P(E_{obs} | \boldsymbol{\theta})$$

③ *c*: (   ) factor

- makes $c \cdot L(\boldsymbol{\theta}) \cdot p(\boldsymbol{\theta})$ a valid PDF

i.e. $\int_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) = \int_{\boldsymbol{\theta}} c \cdot L(\boldsymbol{\theta}) \cdot P(\boldsymbol{\theta}) d\boldsymbol{\theta} =$

$\therefore \; c =$

④ $f(\boldsymbol{\theta})$: (   ) distribution

- represents updated knowledge about $\boldsymbol{\theta}$
- subjective + objective
-

;
;

rare observation available
as more observations are made

◎ Computation of c and posterior statistics

$$\left.\begin{aligned} c &= \left[\int L(\boldsymbol{\theta}) \cdot p(\boldsymbol{\theta}) \cdot d\boldsymbol{\theta} \right]^{-1} \\ \mathbf{M}(\boldsymbol{\theta}) &= \int \boldsymbol{\theta} \cdot f(\boldsymbol{\theta}) d\boldsymbol{\theta} = \int \boldsymbol{\theta} \cdot c \cdot L(\boldsymbol{\theta}) \cdot p(\boldsymbol{\theta}) \cdot d\boldsymbol{\theta} \\ \boldsymbol{\Sigma}_{\boldsymbol{\theta}\boldsymbol{\theta}} &= \int \boldsymbol{\theta}\boldsymbol{\theta}^T f(\boldsymbol{\theta}) d\boldsymbol{\theta} - \mathbf{M}(\boldsymbol{\theta}) \mathbf{M}(\boldsymbol{\theta})^T \end{aligned}\right\} \text{multi-fold integrals}$$

How?

$\left\{ \right.$
- Convenient forms for special distribution (directly update statistics “conjugate”)
- Special numerical algorithms (Geyskens et al. 1993)
- Sampling methods (MCS, importance sampling, $\cdots$)

Probabilistic Shear Strength Models for RC Beams by Bayesian Updating Based on Experimental Observations

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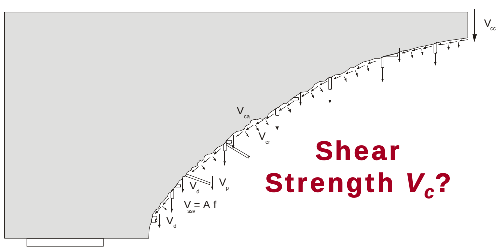
Sungmoon Jung Florida A&M-Florida State University, USA

Probabilistic shear strength models

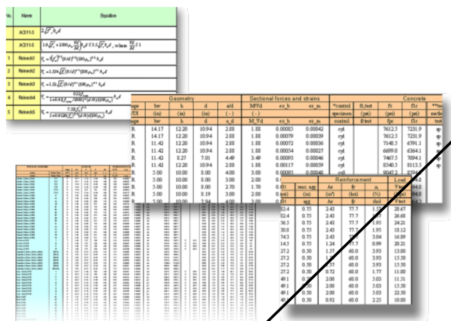


Shear Strength V_c?

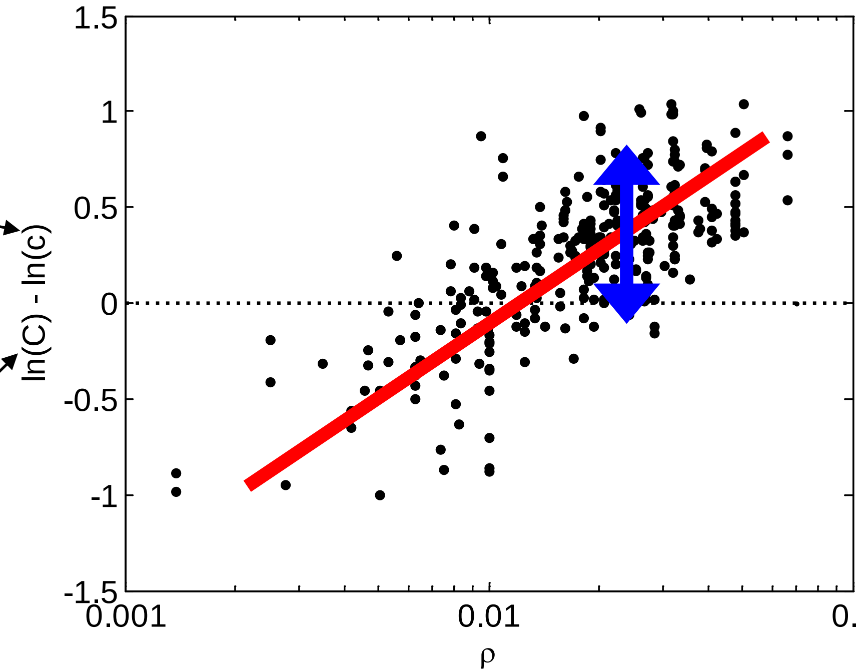
Reinforced Concrete Beam w/o Shear Reinforcement

$$V_c = (1/6)\, f_c'^{0.5} b_w d$$

Empirical Formulas



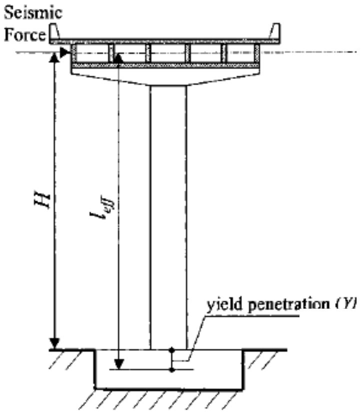
Database of Experimental Observations



- Empirical formulas are widely used for code provisions and designs
  ~ based on simplified mechanics rules and limited amount of experimental observations.
- Inaccurate description of physics & missing variables → **biases** and **scatters**
- Need probabilistic shear strength models that **correct the biases** **and** **quantify the uncertainties** based on comprehensive database of **experimental observations**

Probabilistic models by Bayesian updating*

* Gardoni, P., Der Kiureghian, A., and Mosalam, K.M. (2002)
"**Probabilistic capacity models and fragility estimates for reinforced concrete columns based on experimental observations**"
Journal of Engineering Mechanics, Vol. 128(10)



Bridge Column

$$C(\mathbf{x},\boldsymbol{\Theta}) = c(\mathbf{x}) + \gamma(\mathbf{x},\boldsymbol{\theta}) + \sigma\varepsilon$$

Capacity | **Prediction by deterministic Model** | **Bias-correction** | **Remaining errors**

Assumptions:

- $\sigma\varepsilon$ is independent of input variables ~ "**Homoskedasticity**"
- ε has the normal distribution ~ "**Normality**"

Probabilistic models by Bayesian updating*

* Gardoni, P., Der Kiureghian, A., and Mosalam, K.M. (2002)
"**Probabilistic capacity models and fragility estimates for reinforced concrete columns based on experimental observations**"
Journal of Engineering Mechanics, Vol. 128(10)

Explanatory functions

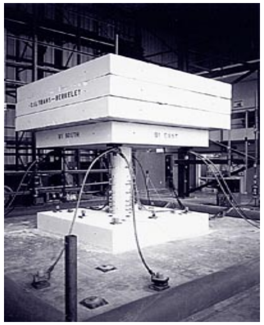
$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \ln[c(\mathbf{x})] + \sum_{i=1}^{p} \boxed{\theta_i} h_i(\mathbf{x}) + \boxed{\sigma\varepsilon}$$

? ?

Nonlinear transformation to achieve "homoskedasticity"

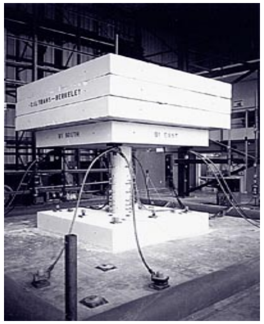
$$f(\boldsymbol{\Theta}) = \kappa L(\boldsymbol{\Theta})\, p(\boldsymbol{\Theta})$$

Bayesian parameter estimation



Database of 106 columns

Step-wise removal process



$$f(\Theta) = \kappa L(\Theta)\, p(\Theta)$$

Bayesian parameter estimation

$$\mu_\theta, \sigma_\theta, \delta_\theta, \rho_{\theta_i \theta_j}$$

- Remove an explanatory terms with the highest c.o.v. (most uncertain)
- Continue until the mean of σ starts increasing significantly

Table 2. Explanatory removing process for joint shear strength, equations (1) and (8)

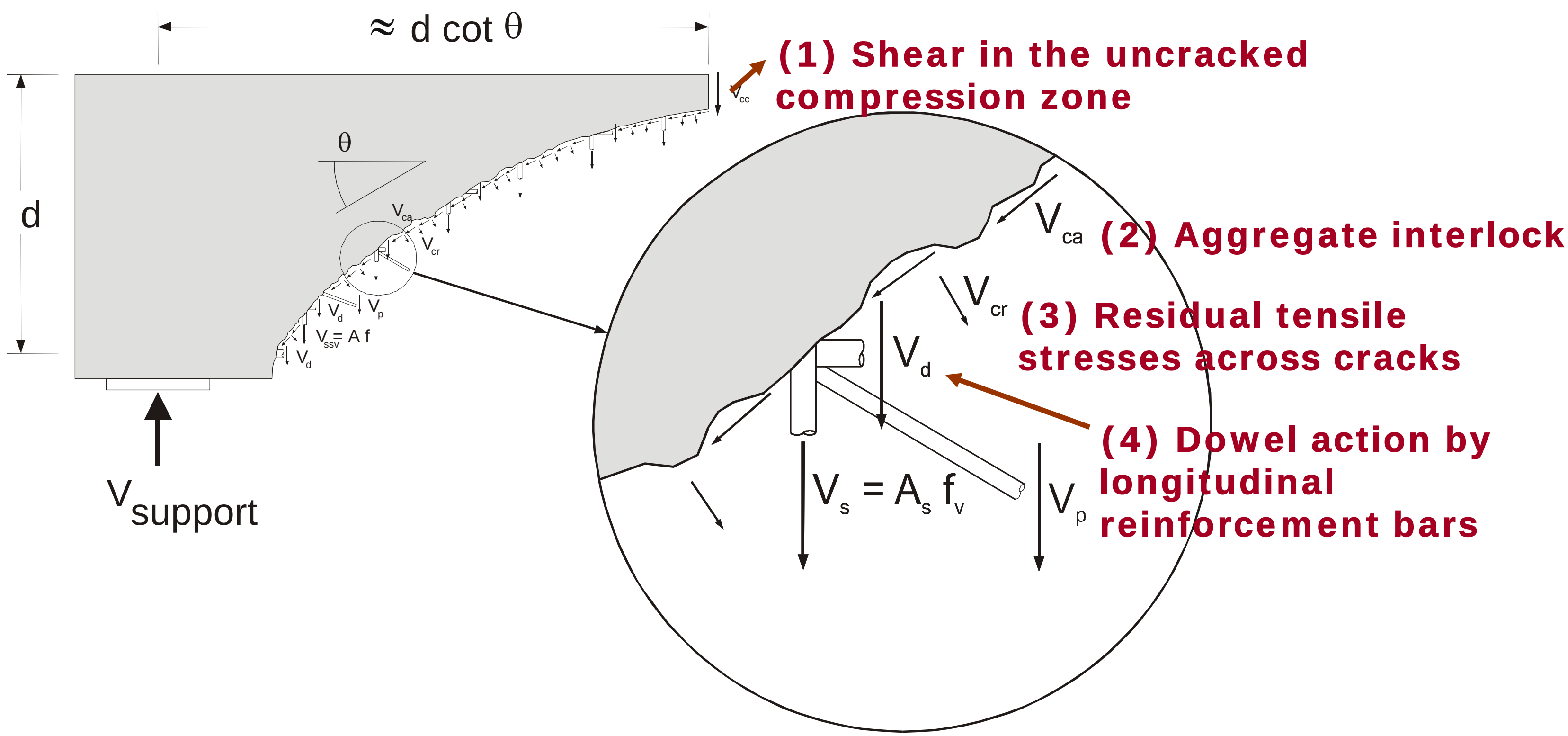
Step	1	2	3	4	5	6	7	8	9	10
f'_c	O	O	O	O	O	O	O	O	O	O
JP	O	O	O	O	O	O	O	O	O	X
BI	O	O	O	O	O	O	O	O	X	X
JI	O	O	O	O	O	O	O	X	X	X
$1 - e/b_c$	O	O	O	O	O	O	X	X	X	X
TB	O	O	O	O	O	X	X	X	X	X
$A_{sh,pro}/A_{sh,req}$	O	O	O	O	X	X	X	X	X	X
h_b/h_c	O	O	O	X	X	X	X	X	X	X
b_b/b_c	O	O	X	X	X	X	X	X	X	X
s_{pro}/s_{req}	O	X	X	X	X	X	X	X	X	X
Mean of σ	0·150	0·150	0·150	0·150	0·151	0·156	0·165	0·186	0·231	0·359

O: Included explanatory term
X: Not-included explanatory term

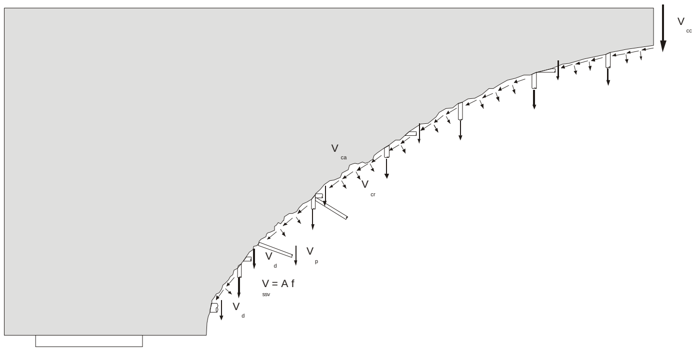
Kim, J., LaFave, J., and Song, J. (2009)
"**Joint Shear Behavior of Reinforced Concrete Beam-Column Connections**"
Magazine of Concrete Research, Vol. 61(**2**), 119-132.

Shear transfer mechanism

Joint ASCE-ACI Committee 426 (1973) & 445 (1998)



Variables affecting shear strengths



$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \ln[c(\mathbf{x})] + \sum_{i=1}^{p} \theta_i h_i(\mathbf{x}) + \sigma\varepsilon$$

$$\mathbf{x} = (f_c', d, a, \rho, \ldots)$$

(1) **Concrete compressive strength:** f_c'

~ tensile strength increases the shear strength (approximated in terms of compressive strength)

(2) **Member depth:** d

~ shear strength decreases as the member depth increases (“size effect”)

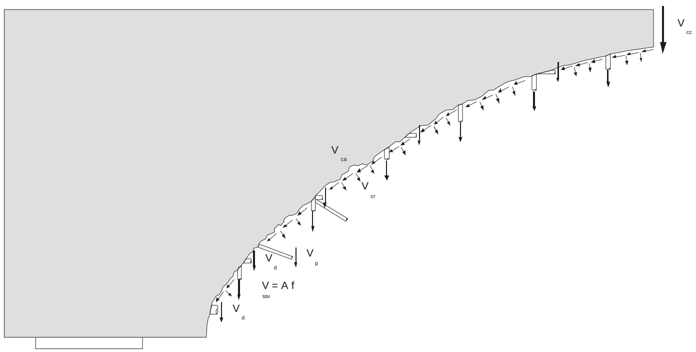
(3) **Shear span-to-depth ratio:** a/d

~ shear strength increases as the ratio decreases (“arch action” of “deep” beam)

(4) **Amount of longitudinal reinforcement:** ρ

~ shear strength increases as the reinforcement increases (“dowel action”)

Empirical shear strength models



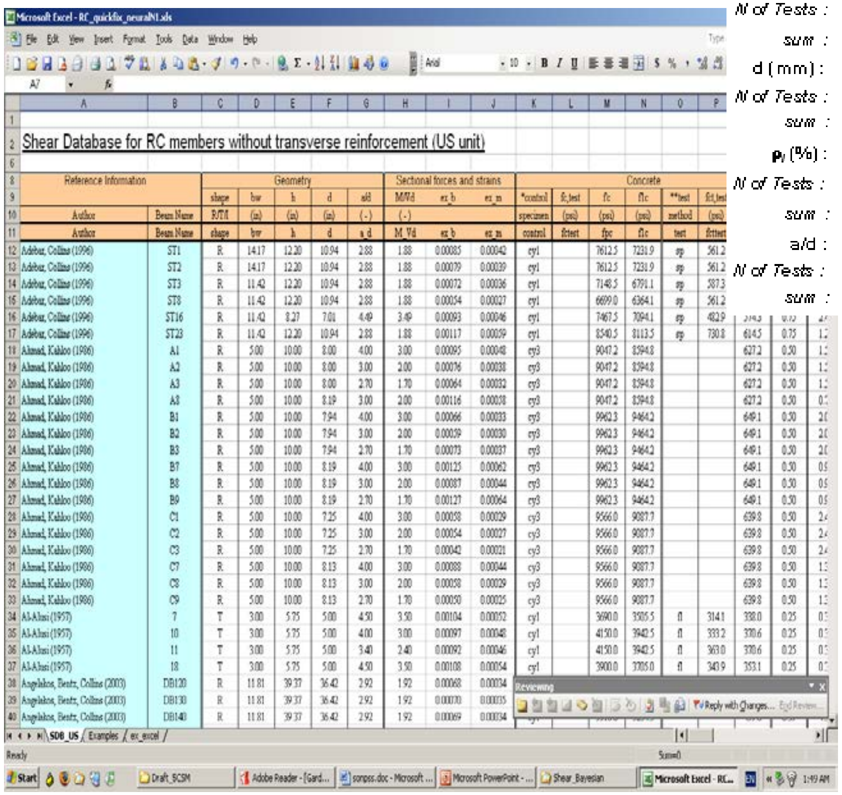
$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \ln[c(\mathbf{x})] + \sum_{i=1}^{p} \theta_i h_i(\mathbf{x}) + \sigma\varepsilon$$

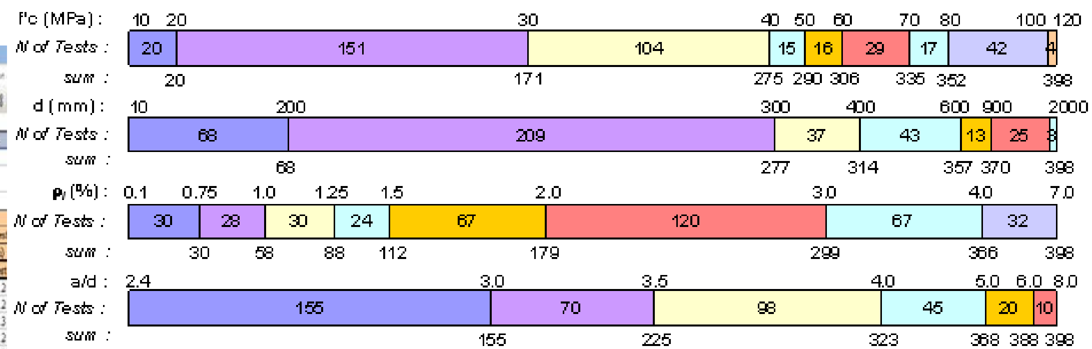
$$\mathbf{x} = (f_c', d, a, \rho, \ldots)$$

Model	Formula	characteristics
ACI 11-3	$V_c = \frac{1}{6}\sqrt{f_c'}\, b_w d$	accounts for compressive strength only
ACI 11-5	$V_c = \left(0.158\sqrt{f_c'} + 17\rho \frac{V_u d}{M_u}\right) b_w d$	compressive strength + ρ
Zsutty	$V_c = 2.2\left(f_c' \rho \frac{d}{a}\right)^{1/3} b_w d$	more accurate than ACI models
Eurocode Draft	$V_c = 0.12k(100\rho f_c')^{1/3} b_w d$	tends to underestimate (conservative)
Okamura & Higai	$V_c = 0.2\frac{(100\rho)^{1/3}}{(d/1000)^{1/4}}(f_c')^{1/3}\left(0.75 + \frac{1.40}{a/d}\right)^{1/3} b_w d$	good without severe biases
Tureyen & Frosch	$V_c = \frac{5}{12}\sqrt{f_c'}\, b_w c$	tends to overestimate for deep beams
Bazant & Yu	$V_c = 1.1044 \cdot \rho^{3/8} b_w \left(1 + \frac{d}{a}\right)\sqrt{\frac{f_c' d_0 d}{1 + d_0/d}}$	mechanics-based, semi-empirical, accurate
Russo et al.	$V_c = 0.72\,\xi\left[\rho^{0.4}(f_c')^{0.39} + 0.5\rho^{0.83} f_y^{0.89}\left(\frac{a}{d}\right)^{-1.2-0.45(a/d)}\right] b_w d$	semi-empirical, large database

Shear strength database

* Reineck, K.H., Kuchma, D.A., **Kim, K.S.**, and Marx, S. (2003)
"**Shear database for reinforced concrete members without shear reinforcement**"
ACI Structural Journal, Vol. 100(2)



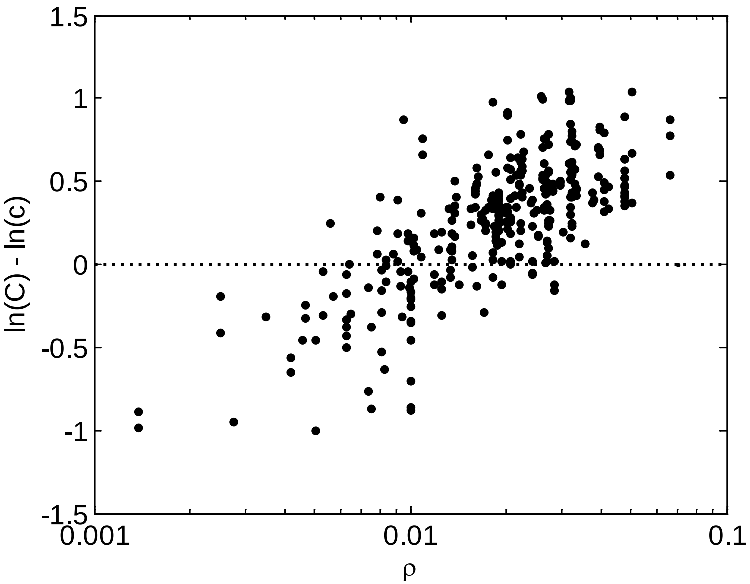


- Checked by various selection criteria discussed by ACI-ASCE Committee 445
- **398** shear strength test data
- Used **341** test data for this study (**57** data: missing aggregate sizes)

Overall errors of the existing models

$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \ln[c(\mathbf{x})] + \theta + \sigma\varepsilon$$

- μ_θ : overall bias of the existing model
- μ_σ : overall scatter of the existing model



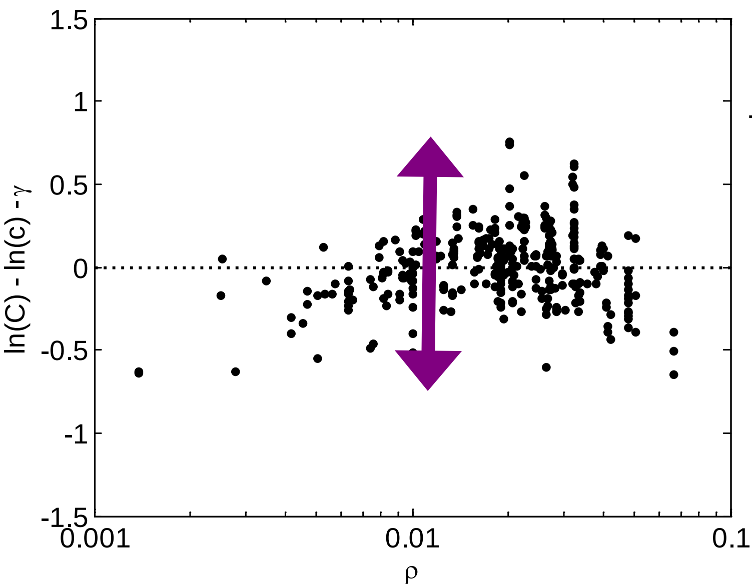
ACI 11-3

Model	Posterior means	
	θ (bias)	σ (scatter)
ACI 11-3	0.257	0.382
ACI 11-5	0.165	0.335
Eurocode Draft	0.456	0.223
Tureyen & Frosch	0.287	0.245
Zsutty	0.0261	0.244
Okamura & Higai	0.116	0.176
Bazant & Yu	0.0142	0.166
Russo et al.	0.00120	0.156

Bayesian updating with bias-correction (H1)

$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \ln[c(\mathbf{x})] + \sum_{i=1}^{p} \theta_i h_i(\mathbf{x}) + \sigma\varepsilon$$

- μ_σ : approximately represents the **uncertainties after the bias correction** (scatter)
- $h_i(\mathbf{x}): 2, \rho, \frac{a}{d}, \frac{E_c}{E_s}, \frac{d_a}{d}, \frac{d}{h}, \frac{b_w}{h}$ dimensionless explanatory terms



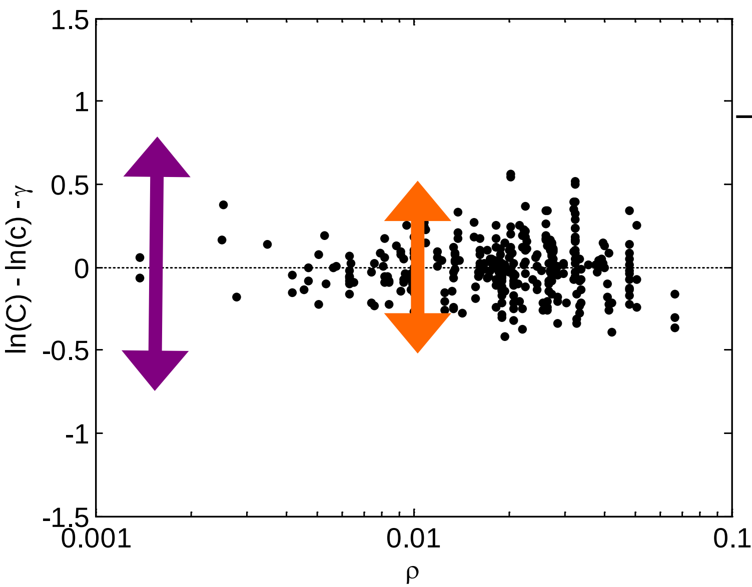
ACI 11-3

Model	Posterior means of σ		
	Constant bias	H_1	H_2
ACI 11-3	0.382	0.222	0.165
ACI 11-5	0.335	0.218	0.177
Eurocode Draft	0.223	0.172	0.165
Tureyen & Frosch	0.245	0.178	0.167
Zsutty	0.244	0.185	0.168
Okamura & Higai	0.176	0.159	0.157
Bazant and Yu	0.166	0.156	0.154
Russo et al.	0.156	0.146	0.146

Bayesian updating with bias-correction (H2)

$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \ln[c(\mathbf{x})] + \sum_{i=1}^{p} \theta_i \ln[h_i(\mathbf{x})] + \sigma\varepsilon$$

- Logarithms are applied to the explanatory functions.
- Consistent with the product forms of the deterministic formulas $C(\mathbf{x},\boldsymbol{\Theta}) = c(\mathbf{x}) h_1(\mathbf{x})^{\theta_1} \cdots h_p(\mathbf{x})^{\theta_p} \exp(\sigma\varepsilon)$



ACI 11-3

Model	Posterior means of σ		
	Constant bias	H_1	H_2
ACI 11-3	0.382	0.222	0.165
ACI 11-5	0.335	0.218	0.177
Eurocode Draft	0.223	0.172	0.165
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Okamura & Higai	0.176	0.159	0.157
Bazant and Yu	0.166	0.156	0.154
Russo et al.	0.156	0.146	0.146

Calibration of existing models

$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \cancel{\ln[c(\mathbf{x})]} + \sum_{i=1}^{p} \theta_i \ln[h_i(\mathbf{x})] + \sigma\varepsilon$$

- Use the fractions of the empirical formulas as the explanatory functions

  e.g. Zsutty's model

$$V_c = 2.2\left(f_c'\rho\frac{d}{a}\right)^{1/3} b_w d$$

$$h_i(\mathbf{x}): 2, f_c', \rho, \frac{a}{d}, b_w, d$$

- Do not drop explanatory terms with large c.o.v.'s

- Explanatory functions **do not have to be dimensionless**
  ~ may be **more effective** in representing the physics than the dimensionless terms

$\mu_\sigma = 0.166 \cong 0.168$ (posterior mean by $\ln[c(\mathbf{x})] + \sum_{i=1}^{p} \theta_i \ln[h_i(\mathbf{x})] + \sigma\varepsilon$)

Construction of new models

- Select some dimensional terms to make the same dimension as quantity and add more non-dimensional terms. Perform the Bayesian parameter estimation by models such as

$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \sum_{i=1}^{p} \theta_i \ln[h_i(\mathbf{x})] + \sigma\varepsilon$$

Product

$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \sum_{i=1}^{l} \theta_i \ln[h_i(\mathbf{x})] + \ln\left[\prod_{i=l+1}^{m} h_i^{\theta_i} + \prod_{i=m+1}^{n} h_i^{\theta_i}\right] + \sigma\varepsilon$$

Product of Sums

- Do not drop “dimensional” explanatory terms
- Useful when

(1) there exist no empirical models that can be used as a base model.
(2) the effects of explanatory terms are not well known.

- Shear strength example: tried 17 explanatory terms

→ Similar forms & parameter values with the two best formulas (with smaller μ_σ)

Zsutty's $\quad V_c = 2.2\left(f_c'\rho\frac{d}{a}\right)^{1/3} b_w d$

Okamura & Higai $\quad V_c = 0.2\frac{(100\rho)^{1/3}}{(d/1000)^{1/4}}(f_c')^{1/3}\left(0.75+\frac{1.40}{a/d}\right)^{1/3} b_w d$

“Probabilistic” Models

- General form

$$\ln[C(\mathbf{x},\boldsymbol{\Theta})] = \ln[\hat{C}(\mathbf{x},\boldsymbol{\theta})] + \sigma\varepsilon \longrightarrow C(\mathbf{x},\boldsymbol{\Theta}) = \hat{C}(\mathbf{x},\boldsymbol{\theta}) \cdot \exp(\sigma\varepsilon)$$

- Capacity ~ follows the **lognormal** distribution
- Mean and c.o.v. are derived as

$$\mu_C(\mathbf{x}) = \hat{C}(\mathbf{x},\boldsymbol{\mu}_\theta) \cdot \exp(\mu_\sigma \varepsilon) \cong \hat{C}(\mathbf{x},\boldsymbol{\theta}) \text{ for } \mu_\sigma << 1$$

$$\delta_C(\mathbf{x}) = \delta_C = [\exp(\mu_\sigma^2) - 1]^{1/2} \cong \mu_\sigma \text{ for } \mu_\sigma << 1$$

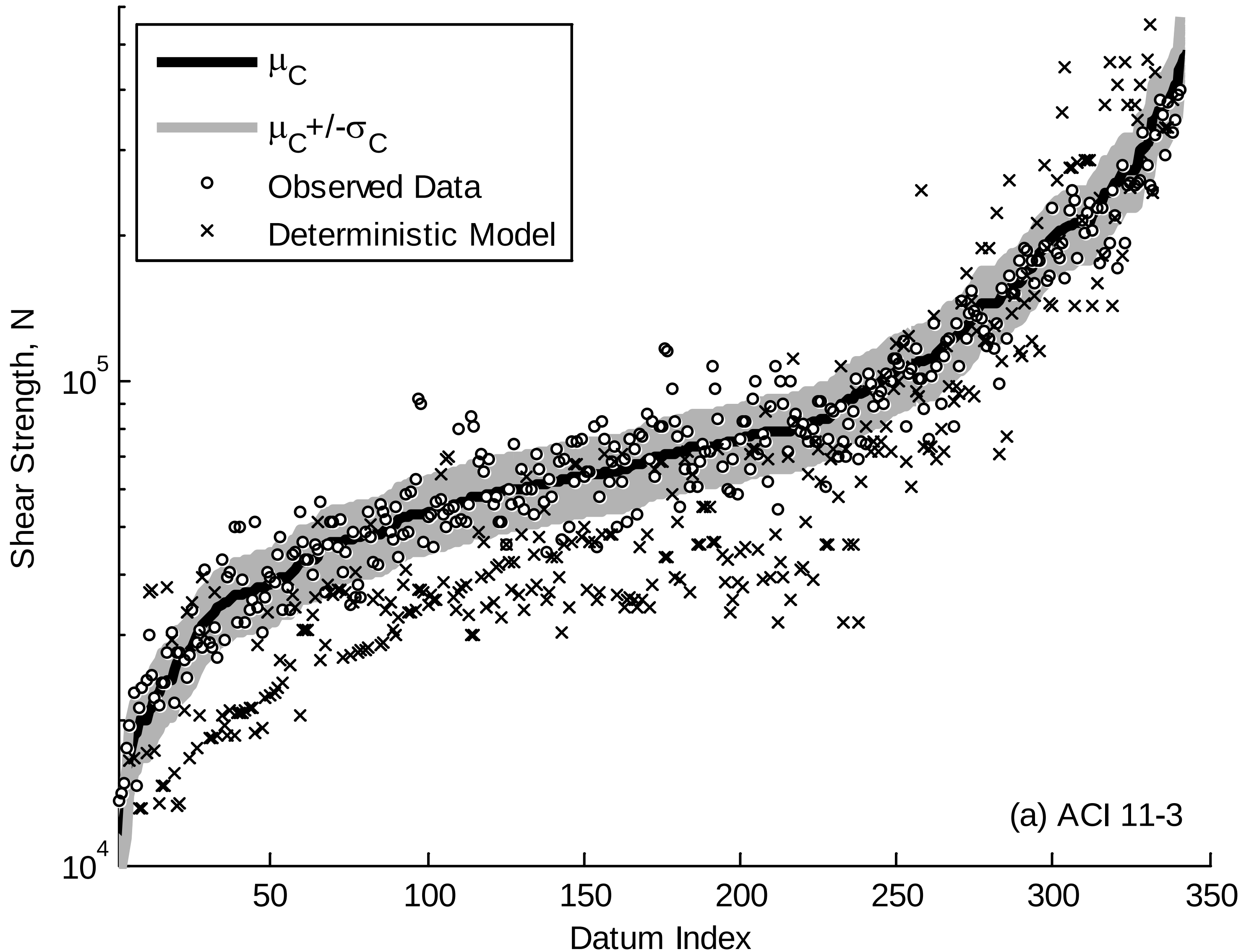
- Conditional pdf of capacity for given $\mathbf{x}$

$$f_C(c \mid \mathbf{x}) = \frac{1}{\sqrt{2\pi}\mu_\sigma c} \exp\left[-\frac{1}{2} \left(\frac{\ln c - \ln \hat{C}(\mathbf{x},\boldsymbol{\mu}_\theta)}{\mu_\sigma} \right)^2 \right]$$

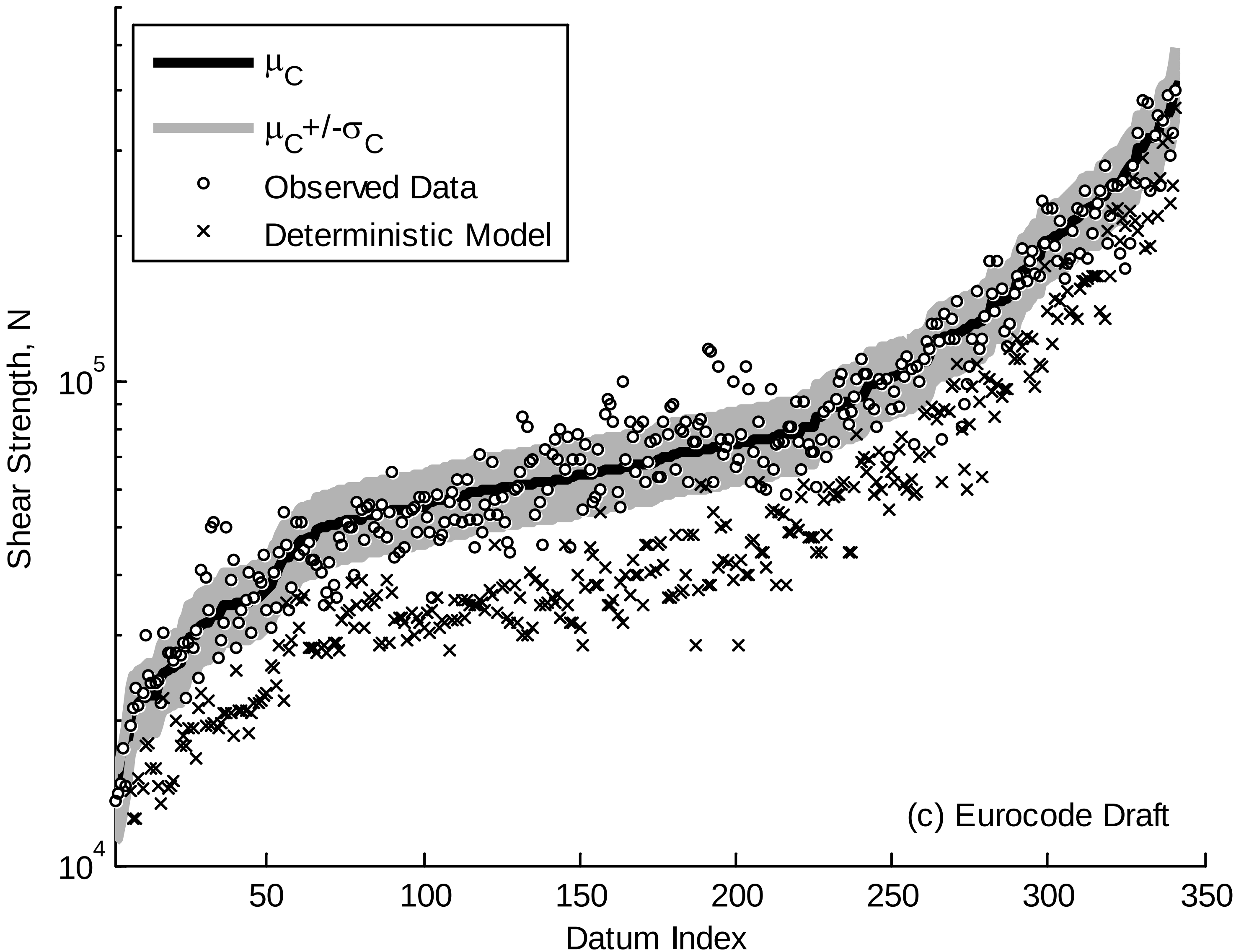
- Predictive pdf of capacity for unknown $\mathbf{x}$

$$f_C(c) = \int_{-\infty}^{\infty} f_C(c \mid \mathbf{x}) \cdot f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

Performance of probabilistic models

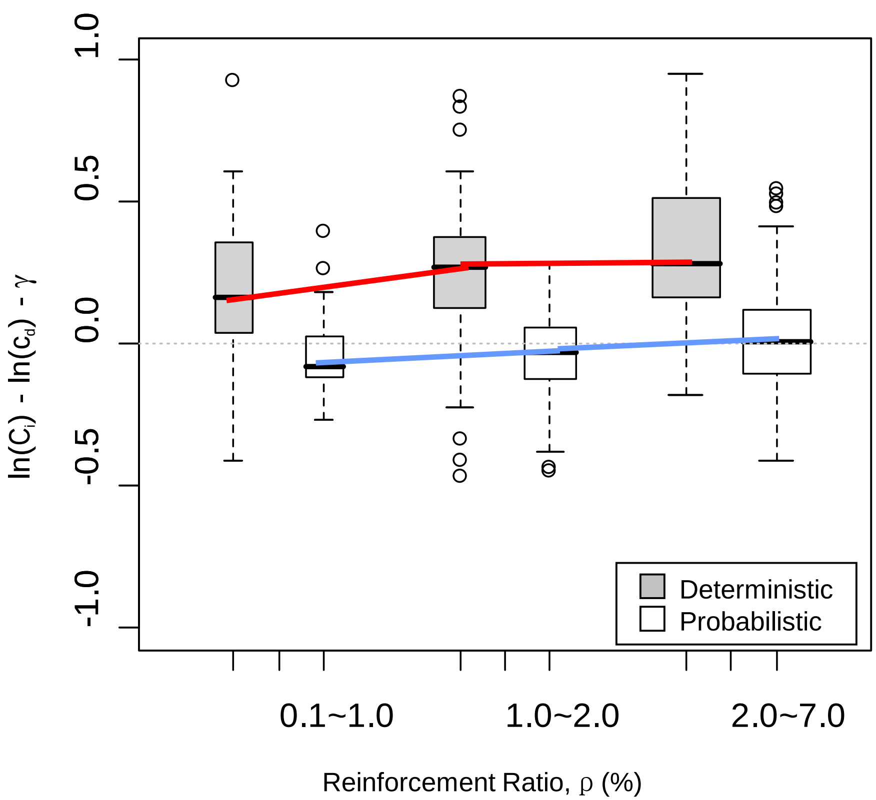


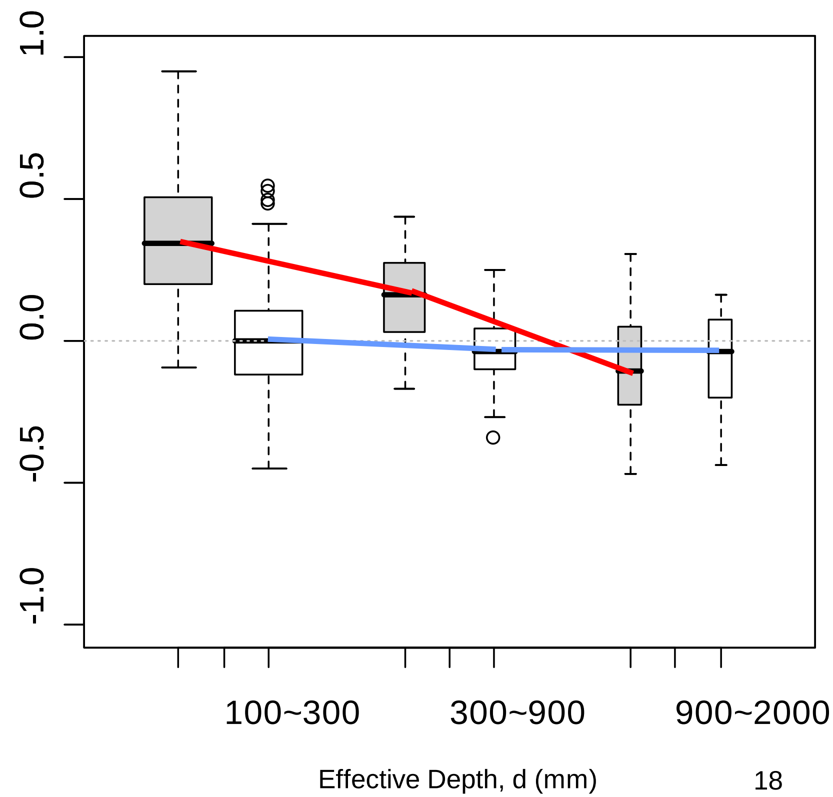
Performance of probabilistic models



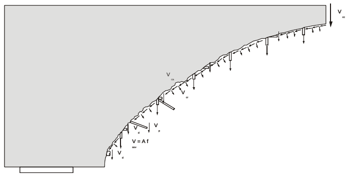
Performance of probabilistic models

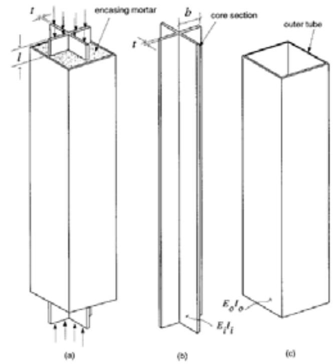
- e.g. Tureyen & Frosch (2003) and a probabilistic strength model developed by this study
- Box plots of errors ~ show that the developed models are unbiased and have consistently good performance for the whole ranges of the parameters.



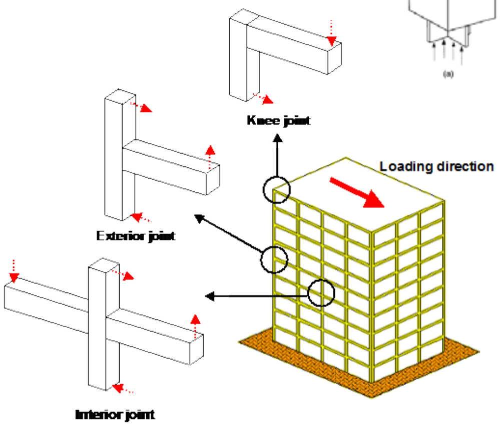


Other Applications

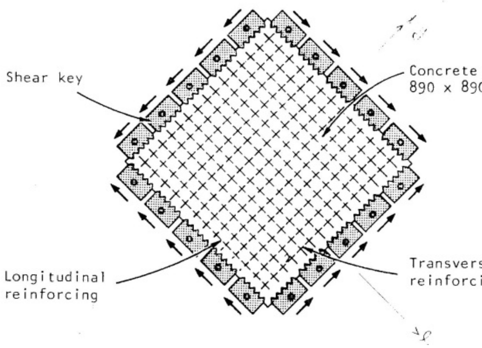




- Shear strengths of RC beams with shear reinforcements (W.-H. Kang, J. Song, and K.S. Kim)
- Seismic strengths of buckling-restrained bracings (B.M. Andrews, J. Song, and L.A. Fahnestock) (Andrews et al. 2009a, 2009b)



- Strengths/ of RC beam-column connections (J. Kim, J.M. LaFave, and J. Song)
- Statistical validation/verification of concrete FEM (H.H. Lee and D.A. Kuchma)
- Shear strengths of RC “deep” beams (strut-and-tie models) (Chetchotisak, P., J. Teerawong, S. Yindeesuk, and J. Song, 2014)



- Course term projects
  - Strengths of concrete-filled tubes (Mark Denavit)
  - Fracture toughness (Tam H. Nguyen)

References

- Gardoni, P., A. Der Kiureghian, and K.M. Mosalam (2002). “Probabilistic capacity models and fragility estimates for reinforced concrete columns based on experimental observations.” *Journal of Engineering Mechanics*, ASCE, 128(10): 1024-1038 [→ Seismic capacity of RC columns]
- Song, J., W.-H. Kang, K.S. Kim, and S. Jung. “Probabilistic shear strength models for reinforced concrete beams without shear reinforcement based on experimental observations,” under review [→ Shear strengths of RC beams without stirrups]
- Kang, W.-H., J. Song, and K.S. Kim (2007). “Probabilistic shear strength models for reinforced concrete beams with shear reinforcements by Bayeisan updating.” *Proc. 18th Engineering Mechanics Division Conference of ASCE (ASCE EMD 2007)*, June 3-6, Blacksburg, VA. [→ Shear strengths of RC beams with stirrups]
- Kim, J., J.M. LaFave, and J. Song (2007). “A new statistical approach for joint shear strength determination of RC beam-column connections subjected to lateral earthquake loading.” *Structural Engineering and Mechanics*, Vol. 27(4), 439-456 [→ Strength of RC beam-column connection ~ strength only]
- Kim, J., J.M. LaFave, and J. Song. Joint shear behavior of RC beam-column connections, under review [→ RC beam-column connections ~ strength & corresponding strain]
- Chetchotisak, P., J. Teerawong, S. Yindeesuk, and J. Song (2014). “New strut-and-tie-models for shear strength prediction and design of RC deep beams.” *Computers and Concrete*, 14(1): 19-40 [→ Shear strength of deep RC beams]

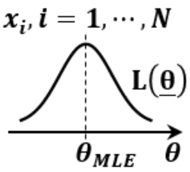
457.646 Topics in Structural Reliability
In-Class Material: Class 21

◎ Likelihood function $L(\boldsymbol{\theta})$ for distribution (statistical) parameters $\boldsymbol{\theta}_f$ (e.g. μ, σ, λ, $\xi\ldots$)

① Measured value are available, $\mathbf{x}_i,\ i=1,\cdots,N$

Assuming the observations are s.i.

$$L(\boldsymbol{\theta}_f) \propto P(\bigcap_{i=1}^{N} \mathbf{X}=\mathbf{x}_i \,|\, \boldsymbol{\Theta}_f=\boldsymbol{\theta}_f)$$
$$= \prod_{i=1}^{N} P(\mathbf{X}=\mathbf{x}_i \,|\, \boldsymbol{\Theta}_f=\boldsymbol{\theta}_f) \qquad (\because s.i.)$$
$$\propto \prod_{i=1}^{N} f_{\mathbf{x}}(\mathbf{x}_i \,|\, \boldsymbol{\theta}_f)$$



e.g. $\mathbf{x}=\{x\}$ uni-variate normal $N(\mu,\sigma^2)$

Two samples observed: 12.3($\leftarrow x_1$), 13.5($\leftarrow x_2$) $f(\boldsymbol{\theta}) = cL(\boldsymbol{\theta})\cdot P(\boldsymbol{\theta})$

$$L(\boldsymbol{\theta}_f) \propto \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{12.3-\mu}{\sigma}\right)^2\right)\times\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{13.5-\mu}{\sigma}\right)^2\right)$$

$$※\ L(\boldsymbol{\theta})\begin{cases}\text{MLE} \qquad \boldsymbol{\theta}_{\text{MLE}} = \arg\max L(\boldsymbol{\theta}) & \frac{\partial L}{\partial \theta}=0, \quad \text{prefer } \frac{\partial \ln L}{\partial \theta}=0 \\ \text{Bayesian Parameter Extimation} \end{cases}$$

$$f(\boldsymbol{\theta}) = c\cdot L(\boldsymbol{\theta})\cdot p(\boldsymbol{\theta})$$

② No direct measurement $\mathbf{x}$ of available, but a set of events that involve $\mathbf{x}$ are available

e.g. no measurement for compressive strength of concrete $f_c^{'}$ ($\leftarrow \mu,\ \sigma,\ \lambda\ldots$) available but spalling observed under a certain condition

Inequality events : $h_i(\mathbf{x}) \le 0,\ i=1,\cdots,N$

Equality events : $h_i(\mathbf{x}) = 0$

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a) Inequality

e.g. $h_i(\mathbf{x}) = -C(\mathbf{x}) + D(\mathbf{x}) \le 0$ no failure observed

$h_i(\mathbf{x}) = C(\mathbf{x}) - D(\mathbf{x}) \le 0$ failure observed

$$L(\boldsymbol{\theta}_f) \propto \prod_{i=1}^{N} P(h_i(\mathbf{x}) \le 0 \,|\, \boldsymbol{\theta}_f)$$

$$= \prod_{i=1}^{N} \int_{h_i(\mathbf{x}) \le 0} f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta}_f) d\mathbf{x} \Rightarrow \text{structural reliability analysis}$$

b) Equality

e.g. $h_i(\mathbf{x}) = a(\mathbf{x}) - a_o = 0$

$a(\mathbf{x})$: fatigue crack growth model, e.g. Paris law

a_o : measured crack size

$$L(\boldsymbol{\theta}_f) \propto \prod_{i=1}^{N} \lim_{\delta \to 0} P[0 < h_i(\mathbf{x}) \le \delta]$$

$$= \prod_{i=1}^{N} \frac{\partial}{\partial \delta} P[h_i(\mathbf{x}) - \delta \le 0]\Big|_{\delta=0}$$

Proof

$$\lim_{\Delta\delta \to 0} \frac{P[h_i(\underline{\mathbf{x}}) - \delta - \Delta\delta \le 0] - P[h_i(\underline{\mathbf{x}}) - \delta \le 0]}{\Delta\delta}\Bigg|_{\delta=0}$$

$$= \lim_{\Delta\delta \to 0} \frac{P[h_i(\underline{\mathbf{x}}) - \Delta\delta \le 0] - P[h_i(\underline{\mathbf{x}}) \le 0]}{\Delta\delta}$$

$$\propto \lim_{\Delta\delta \to 0} P[0 \le h_i(\underline{\mathbf{x}}) \le \Delta\delta]$$

$\nabla_{\boldsymbol{\delta}} P_f \big|_{\delta=0}$: can be considered as parameter sensitivity of P_f w.r.t δ (model parameter)

$\left\{ \begin{array}{l} \text{FORM-based (Madsen, 1987)} \\ \text{Good review \& new development (Straub, 2011)} \end{array} \right.$

↘ a trick to transform equality constraint to ________ constraint

◎ Likelihood function for limit-state model parameters, $L(\boldsymbol{\theta}_g)$

e.g. $g(\mathbf{x};\boldsymbol{\theta}_g) = \underline{V_c(\mathbf{x};\boldsymbol{\theta}_g)} - V_d(\mathbf{x};\boldsymbol{\theta}_g) \le 0$

$\frac{1}{6}\sqrt{f_c^{'}} b_w d$ (ACI 11-3)

① Statistical model (using original deterministic model)

$y = \underline{\hat{g}(\mathbf{x};\boldsymbol{\theta}_g)} + \sigma\varepsilon$ ~ submodel or limit state function

e.g. $\theta_1 f_c^{'\theta_2} b_w d$ (ACI 11-3) $\boldsymbol{\theta}_g = \{\theta_1, \cdots, \theta_n, \sigma\}$

$\mathbf{x}$: observable input parameters ($f_c^{'}$, b_w, $d, \cdots$)

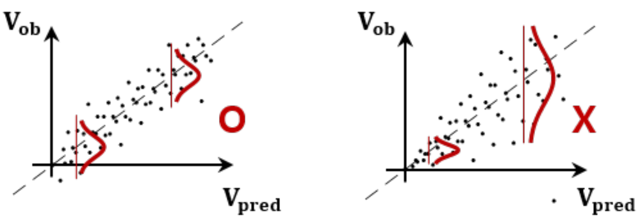
$\mathbf{y}$: observable output parameters (V_c)

$\boldsymbol{\theta}_g$: uncertain model parameters (θ_1, $\theta_2 \cdots$)

$\sigma\varepsilon$: uncertainty due to missing variables and/or inexact mathematical form

- ε : std. normal r.v " " assumption
- σ : magnitude of model error (uncertain parameter)

  → constant over $\mathbf{x}$ " " assumption
- $\mu_\varepsilon = 0$: unbiased model



May achieve H_______________ by a proper nonlinear transformation

e.g. $\ln y = \ln \hat{g}(\mathbf{x}, \boldsymbol{\theta}_g) + \sigma\varepsilon$

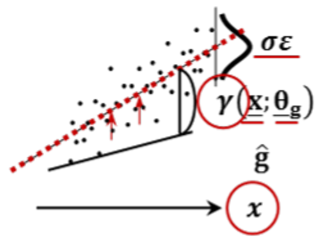
①' Statistical model (based on deterministic model, Gardoni et al. 2002)

$$y = \hat{g}(\mathbf{x}) + \gamma(\mathbf{x}; \boldsymbol{\theta}_g) + \sigma\varepsilon$$

$\hat{g}(\mathbf{x})$: original deterministic model (e.g. $\frac{1}{6}\sqrt{f_c^{'}} b_w d$)

$\gamma(\mathbf{x}; \boldsymbol{\theta}_g)$: corrects the bias

$\sigma\varepsilon$: remaining scatter



e.g. RC beam w/o stirrups shear capacity
(Song et al. 2010, Structural Eng & Mechanics)

$$\ln V = \ln \hat{v}(\mathbf{x}) + \Sigma\theta_g \ln h_i(\mathbf{x}) + \sigma\varepsilon$$

$\hat{v}(\mathbf{x})$: 8 models from codes & papers

$h_i(\mathbf{x})$: explanatory terms from the shear transfer mechanism

Find $\mathbf{M}_{\boldsymbol{\theta}}$, $\boldsymbol{\Sigma}_{\boldsymbol{\theta\theta}}$ $\quad \frac{\sigma_\theta}{\mu_\theta} = \delta_\theta$, $\rho_{\theta_i \theta_j}$

using (Bayesian Parameter Estimation

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② Likelihood function $L(\boldsymbol{\theta}_g)$?

Observed event Equality: $y = y_i,\ i = 1, \cdots, m$ know v_c when failed

Inequality: $\begin{cases} y > a_i & i = m+1, \cdots, m+n \\ y > b_i & i = m+1, \cdots, m+n+N \end{cases}$ No failure up to Vc

Failed but do not know when

Model $Y = \hat{g} + \gamma + \sigma\varepsilon$

a) $P(Y = y_i) = P(\sigma\varepsilon = y_i - \hat{g}(\mathbf{x}) - \gamma(\mathbf{x}, \theta_g))$

$$P(Y = y_i) \propto f_Y(y_i)$$

$$f_Y(y_i) = f_Q(q) \cdot \frac{dq}{dy_i}$$

$$f_Q(q) = f_\varepsilon(\varepsilon) \cdot \frac{d\varepsilon}{dq}$$

$$q = \sigma \cdot \varepsilon$$

$$= f_Q(q_i) \cdot \frac{dq}{dy}$$

$$= f_\varepsilon(\varepsilon_i) \cdot \frac{d\varepsilon}{dq}$$

$$= \frac{1}{\sigma}\varphi\left(\frac{y_i - \hat{g} - \gamma}{\sigma}\right)$$

b) $P(Y > a_i) = P(\hat{g} + \gamma + \sigma\varepsilon > a_i)$

$$= P(\sigma\varepsilon > a_i - \hat{g} - \gamma)$$

$$= \Phi\left(-\frac{a_i - \hat{g} - \gamma}{\sigma}\right)$$

c) $P(Y < b_i) = P(\hat{g} + \gamma + \sigma\varepsilon < b_i)$

$$= P(\sigma\varepsilon < b_i - g - \gamma)$$

$$= \Phi\left(\frac{b_i - g - \gamma}{\sigma}\right)$$

$$\therefore L(\theta_g) = \prod_{i=1}^{m} \frac{1}{\sigma}\varphi\left(\frac{y_i - \hat{g} - \gamma}{\sigma}\right) \times \prod_{i=m+1}^{m+n} \Phi\left(-\frac{a_i - \hat{g} - \gamma}{\sigma}\right) \times \prod_{i=m+n+1}^{m+n+N} \Phi\left(\frac{b_i - \hat{g} - \gamma}{\sigma}\right)$$

※ Matlab codes for "Model Development by Bayesian method"

→ MDB (by Prof. S.Y. Ok at Hankyoung univ. for educational purpose)

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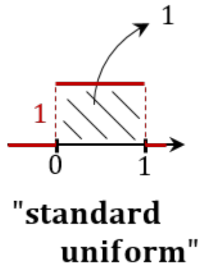
457.646 Topics in Structural Reliability

In-Class Material: Class 22

VI. Simulation Methods

(Ref. CRC Chapter 20 Stochastic Simulation Methods for Engineering Predictions)

◎ Simulating uniform random variable $U(0,1)$



→ Basic in generation of random numbers

→ (   ) sequence from a seed number

→ Desirable to have a (   ) period and (   ) sampling

※ Matlab : rand(   )

→ could choose a random number generation algorithm

→ default: Mersenne Twister (Matsumoto & Nishimura 1997)

→ Period: $2^{19936}-1$

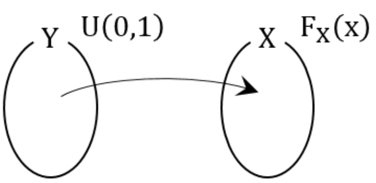
→ "Very fast"

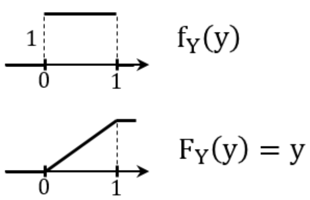
Demo

```
Xv=[100 1000 10000]
for i=1:3
X=rand(Xv(i),1);
subplot(3,1,i)
hist(X,sqrt(Xv(i)));
end
```

◎ Generate random numbers according to CDF

Consider $Y \sim U(0,1)$





$$F_Y(y) \quad = \quad F_X(x)$$
$$\quad = \quad F_X(x)$$

$\therefore\ x =$

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(1) Generate $y_i,\ i=1,\cdots,N$

per $\nwarrow$ $U(0,1)$

(2) Find corresponding x_i, $x_i =$, $i=1,\cdots,N$

◎ Generate general dependent variables

$\mathbf{X}=\{X_1,\cdots,X_n\}^T$ defined by $\begin{cases}\text{joint PDF } f_{\mathbf{X}}(\mathbf{x}) \\ \text{joint CDF } F_{\mathbf{X}}(\mathbf{x})\end{cases}$

cf. Rosenblatt

$\underline{x}_1 \quad \underline{x}_2 \quad \cdots \quad \underline{x}_{N-1} \quad \underline{x}_N$

$$\begin{cases} y_1 = F_{X_1}(x_1) \\ y_2 = F_{X_2|X_1}(x_2|x_1) \\ \quad\vdots \\ y_n = F_{X_n|X_1\cdots X_{n-1}}(x_n|x_1\cdots x_{n-1}) \end{cases}$$

$$\begin{cases} x_1 = F_{X_1}^{-1}(y_1) \\ x_2 = F_{X_2|X_1}^{-1}(y_2|x_1) \\ \quad\vdots \\ x_n = F_{X_n|X_1\cdots X_{n-1}}^{-1}(y_n|x_1\cdots x_{n-1}) \end{cases}$$

(1) Simulate $\{y_1,\cdots,y_n\}^T$

(2) Find $\{x_1,\cdots,x_n\}^T$

Using (←)

◎ Simulation of normally distributed RV's *(Box & Muller 1958)

→ homework

※ **Matlab mvrnd($\mathbf{M},\mathbf{\Sigma},N$)**

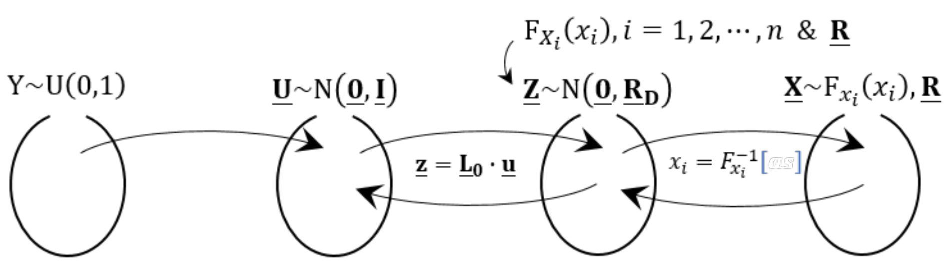
Generate N samples of $\mathbf{X}\sim N(\mathbf{M},\mathbf{\Sigma})$

cf. normrnd

$\mathbf{u}\sim N(\mathbf{0},\mathbf{I})$

$\mathbf{X}=\mathbf{DLu}+\mathbf{M}$

◎ Generate random numbers from Nataf distribution



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i. Find $\mathbf{R}_0$ (Liu & ADK, 1986)

ii. Generate $\mathbf{u}$ from $N(\mathbf{0},\mathbf{I})$ (or $\mathbf{y}$ from $U(0,1)$ & transform)

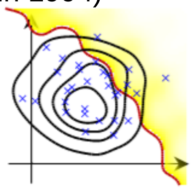
iii. Compute $\mathbf{Z} \sim \mathbf{L}_0\mathbf{u}$ (or $\mathbf{Z} \sim N(\mathbf{0},\mathbf{R}_0)$)

iv. Compute $x_i = F_{X_i}^{-1}(\quad)$, i=1,⋯,n

◎ <u>Monte Carlo</u> Simulation

↖ City in Monaco ("MC project" in 1994)

$$P_f = \int_{(\cup\cap)g(\mathbf{x})\le 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$$



$$= \int \qquad f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$$

$$I(\underline{\mathbf{x}}) = \begin{cases} & g(\underline{\mathbf{x}}) \le 0 \\ & \text{otherwise} \end{cases}$$

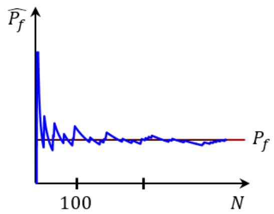
Index Function

$=$ average of index function value (w.r.t $\mathbf{X} \sim F_{\mathbf{X}}(\mathbf{x})$)

Simulate $\mathbf{x}_i$, i=1,⋯,N according to $f_{\mathbf{X}}(\mathbf{x})$

Let $q_i = I(\mathbf{x}_i)$, i=1,⋯,N

$$P_f = \lim_{N\to\infty}$$



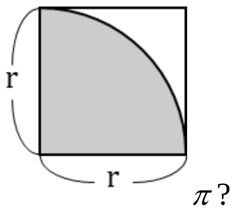
$\hat{P}_f =$ Estimation of P_f using N sample

Compare

mean (rand(3,1))

mean (rand(100000,1))

"MCS is an extremely bad method. It should be used only when all alternative methods are worse" –Alan Sokal (1996)



π?

$$\frac{\frac{1}{4}\pi r^2}{r^2} = \frac{1}{4}\pi$$

$$\therefore \pi = 4 \times \frac{\#}{\# \square}$$

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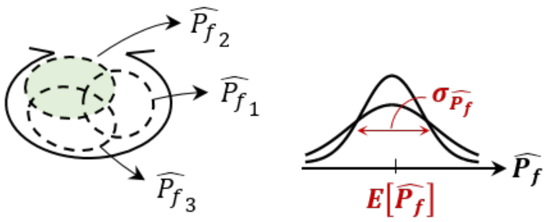
Note: $\hat{P}_f$ is random



↓

How much variability? $\delta_{\hat{P}_f}$

q_i : Bernoulli random variable

$$\begin{cases} 1 & \text{with} \quad p= \\ 0 & \quad\quad\quad 1-p= \end{cases}$$

$E[q_i] =$

$Var[q_i] =$

$=$

$=$

- $E[\hat{P}_f] =$

  "unbiased" estimator of true P_f

- $Var[\hat{P}_f] =$

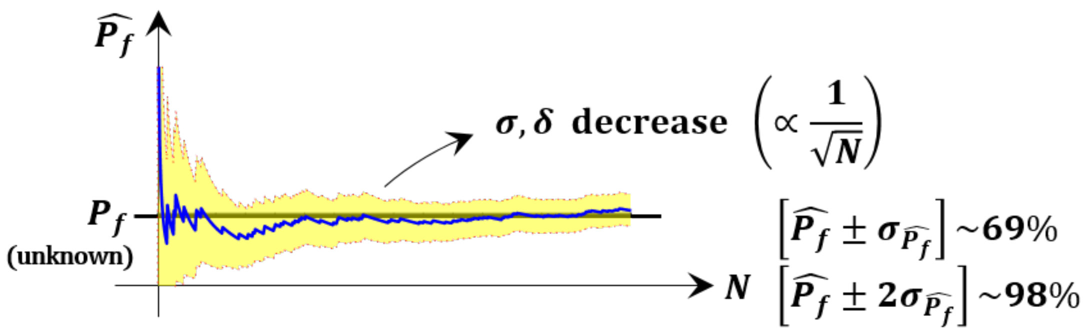
  $=$

  $=$

$\Rightarrow \quad \delta_{\hat{P}_f} = \qquad\qquad\qquad = \dfrac{1}{\sqrt{N}}\sqrt{\dfrac{1-P_f}{P_f}}$

Quantifies variation of $\hat{P}_f$

Used as a measure of convergence



See MCS.m

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※ Minimum No. of Simulation to achieve $\bar{\delta}$

Target c.o.v $\bar{\delta} = \dfrac{1}{\sqrt{N_{\bar{\delta}}}}\sqrt{\dfrac{1-P_f}{P_f}}$

$$\therefore \quad N_{\bar{\delta}} = \frac{1-P_f}{\bar{\delta}^2 \cdot P_f}$$

e.g $P_f = 0.01$

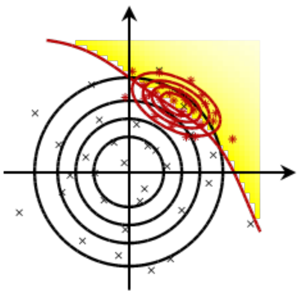
$\bar{\delta}$	$N_{\bar{\delta}}$
0.01	$\simeq 10^6$
0.05	$\simeq 4.0 \times 10^4$
0.10	$\simeq 1.0 \times 10^4$

※ How to improve accuracy of simulation

$$\delta_{P_f} = \frac{\sqrt{Var[\hat{P}_f]}}{E[\hat{P}_f]} = \frac{\frac{1}{\sqrt{N}}\sqrt{Var[q_i]}}{E[q_i]} = \frac{1}{\sqrt{N}} \cdot \delta_{q_i}$$

① Increase N

② Decrease δ_{q_i}



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457.646 Topics in Structural Reliability

In-Class Material: Class 23

◎ Importance sampling

Need to compute integral (in general)

$$I_t = \int g(\mathbf{x})d\mathbf{x}$$
$$= \int \left[\frac{g(\mathbf{x})}{h(\mathbf{x})}\right] h(\mathbf{x})d\mathbf{x}$$
$$=$$

$g(\mathbf{x})$: general function
$h(\mathbf{x})$: sampling PDF having non-
values where g($\mathbf{x}$) is non-

Procedure:

i. Sample $\mathbf{x}_i$, $i = 1, \cdots, N$ according to

ii. Compute q_i

iii. Estimate $\hat{I}_t = \frac{1}{N}\sum_{i=1}^{N} q_i$

To have accuracy (& efficiency), the variance in q must be small. If $g(\mathbf{x}) \geq 0$, $h(\mathbf{x})$ $g(\mathbf{x})$ is the best choice.

※ Application to reliability problem:

$$P_f = \int_x \boxed{I(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})} d\mathbf{x}$$

Sampling density

(non-zero) where $g = I \cdot f \neq 0$

$$= \int_x \left[\frac{I(\mathbf{x})}{\quad} f_{\mathbf{X}}(\mathbf{x})\right] \cdot \qquad d\mathbf{x}$$

$$= E[\qquad\qquad]$$ relative to

$$q_i =$$

Find $h(\mathbf{x})$ such that

$$Var[\frac{I_f}{h}]_h \qquad Var[I]_f$$

c.o.v of $\hat{P}_f$, $\delta_{\hat{P}_f}$ for importance sampling?

$$\hat{P}_f = \frac{1}{N}\sum_{i=1}^{N} q_i$$

X, $x_1, \cdots, x_N$

$$\bar{X} = \frac{1}{N}\{x_1 + \cdots + x_N\}$$

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$$\mu_{\hat{P}_f} = \frac{1}{N}\sum_{i=1}^{N} E[q_i] \rightarrow \bar{\hat{P}}_f = \frac{1}{N}\sum_{i=1}^{N} \bar{Q}$$

↖ Estimate on the of $\hat{P}_f$ (=sample mean of)

$$\sigma^2_{\hat{P}_f} = \frac{1}{N^2}\sum_{i=1}^{N} Var[q_i] \rightarrow S^2_{\hat{P}_f} = \frac{1}{N^2}\sum_{i=1}^{N} S_q^2 = \frac{1}{N} S_q^2$$

↖ Estimate on the variance of $\hat{P}_f = \frac{1}{N} \times$ sample variance of q_i's

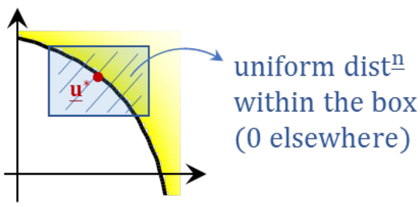
$$\delta_{\hat{P}_f} = \frac{\sqrt{S_{\hat{P}_f}}}{\bar{\hat{P}}_f} = \frac{\frac{1}{\sqrt{N}} S_q}{\bar{Q}} \qquad \left(\frac{If}{h}, \frac{If}{h}, \cdots, \frac{If}{h}\right) \qquad x_i \leftarrow h(\mathbf{x})$$

Importance sampling $P_f = \int_x \left[\frac{If}{h}\right] \cdot \boxed{h(\mathbf{x})} d\mathbf{x} = E\left[\frac{If}{h}\right]$

$$Var\left[\frac{If}{h}\right]_h << Var[I]_f$$

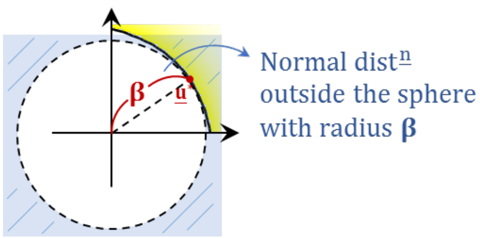
◎ Selection of sampling density

① Shinozuka (1983)



→ not good because zero density assigned to failure cases $q_i = \frac{I \cdot f}{h}$

② Harbitz (1986)



$$h(\mathbf{u}) \begin{cases} c \cdot \varphi_n(\mathbf{u}) & \|\mathbf{u}\| \quad \beta \\ 0 & \text{otherwise} \end{cases}$$

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$$c\int_{\|\mathbf{u}\|\geq\beta}\varphi_n(\mathbf{u})d\mathbf{u} = c\cdot P(\|\mathbf{u}\|\geq\beta) =$$

$$P(\|\mathbf{u}\|\geq\beta) = 1-P(\|\mathbf{u}\|\leq\beta)$$

$$= 1-P(\|\mathbf{u}\|^2\leq\beta^2) = 1-P(u_1^2+\cdots+u_n^2\leq\beta^2)$$
$$= 1-X_n^2(\beta^2)$$

Chi-square distribution n degree of freedom

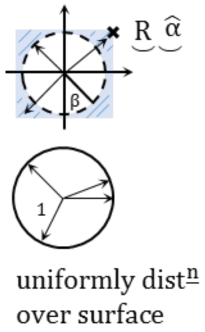
$\therefore\ c=$

$$q_i = \frac{I\cdot f}{h} = \qquad\qquad =$$

How to simulate according to $h(\mathbf{u})$?

i. Simulate $\mathbf{u}_i \sim N(\mathbf{0},\mathbf{I})$

ii. Compute $\hat{\alpha}_i = \dfrac{\mathbf{u}_i}{\|\mathbf{u}_i\|}$

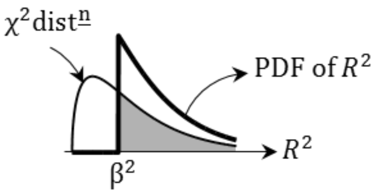


iii. Simulate R^2

$$R^2 = u_1^2+\cdots+u_n^2 \quad (\sim X_n^2(\ ))$$

But truncate $R^2<\beta^2$

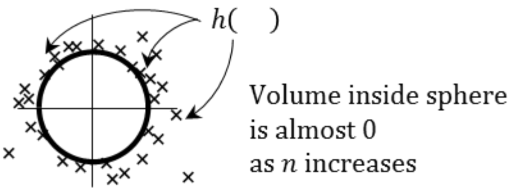
$$F_{R^2}(r^2) = \frac{X_n^2(r^2)}{1-X_n^2(\beta^2)}$$



iv. Compute $R\cdot\hat{\alpha}$

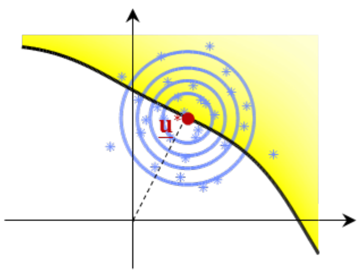
Note: Not effective as n↑

$$1-X_n^2(\beta^2)\cong 1$$

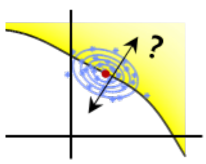


③ Melchers (1989)

$$h(\mathbf{u}) = N(\qquad)$$



e.g. $\mathbf{\Sigma} = \sigma^2\mathrm{I}$ (FERUM's Importance Sampling Option)



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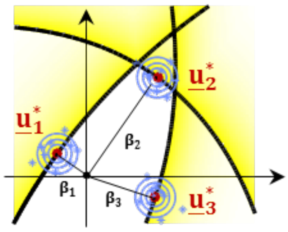
④ Series system

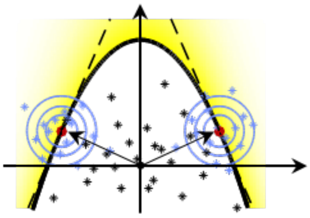


$h(\mathbf{u}) = \sum_i w_i h_i(\mathbf{u})$ where $h_i(\mathbf{u}) \leftarrow N(\mathbf{u}_i^*, \mathbf{\Sigma})$

w_i : weight ($\propto \beta_i^{-m}$, $m > 0$)

◎ **Challenges**

① $h(\mathbf{x}) = 0$ where $\mathrm{I}(\mathbf{x}) \neq 0 \Rightarrow$ does not converge

② 

Multiple design points?

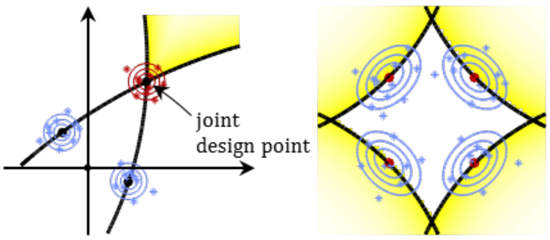
$h(\mathbf{u}) \Rightarrow wh(\mathbf{u}) + w_0\varphi(\mathbf{0}, \mathbf{I})$

ADK & Dakessaian (1998)

③ System problems

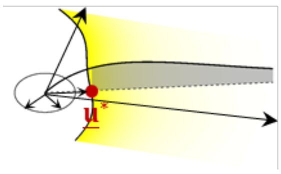
i. Where?

ii. Cost of finding the importance points



※ Adaptive Importance Sampling

(1) Directional Simulation



(2) Sequentially Conditioned Importance Sampling (SCIS) (→multinomal prob. calc.)

(3) Adaptive Importance Sampling based on Cross-Entropy using Gaussian Mixture (Kurtz and Song, 2013); using von Mises Fisher mixture (Wang and Song, 2016)

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457.646 Topics in Structural Reliability

In-Class Material: Class 24

VIII-1. Probability-Based Structural Design Code

→ Cornell. C.A (1969) A probability-based structural code (J. ACI)

→ Ravindara & Galambos (1978) Load & resistance factor design for steel structures

(J. Str. Eng, Div. ASCE)

◎ Load & Resistance Factor Design (LRFD)

Replaced allowable stress design (ASD) (→safety factor)

⇒ Probability-based code

$$\phi R_n \geq \sum \gamma_k Q_{km} = \gamma_D Q_{Dm} + \gamma_L Q_{Lm} \quad \cdots\cdots\cdots\cdots\cdots (1)$$

Dead load   Live load

i. R_n : "   " resistance

→ code formula (e.g. $V_c = \frac{1}{6}\sqrt{f_c^{'}} b_w d$)

→ nominal values used (material & dimension)

: given in "   " force, e.g. bending moment, axial force, shear force

ii. ϕ : "   " Factor ~ ϕ 1

(Dimensionless) conservatism due to the uncertainties in R

iii. Q_m : mean load effect

→ in generalized force (structural analysis)

iv. γ : "Load" factor~ γ 1

Conservatism due to

① Potential overload

② Uncertainty in load effect calculation

v. Limit-State

"U   " limit-states

e.g. frame instability, plastic mechanism formed incremental collapse

"S   " limit-states

e.g. excessive deflection, excessive vibration, premature yielding or slip

LRFD codes suggest formulas for (   ), methods to compute (   ) from loads

provide (   ) & (   )

for each structural element (Q_m) from loads

to satisfy the (   ) reliability level

◎ Measure of (target) reliability

(or conservatism)

⇒ use

$$\beta = \frac{E\left[\ln \frac{R}{Q}\right]}{\sigma_{\ln \frac{R}{Q}}} \overset{FO}{\approx} \qquad \cdots\cdots\cdots\cdots\cdots \text{(2)}$$

$$\geq$$

$$\mu_R \geq$$

Want to split so that factors for R & Q can be determined independently

※ Lind (1971) $\sqrt{\delta_R^2 + \delta_Q^2} \approx \bar{\alpha}(\delta_R + \delta_Q)$ where $\bar{\alpha} = 0.75$

$$\therefore \qquad \geq \qquad \cdots\cdots\cdots\cdots\cdots \text{(3)}$$

$(\mu_R,\ \mu_Q,\ \delta_R,\ \delta_Q)$?

◎ Uncertainties in the Resistance, R

$$R = R_n \cdot M \cdot F \cdot P \qquad \cdots\cdots\cdots\cdots\cdots \text{(4)}$$

R_n : nominal resistance by codes

M : “M”aterial ~

F : “F”abrication ~

P : “P”rofessional ~

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① $\mu_R \overset{FO}{\square}$

② δ_R ? $\ln R =$

$Var[\ln R] = \xi_R^2 =$

Note $\xi_X^2 \square \delta_R^2$ when $\delta \square 1$

$\therefore \; \delta_R \cong$

◎ Uncertainties in Loads, Q

$$Q = E(C_D AD + C_L BL) \qquad \cdots\cdots\cdots\cdots\cdots (5)$$

① $\mu_Q \; \square$

②
$$\delta_Q \cong \delta_E^2 + \delta_{c_{DAD}+c_L BL}^2$$
$$= \delta_E^2 + \frac{c_D^2 \mu_A^2 \mu_D^2 (\delta_A^2 + \delta_D^2) + c_L^2 \mu_B^2 \mu_L^2 (\delta_B^2 + \delta_L^2)}{(c_D \mu_A \mu_D + c_L \mu_B \mu_L)^2}$$

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◎ Finding target reliability index β

Initially, Eq. (3) & $\mu_R, \mu_Q, \delta_R, \delta_Q$ → existing, e.g. allowable stress code

→ can back-calculate target reliability index β embedded in the existing code

For example, 1969 AISC simply supported beams:

$\beta \cong 3.0$ (member), $\beta \cong 4.5$ (connections)

→ Provided starting points (and calibrated later)

◎ Load & Resistance Factors for given target β

Eq. (1) $\phi R_n \geq \sum_k \gamma_k Q_{km} = \gamma_E (\gamma_D C_D \mu_D + \gamma_L C_L \mu_L)$

Eq. (3) $\exp(-\bar{\alpha} \cdot \beta \cdot \delta_R) \cdot \mu_R \geq \exp(\bar{\alpha} \cdot \beta \cdot \delta_Q) \cdot \mu_Q$ ← expressions derived for $\mu_R, \mu_Q, \delta_R, \delta_Q$

From the LHS of Eq. (1) and Eq. (3): $\phi = \exp(-\alpha\beta\delta_R)\dfrac{\mu_R}{R_n}$ where $\alpha = 0.55$

From the RHS:

$$\begin{cases} \gamma_E = \exp(\alpha\beta\delta_E) \\ \gamma_D = 1 + \alpha\beta\sqrt{\delta_A^2 + \delta_D^2} \\ \gamma_L = 1 + \alpha\beta\sqrt{\delta_B^2 + \delta_L^2} \end{cases}$$

i) If $\beta \uparrow$ $\begin{cases} \phi \\ \gamma \end{cases}$

ii) $\dfrac{\mu_R}{R_n} > 1$, If $\dfrac{\mu_R}{R_n} \uparrow$, ϕ

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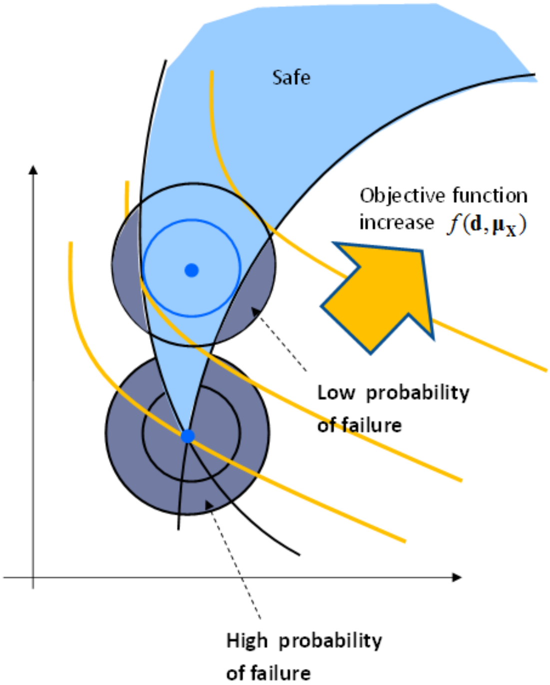
Review in Nguyen, Song & Paulino (2010)

VIII-2. Reliability-Based Design Optimization (RBDO)

◎ RBDO formulation



$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{x}} f(\mathbf{d}, \boldsymbol{\mu}_\mathbf{x})$$

s.t. $P[g(\mathbf{d}, \boldsymbol{\mu}_\mathbf{x}) \le 0] \le P_f^t$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^u$$

$$\boldsymbol{\mu}_\mathbf{x}^L \le \boldsymbol{\mu}_\mathbf{x} \le \boldsymbol{\mu}_\mathbf{x}^u$$

Where

$f(\mathbf{d}, \boldsymbol{\mu}_\mathbf{x})$

$\mathbf{d}$

$\mathbf{x}$

$\boldsymbol{\mu}_\mathbf{x}$

P_f^t

$\mathbf{d}^L, \mathbf{d}^u$

$\boldsymbol{\mu}_\mathbf{x}^L, \boldsymbol{\mu}_\mathbf{x}^u$

◎ Reliability Index Approach (RIA; Enevaldsen & Sorensen 1994)

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{x}} f(\mathbf{d}, \boldsymbol{\mu}_\mathbf{x})$$

s.t. β $\qquad\qquad$ β^t

$\beta^t \leftarrow$ target reliability index $\;-\Phi^{-1}[P_f^t]$

$\beta \leftarrow$ generalized reliability index

$\beta = -\Phi^{-1}[\qquad\qquad\qquad\qquad]$

↖ By FORM analysis (or others)

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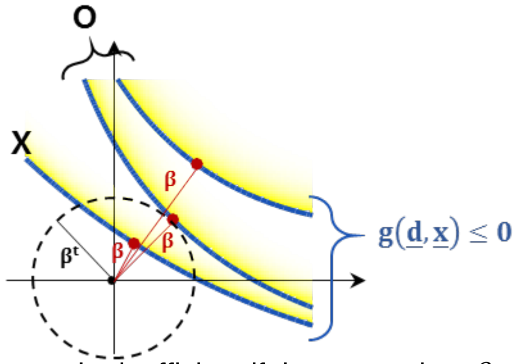
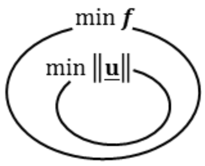
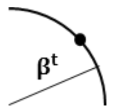


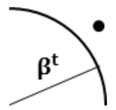
⇒ compute P_f for each interation of $\mathbf{d}$ to check if the constraint is satisfied

⇒ double loop approach



⇒ can be inefficient if the constraint $\beta \ge \beta^t$ is inactive





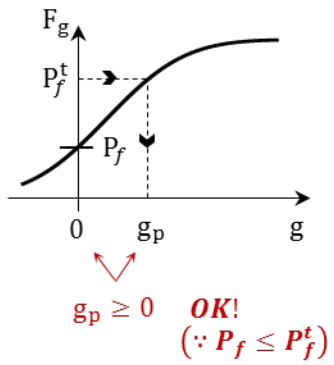
⇒ may not be able to provide an optimal solution if the failure does not occur in the feasible domain

◎ Performance Measure Approach (PMA; Tu et al., 1999) ※ double-loop

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{x}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{x})$$

s.t. $\underline{g_p} = F_g^{-1}\left[\Phi\left(-\beta^t\right)\right] \ge 0$ $(\Phi^{-1}[-\beta^t] = P^t)$

"Performance function" = quantile of g at P^t

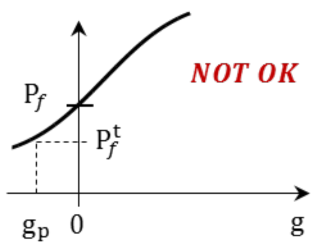


$g_p \ge 0 \Leftrightarrow P_f \le P_f^t$

$\Leftrightarrow \beta \ge \beta^t$

Equivalent RBDO

How to find g_p?



They proposed (instead of solving FORM target β)

$$g_p = \min_{\mathbf{u}} G(\mathbf{d},\mathbf{u}) \qquad \cdots\cdots\cdots\cdots\cdots (1)$$

s.t. $\|\mathbf{u}\| = \beta^t$ ⇒ Minimizes g instead of $\|\mathbf{u}\|$

~ facilitates gradient-based optimization (using $\dfrac{\partial g}{\partial \mathbf{d}}$)

⇒ Overcomes the problems in RIA

Is this $\overset{(\uparrow)}{g_p}$ really $F_g^{-1}\left[P_f^t\right]$?

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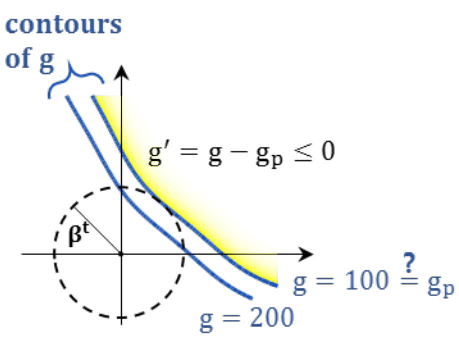


Set a new limit-state function

$$g^{'}(x) = g(\mathbf{x}) - g_p$$

$$P(g' \le 0) \cong \Phi(-\beta^t) = \boxed{P_f^t}$$

$$||$$

$$P(g \le g_p)$$

$$||$$

$$\boxed{F_g(g_p)} \qquad g_p = F_g^{-1}[P_f^t]$$

◎ Single-Loop PMA (Liang et al., 2004)

Replace the optimization in (1) with an approximation (but non-iterative)

system equation, i.e, Karush-Kuhn-Tucker (KKT)

condition

$$\nabla_{\mathbf{u}} G(\mathbf{d},\mathbf{u}) + \lambda \nabla_{\mathbf{u}} (\|\mathbf{u}\| - \beta^t) = 0$$ (λ →Lagrange Multiplier)

$$\|\mathbf{u}\| - \beta^t = 0$$

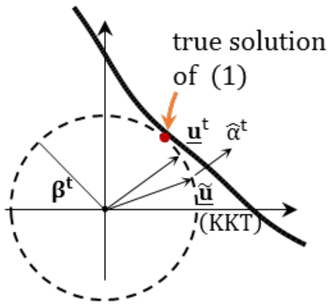
i. Solve KKT to get $\mathbf{u} = \tilde{\mathbf{u}}$

ii. Evaluate $\hat{\alpha}$ at $\mathbf{u} = \tilde{\mathbf{u}}$

iii. Approximate design point by

$$\mathbf{u}^t = \beta^t \cdot \hat{\alpha}^t$$

iv. Check $g(\mathbf{u}^t) \square g_p \ge 0$



Single loop RBDO

$$\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{x}}} \quad f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{x}})$$

s.t. $g_p \square g(\mathbf{d}, \mathbf{x}(\mathbf{u}^t)) \ge 0$

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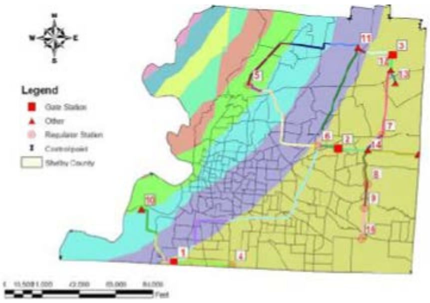
457.646 Topics in Structural Reliability

In-Class Material: Class 25

VIII-3. Random fields

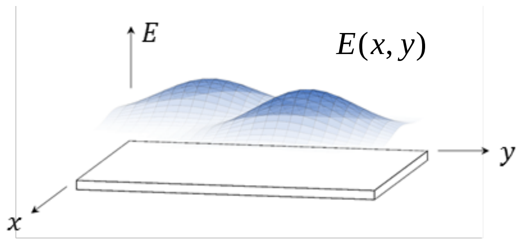
~ Random quantity distributed over _______________ field (space or time)

Ex1) Spatial Distribution of Random Ground Motion Intensity)



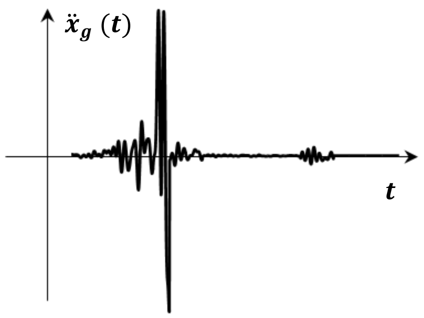
(Song & Ok, 2010)

Ex2) Spatial distribution of material property (Young's Modulus)



Ex3) Ground acceleration time history $\ddot{x}_g(t)$

e.g.

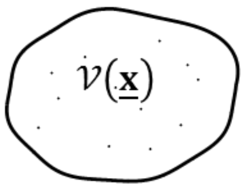


random process, stochastic process

⇒ ( ) # of random variables

⇒ ( ) representation is required

◎ Discretization of Random field → Random vector



$$\rightarrow \quad \mathbf{v} = \{v_1, \cdots, v_n\}^T$$

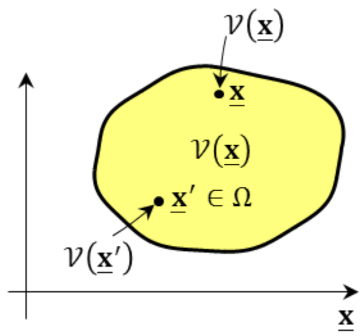
Random n-vector

$$? \begin{cases} \mathbf{M_v} = E[\mathbf{v}] = \{\mu_{v_i}\} \\ \mathbf{\Sigma_{vv}} = E[(\mathbf{v}-\mathbf{M}_v)(\mathbf{v}-\mathbf{M}_v)^T] \\ \quad = \mathbf{D_v R_{vv} D_v} \qquad\qquad \text{covariance matrix} \\ \text{where} \qquad D_v = \ \text{diag}[\sigma_{v_i}] \\ \qquad\qquad R_{vv} = [\rho_{v_i v_j}] \\ f_\mathbf{v}(\mathbf{v}) \rightarrow \text{joint PDF of } \mathbf{v} \end{cases}$$

◎ Theoretical Representation of R.F

$v(\mathbf{x}),\ \mathbf{x} \in \Omega$ random field in domain Ω

Partial descriptors:



$$\begin{cases} \mu(\mathbf{x}) : \text{mean function} \quad E[v(\mathbf{x})] \\ \sigma^2(\mathbf{x}) : \text{variance function} \quad E[v^2(\mathbf{x})] - \mu^2(x) \\ \rho(\mathbf{x}, \mathbf{x}') : \text{correlation coefficient function} \quad \rho_{v(\mathbf{x})v(\mathbf{x}')} \end{cases}$$

For Gaussian R.F. the above gives a complete specification

For Nataf R.F., also specify $F_v(v; \mathbf{x})$

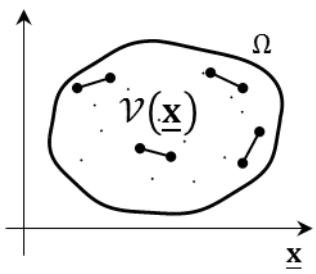
For general RF's, specify joint PDF of ( ) and ( )

for, $x, x' \in \Omega$, $f_{vv}(v(x), v(x'))$

e.g. ______________ Random field

~ ______________ does not change over the domain Ω

$v(\mathbf{x})$, $x \in \Omega$

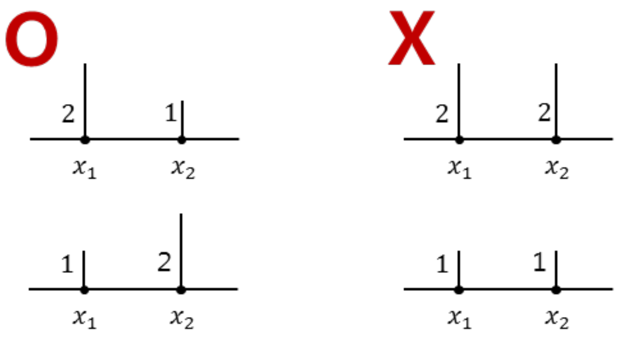


$$\begin{cases} \mu(\mathbf{x}) = \\ \sigma^2(\mathbf{x}) = \\ \rho(\mathbf{x}, \mathbf{x}') = \\ F(v; \mathbf{x}) = \end{cases}$$

Note; This doesn't mean $v(\mathbf{x}) = v$ (not constant over the domain)

Scenario 1

Scenario 2

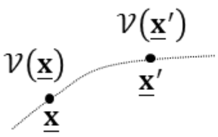


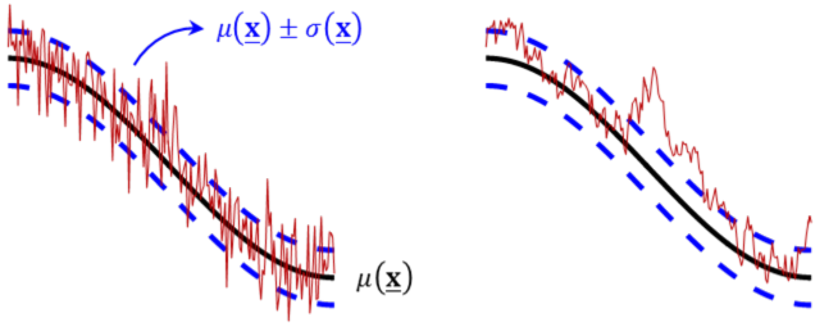
$\mu_{x_1} = 1.5$ $\mu_{x_2} = 1.5$

$\mu(\mathbf{x}) = \mu = 1.5$

$v(\mathbf{x}) = v$

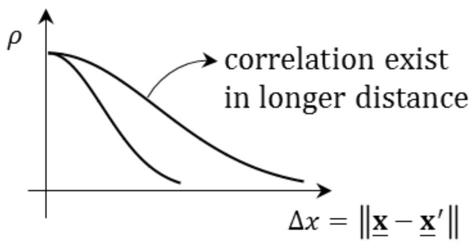
Q: Correlation Function $\rho(\mathbf{x}, \mathbf{x}')$ ←meaning?



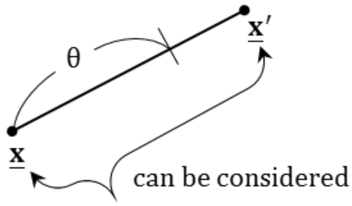


How to capture this from $\rho(\mathbf{x}, \mathbf{x}')$?

◎ Correlation length



$$\theta = \int_0^\infty \rho(\Delta x) dx$$



~ measure of the distance over which significant loss of correlation occurs

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Examples

- $\rho(\Delta x) = \exp\left(-\frac{\Delta x}{a}\right)$

$$\theta = \int_0^\infty \exp\left(-\frac{\Delta x}{a}\right) d\Delta x$$
$$= -a \exp\left(-\frac{\Delta x}{a}\right)\Big|_0^\infty = a$$

- $\rho(\Delta x) = \exp\left(-\frac{\Delta x^2}{a^2}\right)$

$$\theta = \int_0^\infty \exp\left(-\frac{\Delta x^2}{a^2}\right) d\Delta x$$
$$= \frac{1}{2}\int_{-\infty}^\infty \exp\left(-\frac{\Delta x^2}{a^2}\right) d\Delta x$$
$$= \frac{1}{2}\sqrt{\pi} a \qquad \theta \propto a$$

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457.646 Topics in Structural Reliability

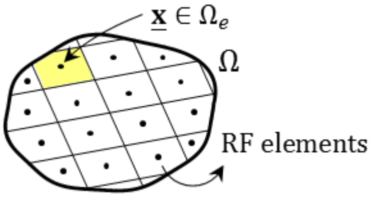
In-Class Material: Class 26

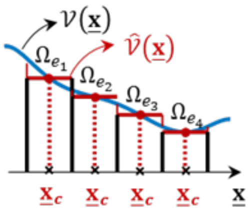
◎ Discrete Representation of RFs (Summary: Sudret & ADK 2000; 2002 PEM)

① Mid-point method

$$v(\mathbf{x}) \simeq \hat{v}(\mathbf{x})$$
$$= v(\mathbf{x}_c),\ \mathbf{x} \in \Omega_e$$

(constant in each Ω_e)





- Represented by a constant r.v.

  over each RF element

- Positive definiteness problem of $\mathbf{R}$ … if RF element size is small relative to θ

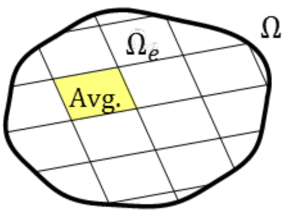
Recommended size of RF element size

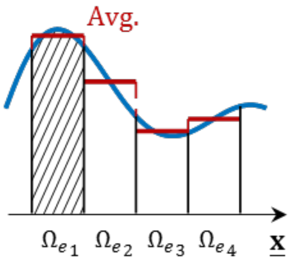
$$\frac{\theta}{10} \sim \frac{\theta}{15} \leq \text{ RF size } \leq \frac{\theta}{3} \sim \frac{\theta}{5}$$

Numerical stability (Positive definiteness) — Accurate representation

② Spatial averaging method

$$\hat{v}(\mathbf{x}) = \frac{\int_{\Omega_e} v(\mathbf{x}) d\Omega}{\int_{\Omega_e} d\Omega}, \quad \mathbf{x} \in \Omega_e$$





- Represented by a single r.v per Ω_e
- Variances are ( ) → _____-estimate P_f
- Positive definiteness problem

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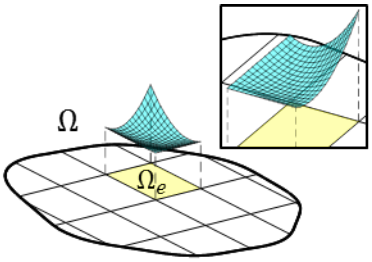
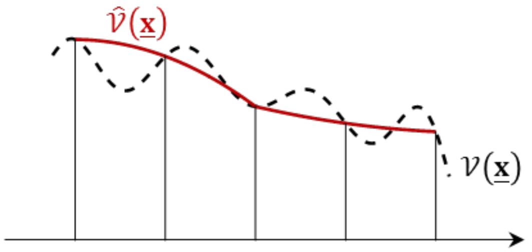
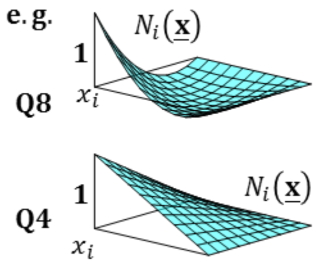
③ Shape function method (←motivated by FE people)

$$v(\mathbf{x}) \simeq \hat{v}(\mathbf{x}) = \sum_{\substack{\text{element}\\ \text{nodes}}} N_i(\mathbf{x}) v(\mathbf{x}_i)$$



• Represented by continuous function





$$N_i(\mathbf{x}_j) = \delta_{ij}$$

to guarantee $\hat{v}(\mathbf{x}_i) = v(\mathbf{x}_i)$

④ Karhunen-Loève (KL) expansion (Gaussian RFs)

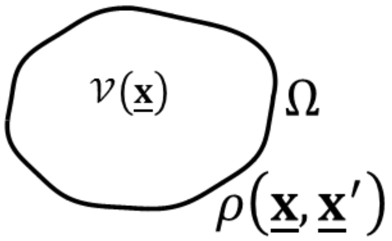
→ Describe RF in terms of finite # of shape functions

defined over ________ domain

(no geometric discretization)

→ Discretization based on

______________ structure $\rho(\mathbf{x}, \mathbf{x}')$



Goal: Want to descrive $\rho(\mathbf{x}, \mathbf{x}')$ by

$$\rho(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^{\infty} \lambda_i \varphi_i(\mathbf{x}) \varphi_i(\mathbf{x}')$$

Orthogonal shape (base) functions

Can find λ, φ by solving an integral eigenvalue problem, i.e.

$$\int_{\Omega} \rho(\mathbf{x}, \mathbf{x}') \varphi_i(\mathbf{x}') d\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x})$$ (Fredholem integral eqn – 2nd kind)

Note $\rho(\mathbf{x}, \mathbf{x}')$ is bounded, symmetric, (+) definite.

If so, one can find

$\varphi_i(\mathbf{x})$: orthogonal $\int \varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}) d\mathbf{x} = \delta_{ij}$

λ_i: real & positive

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Can drop λ_i's if $\lambda_r \cong 0$

Then using $\varphi_i(\mathbf{x})$, and λ_i, $i=1,\ldots,r$, one can describe Gaussian RF $v(x)$ by

KL expansion of Gaussian RF

$$\boxed{v(\mathbf{x}) \simeq \hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \sigma(\mathbf{x})\sum_{i=1}^{r}(u_i\sqrt{\lambda_i}\varphi_i(\mathbf{x})), \quad x \in \Omega} \Rightarrow v(\mathbf{x}) \Rightarrow \{u_1,\cdots,u_r\}$$

$u_i \to N(0,1),\ u_i$ s.i

Let's check!

i. Gaussian? Yes, function of u_i's

ii. $E[\hat{v}(\mathbf{x})] = \mu(\mathbf{x})$? $E[\hat{v}(\mathbf{x})] =$

iii. $Var[\hat{v}(\mathbf{x})] = E[(\qquad)^2]$

$$= E[\qquad \sum_{i=1}^{r}\sum_{j=1}^{r} \qquad]$$

$$= \sigma^2(\mathbf{x})\sum_{i=1}^{r}\sum_{j=1}^{r} \qquad \sqrt{\lambda_i}\sqrt{\lambda_j}\varphi_i(\mathbf{x})\varphi_j(\mathbf{x})$$

$$= \sigma^2(\mathbf{x})\sum_{i=1}^{r}\lambda_i\varphi_i^2(\mathbf{x})$$

$$= \sigma^2(\mathbf{x})$$

(because $\rho(\mathbf{x},\mathbf{x}) = \qquad = \qquad$)

iv. $\rho_{\hat{v}\hat{v}}(\mathbf{x},\mathbf{x}') \overset{?}{=} \rho(\mathbf{x},\mathbf{x}')$

$$= E[(\hat{v}(\mathbf{x}) - \mu(\mathbf{x}))(\hat{v}(\mathbf{x}') - \mu(\mathbf{x}'))] / \sigma(\mathbf{x})\sigma(\mathbf{x}')$$

$$= E[\sum_{i=1}^{r}\sum_{j=}^{r} u_i\sqrt{\lambda_i}\varphi_i(\mathbf{x})u_j\sqrt{\lambda_j}\varphi_j(\mathbf{x}')]$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} E[\qquad]\sqrt{\lambda_i}\sqrt{\lambda_j}\varphi_i(\mathbf{x})\varphi_j(\mathbf{x}')$$

$$= \sum_{i=1}^{r}\lambda_i\varphi_i(\mathbf{x})\varphi_i(\mathbf{x}')$$

$$= \varphi(\mathbf{x},\mathbf{x}')$$

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- # of RV's:
- Represented by function
- No necessary
- Most efficient (in terms of # of)
- Requires solution of an integral eigenvalue problem.

⑤ Orthogonal expansion (eigen-expansion, but correlated rv's)

⑥ Optimal linear estimation (OLE)~ linear regression

⑦ Expansion OLE

⋮ See Sudret & ADK (2000)

◎ Nataf RF

$v(\mathbf{x}) \Rightarrow F(v, \mathbf{x})$, $\rho_{ZZ}(\mathbf{x}, \mathbf{x}')$

$v(\mathbf{x}) = F_v^{-1}\{\Phi(\hat{Z}(\mathbf{x}))\}$, $Z(\mathbf{x}) \sim N(\mathbf{0}, \rho_{ZZ}(\mathbf{x}, \mathbf{x}'))$ ($Z(\mathbf{x}) \rightarrow$ Gaussian RF)

⇒ Construct $Z(\mathbf{x})$ and discrete to $\hat{Z}(\mathbf{x})$

⇒ $v(\mathbf{x}) = F^{-1}\{\Phi(\hat{Z}(\mathbf{x}))\}$

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VIII-4. Response Surface Method (CRC Ch.19 & Mike Tipping's chapter)

◎ Reliability Analysis, Uncertainty Quantification & Response Surface

Reliability Analysis

$P_f = \int_{g(\mathbf{x})\le 0} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$ → e.g. FORM/SORM $g(\mathbf{x}_i)$, $\nabla g(\mathbf{x}_i)$

→ e.g. Sampling $q_i = I(\mathbf{x}_i)$ or $\dfrac{I(\mathbf{x}_i)\cdot f(\mathbf{x}_i)}{h(\mathbf{x}_i)}$

where $I(\mathbf{x}_i) = \begin{cases} 1 & g(\mathbf{x}_i) \le 0 \\ 0 & g(\mathbf{x}_i) > 0 \end{cases}$

Uncertainty Quantification

"Process of determining the effect of input uncertainties"

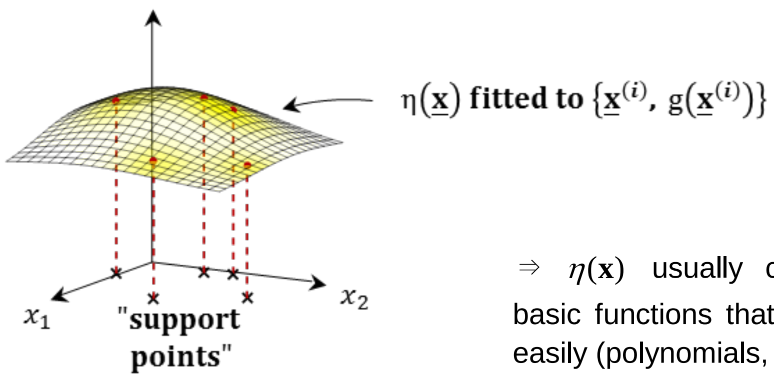
on response metrics of interest (Eldred et al. 2008)

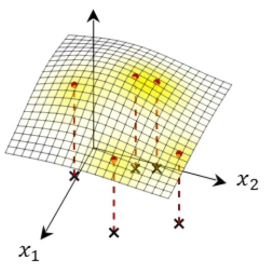
e.g. $E[g(\mathbf{x})^m] = \int_{\mathbf{x}} g(\mathbf{x})^m f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$

① $g(\mathbf{x})$ Sometimes

- Computationally costly for MCS
- No analytical gradients but many RVs
  ⇒ FORM/SORM difficult
- Experiments expensive (statistical analysis of experiment data infeasible)

② Idea: $g(\mathbf{x}) \simeq \eta(\mathbf{x})$ ($\eta(\mathbf{x})$ ← "response surface" or "surrogate" model)





⇒ $\eta(\mathbf{x})$ usually constructed in terms of basic functions that can be computed more easily (polynomials, exp, ⋯)

⇒ Should fit $g(\mathbf{x}^{(i)})$ sufficiently well especially in the region that contributes most to P_f or $E[g(\mathbf{x})^m]$

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③ History

- Box and Wilson (1954): influential
- Applied mostly in chemical, industrial eng. etc.

  (Mostly for “experimental design”)
- Rackwitz (1982) ⇒ Use RS for Structural Reliability Analysis
- Has been applied to random field, nonlinear structural dynamics, etc.

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457.646 Topics in Structural Reliability

In-Class Material: Class 27

◎ Basic formulation of RS models

Two approaches: regression $\Rightarrow$ use assumed mathematical model & fit it to data

e.g. $\eta(\mathbf{x}) = \sum_{i=1}^{p} \theta_i x_i^m$

Interpolation $\Rightarrow$ Interpolate using nearby data points

e.g. K-nearest points

<u>Regression</u>

True response of $g(\mathbf{x})$: $Z(\mathbf{x})$

$$Z(\mathbf{x}) = \eta(\underbrace{\theta_1, \cdots, \theta_p}_{\text{Model parameters}}; \underbrace{\mathbf{x}}_{\text{Input}}) + \underbrace{\varepsilon}_{\text{Zero mean (random) error term}}$$

Zero mean (random) error term

$\Rightarrow E[z-\eta] = E[\varepsilon] = 0$

"unbiased" model

How to find $\boldsymbol{\theta}$? What do data tell us?

Ref: Tipping, M.E. (2004)

"Bayesian inference: an introduction to principles and practice in machine learning"
Advanced lectures on machine learning, pp.41-62

(Free codes and papers at miketipping.com)

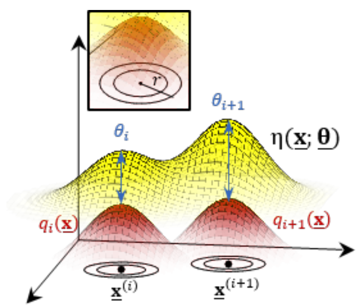
$$\eta = \theta_1 \exp(x) + \theta_2 \ln x + \theta_3 \cdots$$

◎ <u>Linear</u> models (Linear in)

Find $Z = \eta(\mathbf{x}; \boldsymbol{\theta}) + \varepsilon$

$$= \sum_{i=1}^{p} \underbrace{\theta_i}_{\text{Model Parameter}} \underbrace{q_i(\mathbf{x})}_{\text{Basis Function (Shape function)}} + \varepsilon$$

from $\{\mathbf{x}^{(i)}, Z^{(i)}\}$, $i = 1, \cdots, m$



e.g. $q_i(\mathbf{x}) \propto$ PDF of $N(\mathbf{x}^{(i)}, r^2\mathbf{I})$

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$\mathbf{Z} = \mathbf{Q}\boldsymbol{\theta} + \boldsymbol{\varepsilon}$

!!!

$$\underset{m\times 1}{\begin{Bmatrix} Z^{(1)} \\ Z^{(2)} \\ \vdots \\ Z^{(m)} \end{Bmatrix}} = \underset{m\times p}{\begin{bmatrix} q_1(\mathbf{x}^{(1)}) & \cdots & \cdots & q_p(\mathbf{x}^{(1)}) \\ \vdots & & & \\ q_1(\mathbf{x}^{(m)}) & \cdots & \cdots & q_p(\mathbf{x}^{(m)}) \end{bmatrix}} \underset{p\times 1}{\begin{Bmatrix} \theta_1 \\ \vdots \\ \theta_p \end{Bmatrix}} + \underset{m\times 1}{\begin{Bmatrix} \varepsilon^{(1)} \\ \vdots \\ \varepsilon^{(m)} \end{Bmatrix}}$$

Five approaches (Tipping 2004)

① "Least-Square" Approximation (classic)

⇒ Minimize sum of squared errors

$$E_D = \frac{1}{2}\sum_{i=1}^{m}(Z^{(i)} - \eta(\mathbf{x}^{(i)},\theta))^2$$
$$= \frac{1}{2}(\mathbf{Z}-\mathbf{Q}\boldsymbol{\theta})^T(\mathbf{Z}-\mathbf{Q}\boldsymbol{\theta})$$
$$= \frac{1}{2}\mathbf{Z}\mathbf{Z}^T + \frac{1}{2}(\mathbf{Q}\boldsymbol{\theta})^T(\mathbf{Q}\boldsymbol{\theta}) - \mathbf{Z}^T\mathbf{Q}\boldsymbol{\theta}$$

$$\frac{\partial E_D(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -\mathbf{Z}^T\mathbf{Q} + (\mathbf{Q}\boldsymbol{\theta})^T\mathbf{Q} = 0$$

Solve for $\boldsymbol{\theta}$,

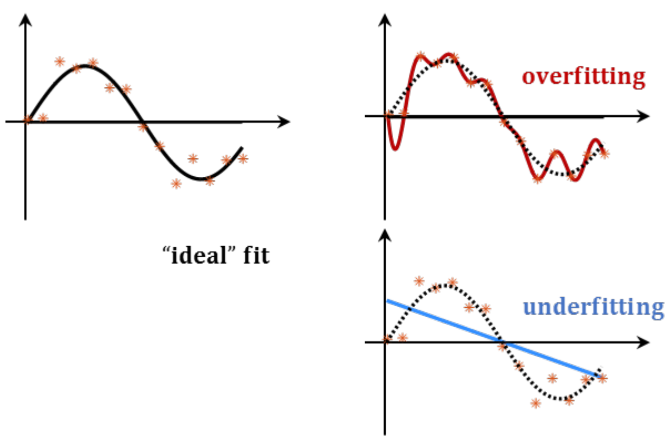
$$\boxed{\boldsymbol{\theta}_{LS} = (\mathbf{Q}^T\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{Z}}$$

※ over-fitting?

e.g. $Z = \sin x + \varepsilon$

$\sin x \rightarrow$ true model, $\varepsilon \rightarrow$ noise

Figure 1 in Tipping (2004)



② Regularization (by giving penalty on large $\boldsymbol{\theta}$)

$$\hat{E}(\boldsymbol{\theta}) = E_D(\boldsymbol{\theta}) + \lambda \underline{E_W(\boldsymbol{\theta})}$$

Standard choice

$$E_W(\boldsymbol{\theta}) = \frac{1}{2}\sum_{i=1}^{p}\theta_i^2$$

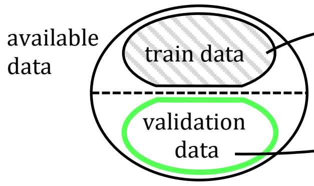
regularization parameter $\lambda \uparrow$ Discourage large value of $\boldsymbol{\theta}$

$\Rightarrow$ Smooth function

$$\frac{\partial E_D(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0 \Rightarrow \boxed{\boldsymbol{\theta}_{PLS} = (\mathbf{Q}^T\mathbf{Q} + \lambda\mathbf{I})^{-1}\mathbf{Q}^T\mathbf{Z}}$$

※ Appropriate value of λ?

A common approach: Use "validation" data



Find $\boldsymbol{\theta}_{PLS}$ for a given λ, $\boldsymbol{\theta}_{PLS}(\lambda)$

Construct surrogate $\eta(\mathbf{x};\lambda) = \sum_{i=1}^{p}\theta_i^{PLS}(\lambda)q_i(\mathbf{x})$

Compute error against validation data
$\varepsilon = Z - \eta$ at each $\mathbf{x}^{(i)} \in$ validation data set

and choose λ that makes $\frac{1}{2}\Sigma(Z-\eta)^2$ minimum

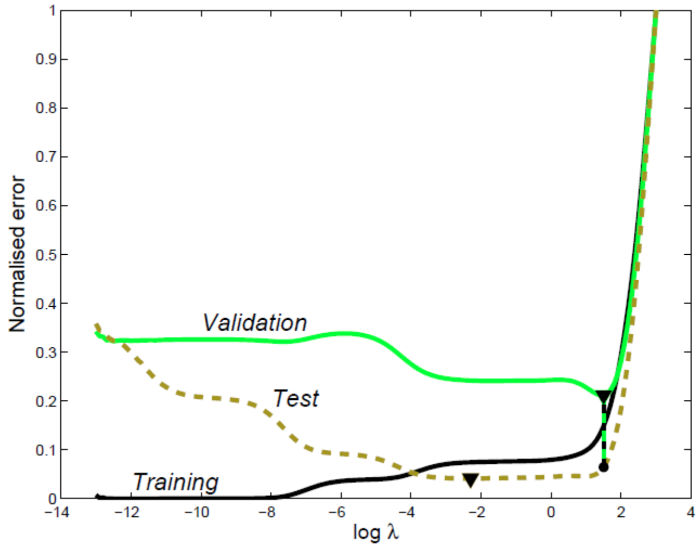


Fig. 3. Plots of error computed on the separate 15-example training and validation sets, along with 'test' error measured on a third noise-free set. The minimum test and validation errors are marked with a triangle, and the intersection of the best λ computed via validation is shown.

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※ Probabilistic Regression

$Z = \eta + \varepsilon$!

e.g. $\varepsilon \sim N(0, \sigma^2)$ $\quad\quad \therefore Z \sim N(\eta, \sigma^2)$

Using this information one can construct likelihood function

$$L(\mathbf{Z} \,|\, \mathbf{x}, \boldsymbol{\theta}, \sigma^2) = \prod_{i=1}^{n} f(Z^{(i)} \,|\, \mathbf{x}^{(i)}, \boldsymbol{\theta}, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\{Z^{(i)} - \eta(\mathbf{x}^{(i)}; \boldsymbol{\theta})\}^2}{2\sigma^2} \right]$$

③ Maximum Likelihood Estimation

Find $\boldsymbol{\theta}$ that maximizes L( ) ⇔ Find $\boldsymbol{\theta}$ that minimizes –lnL( )

$$-\ln L(\quad) = \frac{n}{2}\ln(2\pi\sigma^2) + \boxed{\frac{1}{2\sigma^2}} \boxed{\sum_{i=1}^{n}\{Z^{(i)} - \eta(\mathbf{x}^{(i)}, \boldsymbol{\theta})\}^2}$$

$E_D(\boldsymbol{\theta})$
⇒ error measure
for $\underline{\boldsymbol{\theta}}_{LS}$

Therefore, MLE based on s.i. _________ error assumption (i.e. $\varepsilon \sim N(\quad)$)

Gives $\boxed{\boldsymbol{\theta}_{MLE} = \boldsymbol{\theta}_{LS}}$

(cf. Assuming errors are dependent? $\varepsilon \sim N(\mathbf{0}, \boldsymbol{\Sigma})$)

$$\rho_{ij} = \exp\left(-\frac{\left\| \mathbf{x}^{(i)} - \mathbf{x}^{(j)} \right\|}{L} \right) \Rightarrow$$ "Kriging" Method (Satner et al. 2003)

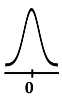
※ Bayesian Methods $f = c \cdot L \cdot p$

Introduce a prior distribution

$$p(\boldsymbol{\theta} \,|\, \alpha) = \prod_{i=1}^{p} \left(\frac{\alpha}{2\pi} \right)^{1/2} \exp\left\{ -\frac{\alpha}{2}\theta_i^2 \right\}$$

$$= \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi}\, 1/\sqrt{\alpha}} \exp\left\{ -\frac{\theta_i^2}{2(1/\alpha)} \right\}$$

(degree of belief about smooth model)

$\alpha \uparrow$ Variability reduces  ⇒ certain that θ is around 0

⇒Become smooth

$\therefore \alpha \propto \lambda$

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④ Maximum a posteriori (MAP) estimation (a Bayesian "shortcut")

$$f = c \cdot L \cdot p$$

$$P(\boldsymbol{\theta} | \mathbf{Z}, \alpha, \sigma^2) \; = \; c \; \cdot \; L(\mathbf{Z} | \boldsymbol{\theta}, \sigma^2) \; \cdot \; p(\boldsymbol{\theta} | \alpha)$$

Posterior   Likelihood function   prior

Find $\boldsymbol{\theta}$ where $P(\boldsymbol{\theta} | \mathbf{Z}, \alpha, \sigma^2)$ is maximum

e.g. Normal s.i errors ε, $Z \sim N(\eta, \sigma^2)$

$$-\ln(f) = \frac{1}{2\sigma^2} \sum_{i=1}^{n} \{Z^{(i)} - \eta(\mathbf{x}^{(i)}; \boldsymbol{\theta})\}^2 + \frac{\alpha}{2} \sum_{i=1}^{p} \theta_i^2$$

$\lambda = \alpha\,\sigma^2$

$$-\sigma^2 \ln(f) \;=\; \underbrace{\frac{1}{2} \sum_{i=1}^{n} \{Z^{(i)} - \eta(\mathbf{x}^{(i)}; \boldsymbol{\theta})\}^2}_{E_D(\boldsymbol{\theta})} \;+\; \underbrace{\frac{\alpha\sigma^2}{2}}_{=\frac{1}{2}\lambda} \underbrace{\sum_{i=1}^{p} \theta_i^2}_{E_W(\boldsymbol{\theta})}$$

the same as

※ α, σ^2 ? no need to bother w/ Bayesian?

⑤ **Full** Bayesian ("Marginalization")   integrate $P(\mathbf{Z} | \boldsymbol{\theta}, \alpha, \sigma^2)$ over all $\boldsymbol{\theta}$

$$P(\mathbf{Z}) = \int P(\mathbf{Z} | \boldsymbol{\theta}) \cdot P(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

Focus on

Total probability theorem

$$P(\mathbf{Z} | \boxed{\alpha, \sigma^2}) = \int P(\mathbf{Z} | \boldsymbol{\theta}, \boxed{\alpha, \sigma^2}) \cdot P(\boldsymbol{\theta} \,\boxed{| \alpha, \sigma^2}) d\boldsymbol{\theta}$$

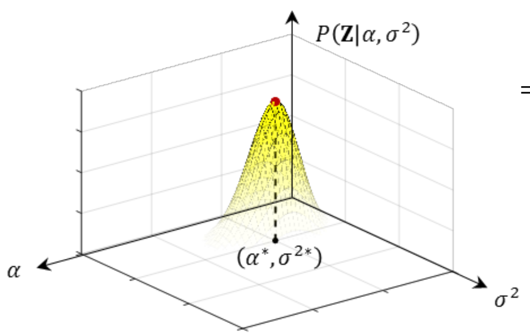
Simplified to

$$= \int P(\mathbf{Z} | \boldsymbol{\theta}, \sigma^2) \cdot P(\boldsymbol{\theta} | \alpha) d\boldsymbol{\theta}$$

→ Closed-form available:

$f_N(\mathbf{Z}, \alpha, \sigma^2)$ (Eq. 23 in Tipping, 2004)

※ $P(\mathbf{Z} | \alpha, \sigma^2)$: Probability that you will observe $\mathbf{Z}$ for given α, σ^2



$\Rightarrow$ Find $\alpha \,\&\, \sigma^2$ that maximizes $P(\mathbf{Z} | \alpha, \sigma^2)$

(i.e. Let data $\mathbf{Z}$ tell us the optimal point α^*, σ^{2*})

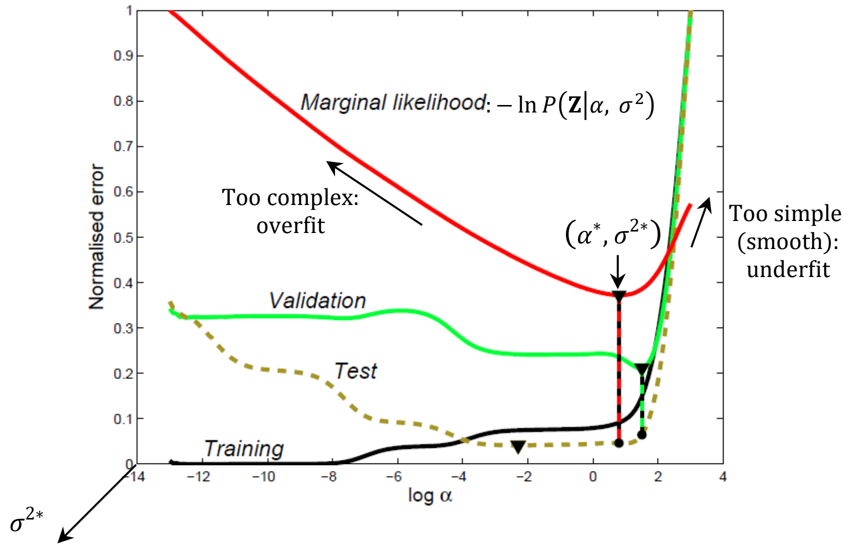


Fig. 5. Plots of the training, validation and test errors of the model as shown in Figure 3 (with the horizontal scale adjusted appropriately to convert from λ to α) along with the negative log marginal likelihood evaluated *on the training data alone* for that same model. The values of α and test error achieved by the model with highest marginal likelihood (smallest negative log) are indicated.

☆ **Okham's Razar** (or the law of parsimony)**:**

"model should be no more complex than is sufficient to explain the data"

CRC CH.19 RS
→DOE
→$q_i(\mathbf{x})$

◎ Other RS or UQ methods

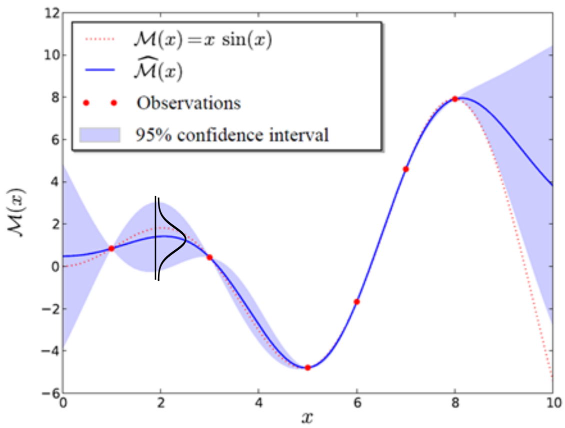
① Kriging (Santner et al. 2003)

(Dubourg et al. 2010 IFIP)

$\varepsilon \sim N(\mathbf{0}, \mathbf{\Sigma})$

e.g. $\rho_{ij} = \exp\left(-\dfrac{\left\|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\right\|}{L}\right)$

- coincides at each point
- Interpolate b/w each point
- Can quantify confidence
- Regularization



(Dubourg et al. 2011)

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② Dimension Reduction (Rahman & Xu, 2004; Xu & Rahman 2004)

$$g(\mathbf{x}) \rightarrow g(\hat{\mathbf{x}}) = \sum_{i=1}^{n} g(\mu_1, \cdots, \mu_{i-1}, x_i, \mu_{i+1}, \cdots, \mu_n) - (n-1) g(\mu_1, \cdots, \mu_n)$$

$\Downarrow$

$$E[(g(x))^m] \cong E[(\hat{g}(x))^m]$$

$$= \int (\hat{g}(x))^m \underline{f_{\mathbf{x}}(\mathbf{x})} d\mathbf{x} \quad \nearrow \quad \Pi \varphi(x_i)$$

Transform to s.i. space; Multivariate Integral $\Rightarrow$ Multiple univariate Integral

③ Polynomials chaos (a good review by Eldred et al. 2008)

$$R = a_0 B_0 + \sum_{i_1=1}^{\infty} a_{i1} B_1(\zeta_{i1})$$

$$+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} a_{i1,i2} B_2(\zeta_{i1} \zeta_{i2}) + \cdots$$

$$= \sum_{j=0}^{p} \alpha_j \psi_j(\zeta)$$ $\rightarrow$ Orthogonal bases for given types of r.v's distribution

$$\alpha_j = \frac{< R, \psi_j >}{< \psi_j^2 >} = \frac{\int R \psi_j f(\zeta) d\zeta}{< \psi_j^2 >}$$ $\rightarrow$ Important sampling, etc.

$\rightarrow$ closed form available

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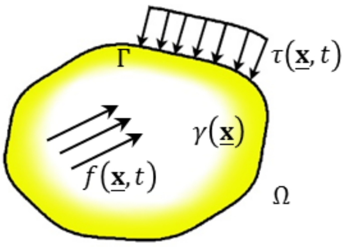
457.646 Topics in Structural Reliability

In-Class Material: Class 28

VII-4. Finite Element Reliability Analysis (Haukaas, 2006)

→ summary and good findings

◎ Equations of Motion and Randomness



"Weak" form of equilibrium:

$$\int_{\Omega}\delta u_i\gamma\ddot{u}_i d\Omega+\int_{\Omega}\delta u_{i,j}\sigma_{ij}d\Omega-\int_{\Omega}\delta u_i f_i d\Omega-\int_{\Gamma}\delta u_i\tau_i d\Gamma=0$$

γ : density, $\ddot{u}_i$: acc, $u_{i,j}$: strain, σ_{ij} : stress, f_i : body force, τ_i : traction

① Basic random fields

$C_{ijkl}(\mathbf{x})$, $\gamma(\mathbf{x})$: material properties (constants)

Tensor of material elastic constants, $\sigma_{ij}=C_{ijkl}\varepsilon_{kl}$

$f_i(\mathbf{x},t)$, $\tau_i(\mathbf{x};t)$: loads

Ω, Γ : geometry

⇒ Discretized to a random vector $\mathbf{v}$

② Derived response is a function of $\mathbf{v}$

$u_i(\mathbf{x},t,\mathbf{v})$: displacement

$\varepsilon_{ij}(\mathbf{x},t,\mathbf{v})$: strain

$\varepsilon_{ij}^P(\mathbf{x},t,\mathbf{v})$: plastic strain

$\sigma_{ij}(\mathbf{x},t,\mathbf{v})$: stress

$$g(S(\mathbf{v}),\mathbf{v})\le 0$$

⋮

$\mathbf{S}(\mathbf{x},t,\mathbf{v})$: generic response vector

③ FE models and r.v's

i. Nonlinear & Dynamic problem

$$\mathbf{M}(\mathbf{v})\ddot{\mathbf{u}}(t,\mathbf{v})+\mathbf{C}(\mathbf{v})\dot{\mathbf{u}}(t,\mathbf{v})+\mathbf{R}(\mathbf{u}(t,\mathbf{v}),\mathbf{v})=\mathbf{P}(t,\mathbf{v})$$

ii. Static problem

$$\mathbf{R}(\mathbf{u}(t,\mathbf{v}),\mathbf{v})=\mathbf{P}(t,\mathbf{v})$$

iii. Linear Static problem

$$\mathbf{K}(\mathbf{v})\cdot\mathbf{u}(\mathbf{v})=\mathbf{P}(\mathbf{v})$$

④ FE reliability analysis

i. MCS $\mathbf{v}_i,\ \ i=1,\cdots,N$

ii. Importance Sampling

iii. Response Surface $g\approx\eta(\mathbf{x})$

iv. Form (HLRF)

Initialize $\mathbf{u}_1=\mathbf{u}(\mathbf{v}_1)$

$\downarrow$

$\mathbf{v}_i=\mathbf{v}(\mathbf{u}_i)$ skip if $i=1$

$G(\mathbf{u}_i)=g(\mathbf{S}(\mathbf{v}_i),\mathbf{v}_i)$

$$\nabla_{\mathbf{u}}G(\mathbf{u}_i)=\nabla_{\mathbf{v}}g(\mathbf{v})J_{\mathbf{v},\mathbf{u}}=(\nabla_{\mathbf{s}}g\cdot J_{\mathbf{s},\mathbf{v}}+\nabla_{\mathbf{v}}g)\cdot J_{\mathbf{v},\mathbf{u}}$$

$\leftarrow$ $\mathbf{S}(\mathbf{v}_i), J_{\mathbf{s},\mathbf{v}}$ — FE code

$\downarrow$ The same procedure

$\vdots$

e.g. FERUM-ABAQUS

(Young Joo, Lee, 2012)

◎ **Gradient** $J_{\mathbf{s},\mathbf{v}}$ **?**

e.g. $\dfrac{\partial u_i}{\partial E},\dfrac{\partial\sigma_i}{\partial P},\cdots$

Methods to get sensitivity $J_{\mathbf{s},\mathbf{v}}$

e.g. Linear Static Problem (suppose there is only one r.v. $\mathbf{v}=v$)

$$\underbrace{\mathbf{K}(v)}_{\text{Stiffness}}\cdot\underbrace{\mathbf{u}(v)}_{\text{displacement}}=\underbrace{\mathbf{P}(v)}_{\text{loads}}\quad\rightarrow\quad\mathbf{u}=\mathbf{K}^{-1}(v)\cdot\mathbf{P}(v)$$

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① Finite Difference Method ("FFD" option of FERUM)

$$\mathbf{u}(v) = \mathbf{K}^{-1}(v) \cdot \mathbf{P}(v) \quad \text{original FE}$$

$$\mathbf{u}(v+\Delta v) = \mathbf{K}^{-1}(v+\Delta v) \cdot \mathbf{P}(v+\Delta v) \quad \text{(i.e. additional FE for each } v_i \text{ in } \mathbf{v}\text{)}$$

$$\frac{\partial \mathbf{u}}{\partial v} \cong \frac{\mathbf{u}(v+\Delta v) - \mathbf{u}(v)}{\Delta v}$$

⇒ Need to solve FE again (for each r.v)

⇒ Can cause numerical errors

② Perturbation Method

$$\mathbf{Ku} = \mathbf{P}$$

$$\Delta\mathbf{K} = \mathbf{K}(v+\Delta v) - \mathbf{K}(v)$$
$$\Delta\mathbf{P} = \mathbf{P}(v+\Delta v) - \mathbf{P}(v)$$

$$(\mathbf{K} + \Delta\mathbf{K})(\mathbf{u} + \Delta\mathbf{u}) = \mathbf{P} + \Delta\mathbf{P}$$

$$\mathbf{Ku} + \mathbf{K}\Delta\mathbf{u} + \Delta\mathbf{K}\mathbf{u} + \Delta\mathbf{K}\Delta\mathbf{u} = \mathbf{P} + \Delta\mathbf{P}$$

$$\therefore \Delta\mathbf{u} \cong \mathbf{K}^{-1}(\Delta\mathbf{P} - \Delta\mathbf{K}\mathbf{u})$$

⇒ Do not have to re-solve FE

⇒ Error ($\Delta\mathbf{K}\Delta\mathbf{u} \approx 0$)

③ Direct Differentiation Method ('DDM' option for FERUM)

$$\mathbf{Ku} = \mathbf{P}$$

$$\frac{\partial \mathbf{K}}{\partial v}\mathbf{u} + \mathbf{K}\frac{\partial \mathbf{u}}{\partial v} = \frac{\partial \mathbf{P}}{\partial v}$$

$$\frac{\partial \mathbf{u}}{\partial v} = \mathbf{K}^{-1}\left(\frac{\partial \mathbf{P}}{\partial v} - \frac{\partial \mathbf{K}}{\partial v}\mathbf{u}\right)$$

→ Do not need to solve FEM again

→ No error

→ $\dfrac{\partial \mathbf{K}}{\partial v} = \sum_e \dfrac{\partial \mathbf{K}^e}{\partial v}$ ($\dfrac{\partial \mathbf{K}^e}{\partial v}$ ← direct stiffness method)

→ Nonlinear static, nonlinear dynamic

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④ Adjoint method

➔ Tutorial by Prof. Andrew M. Bradley at Stanford University:
http://cs.stanford.edu/~ambrad/adjoint_tutorial.pdf)

$x \in \Re^{n_x},\ p \in \Re^{n_p},\ f\big(x(p)\big): \Re^{n_x} \to \Re$

Subject to $h(x(p), p) = 0$ for $h: \Re^{n_x} \times \Re^{n_p} \to \Re$

e.g. h=0 → PDE equilibrium (mass ∈ p, displacement ∈ x, member force = f)

$d_p f$? (total derivative of f w.r.t p)

Consider the Lagrangian

$$L(x, p, \lambda) = f(x(p)) + \lambda^{\mathrm{T}} h(x(p), p)$$

$$\begin{aligned} d_p f &= d_p L \qquad (\because \text{ only on } h = \qquad\qquad) \\ &= \partial_x f d_p x + d_p \lambda^{\mathrm{T}} h + \lambda^{\mathrm{T}} (\partial_x h d_p x + \partial_p h) \\ &= f_x x_p + \lambda^{\mathrm{T}} (h_x x_p + h_p) \qquad (\because \qquad\qquad) \\ &= (f_x + \lambda^T h_x) x_p + \lambda^{\mathrm{T}} h_p \end{aligned}$$

Choose λ such that $h_x^{\mathrm{T}} \lambda = -f_x^{\mathrm{T}}$ ("adjoint equation") → λ*

Then we can avoid calculating ( )

Then compute $d_p f$ as ___________

⇒ Used for RBTO of structures under stochastic excitations
(Chun, Song and Paulino, 2016)

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VI. Simulation methods (contd.)

◎ Latin Hypercube Sampling (Mckay et al. 1979)

Extension of "Latin Square" – appearing exactly once in each row and exactly once in each column)

(←) 7x7 Latin Square stained glass honoring R.A. Fisher's work on DOE

Evenly distribute sampling points to promote early convergence

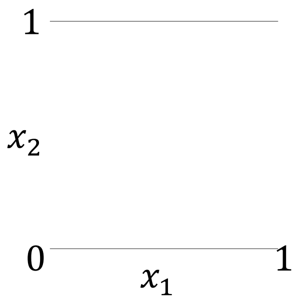
e.g. $\mathbf{X} = \{X_1, X_2\}$ uniform (0,1), s.i

⇒ 4 samples

- Brute force MCS:

  Samples are generated independently

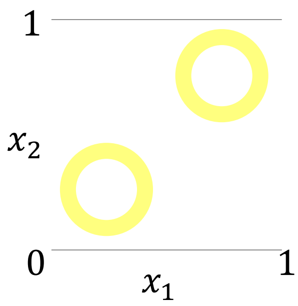
  No memory



- Latin Hypercube Sampling:

  There is only one sample in each row and column

  (w/ memory)



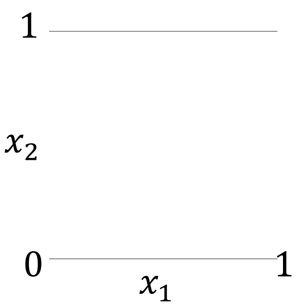
Possible LHS combination?

$$\left.\prod_{n=0}^{M-1} (M-n)^{N-1}\right\} \quad M = 4,\ N = 2 \quad \therefore 24 \text{ cases}$$

- Orthogonal Sampling:

  LHS + subspace sampled

  w/ same frequency



choose LHS combinations
that satisfy orthogonal

Example) Y.S. Kim et al. (2009)

→ Seismic Performance Assessment of Interdependent Lifeline Systems

⇒ Generated random samples of post-disaster conditions of network components using LHS

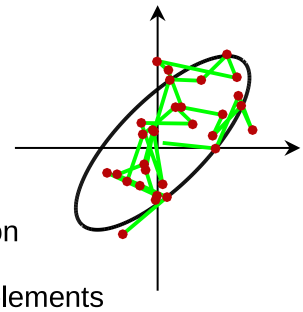
◎ Markov Chain Monte Carlo Simulation (MCMC)

$P(\mathbf{Z}^{(m+1)} | \mathbf{Z}^{(m)})$ transition prob.

→ Use MCS to generate samples as a Markov chain (good for high-dimensional problem)

① Metropolis-Hastings algorithm (Hastings, 1970)

~Accept/reject w/ probability (see next page)



② Gibbs sampling (Geman & Geman 1984)

See next page: Sample "one" element each time based on

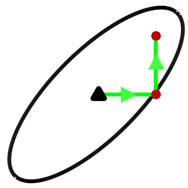
Conditional distribution given the outcomes of the other elements

e.g. $P(Z_1, Z_2, Z_3)$ $\mathbf{Z}$

sample $Z_1^{\tau+1}$ by $P(Z_1 | Z_2^{\tau}, Z_3^{\tau})$

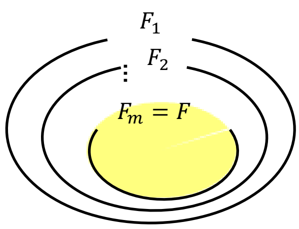
$Z_2^{\tau+1}$ by $P(Z_2 | Z_1^{\tau}, Z_3^{\tau})$

$Z_3^{\tau+1}$ by $P(Z_3 | Z_1^{\tau}, Z_2^{\tau})$



③ Subset Simulation (Au & Beck, 2001)

$F_1 \supset F_2 \supset \cdots \supset F_m = F$ event of interest



$P(F) = P(F_m)$ too low

e.g. $F_i = \{D > C_i\}$

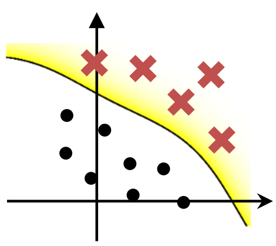
$C_1 < C_2 < \cdots < C_m = C$

$$P(F) = P(F_m) = P(\bigcap_{i=1}^{m} F_i)$$

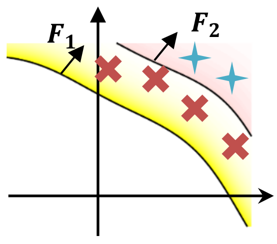
$$= P(F_m | \bigcap_{i=1}^{m-1} F_i) \cdot P(\bigcap_{i=1}^{m-1} F_i)$$

$$= P(F_m | f_{m-1}) \cdot P(\bigcap_{i=1}^{m-1} F_i)$$

$$= \underbrace{P(F_1) \times P(F_2 | F_1)}_{\text{Each larger than } P(F)} \times \cdots \times P(F_m | F_1 \cdots F_{m-1})$$



$P(F_1)$



$P(F_2|F_1)$

Each larger than $P(F)$

Use MCMC algorithm to compute $P(F_{i+1} | F_i)$

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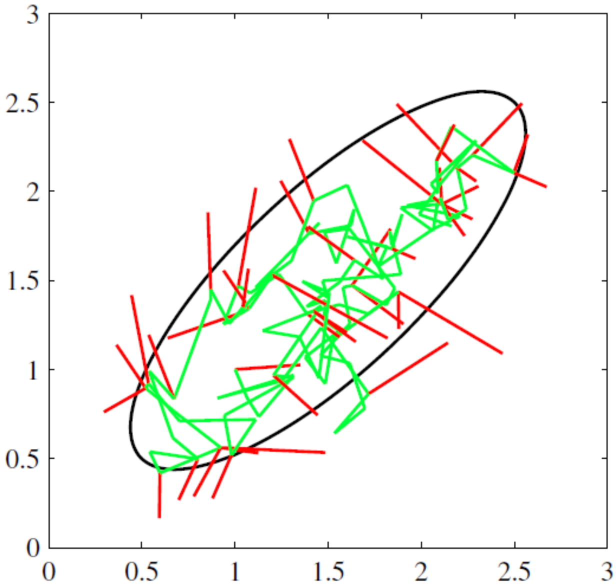
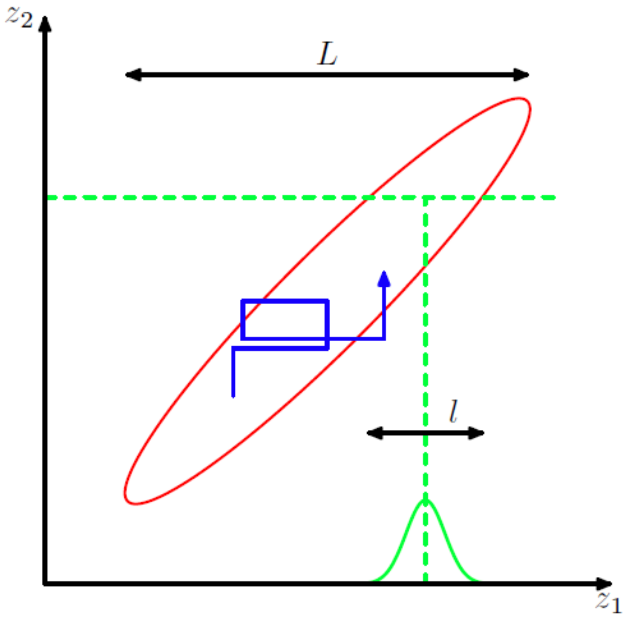
* Generating samples of a bi-variate Gaussian distribution using Metropolis algorithm

A simple illustration using Metropolis algorithm to sample from a Gaussian distribution whose one standard-deviation contour is shown by the ellipse. The proposal distribution is an isotropic Gaussian distribution whose standard deviation is 0.2. Steps that are accepted are shown as green lines, and rejected steps are shown in red. A total of 150 candidate samples are generated, of which 43 are rejected.



* Generating samples of a bi-variate Gaussian distribution using Gibbs sampling

Illustration of Gibbs sampling by alternate updates of two variables whose distribution is a correlated Gaussian. The step size is governed by the standard deviation of the conditional distribution (green curve), and is $O(l)$, leading to slow progress in the direction of elongation of the joint distribution (red ellipse). The number of steps needed to obtain an independent sample from the distribution is $O((L/l)^2)$.



Reference: “Pattern Recognition and Machine Learning” by Christopher M. Bishop (2006)

◎ Extrapolation-based MCS (Naess et al. 2009)

$g(\lambda) = g - \mu_g(1-\lambda)$ $\qquad \lambda = 0: \quad g(\lambda) = g - \mu_g \quad P_f \approx 50\%$

$0 \le \lambda \le 1$ $\qquad \lambda = 1: \quad g(\lambda) = g \quad P_f \approx 1$

Generate samples $\{g_1, \cdots, g_n\}$ and use to estimate

$\tilde{P}_f(\lambda) = \dfrac{N_f(\lambda)}{N}$ while varying λ

Fitted to $\underset{\lambda \to 1}{\cong} q(\lambda) \cdot \exp\{-a(\lambda - b)^c\}$ (can assume constant q), i.e.

$$\tilde{P}_f(\lambda) \underset{\lambda \to 1}{\cong} q^* \cdot \exp\{-a^*(\lambda - b^*)^{c^*}\}$$

Find a, b, c, q by fitting and extrapolate as $\tilde{P}_f(\lambda)$ as $\lambda \to 1$

$\Rightarrow$ Has been applied to component/system (Naess et al. 2009)

and large-size system problems (Naess et al. 2010)

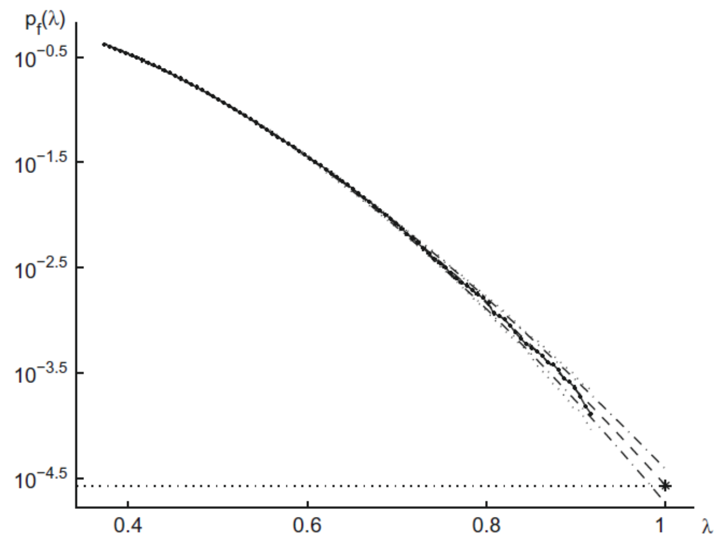


Fig. 9. Plot of $\log \hat{p}_f(\lambda_j)$ for Example 4: Monte Carlo ($\cdot$); fitted optimal curve ($--$); reanchored empirical confidence band ($\cdots$); fitted confidence band ($-\cdot$). $\log q = -0.303$, $a = 16.231$, $b = 0.252$, $c = 1.591$.

Many thanks for your hard work in this semester to learn theories of structural reliability and their applications. I wish you the very best on your course work, research and future career.

Cheers,
Junho

2017