SEOUL NATIONAL UNIVERSITY

NuTHEL
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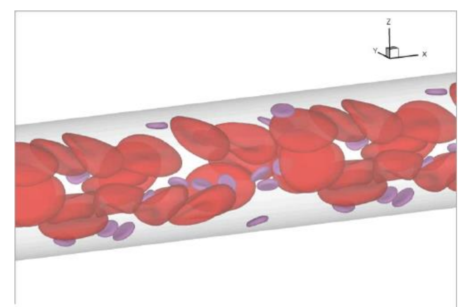
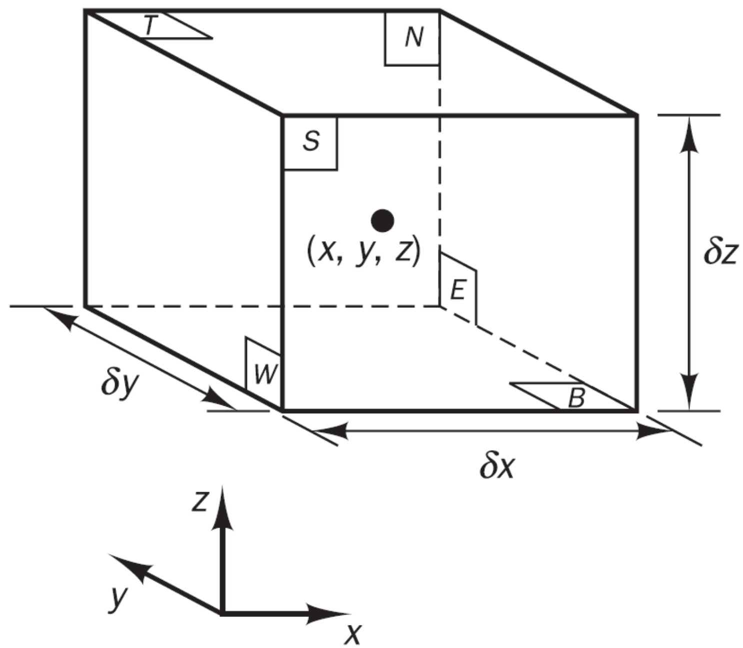
Governing Equations of Fluid Flow and Heat Transfer

- The governing equations of fluid flow represent mathematical statements of the **conservation laws of physics**.
  - ✓ **The mass of fluid is conserved.**
  - ✓ **The rate of change of momentum equals the sum of the forces on a fluid particle**
    - Newton's second law
  - ✓ **The rate of change of energy is equal to the sum of the rate of heat addition to and the rate of work done on a fluid particle**
    - First law of thermodynamics
  - ✓ **Continuum assumption**
    - Fluid flows at macroscopic length scales > 1 μm
    - The molecular structure and motions may be ignored.
    - Macroscopic properties
      - Velocity, pressure, density, temperature
      - Averaged over suitably large numbers of molecules
    - Fluid particle
      - The smallest possible element of fluid whose macroscopic properties are not influenced by individual molecules.



Governing Equations of Fluid Flow and Heat Transfer

- Control volume
  - ✓ **Six faces: N, S, E, W, T, B**
  - ✓ **The center of the element: (x, y, z)**



- Properties at the volume center

$$\rho = \rho\,(x, y, z, t)$$

$$p = p\,(x, y, z, t)$$

$$T = T\,(x, y, z, t)$$

$$\boldsymbol{u} = \boldsymbol{u}\,(x, y, z, t)$$

- Fluid properties at faces are approximated

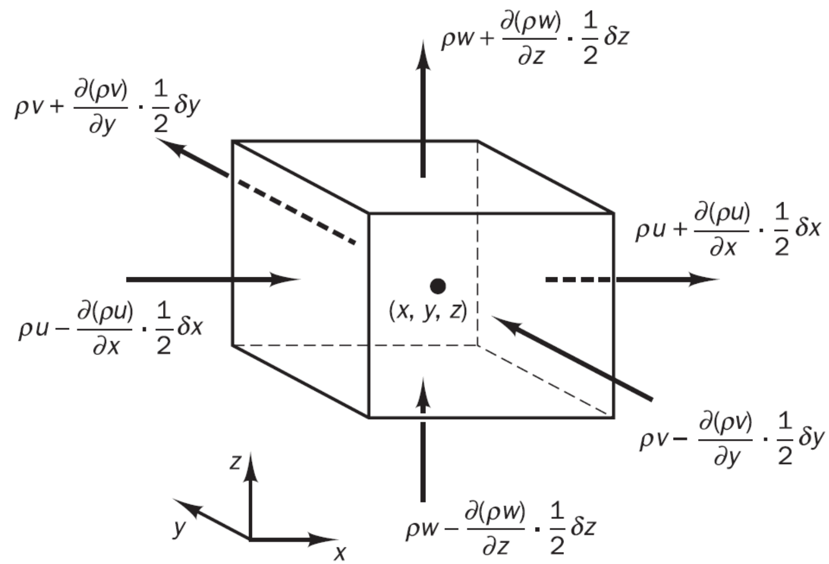
  by means of the two terms of the Taylor series.
  - ✓ The pressure at the W and E faces

$$p - \frac{\partial p}{\partial x}\frac{1}{2}\delta x \quad p + \frac{\partial p}{\partial x}\frac{1}{2}\delta x$$

Governing Equations of Fluid Flow and Heat Transfer

❖ Mass Conservation in Three Dimensions

$$\begin{pmatrix}\text{Rate of increase}\\ \text{of mass in}\\ \text{fluid element}\end{pmatrix} = \begin{pmatrix}\text{Net rate of flow}\\ \text{of mass into}\\ \text{fluid element}\end{pmatrix}$$



✓ **Rate of increase of mass**

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = \frac{\partial \rho}{\partial t}\delta x\,\delta y\,\delta z$$

✓ **Net rate of flow of mass into the element**

$$\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\,\delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\,\delta z$$
$$+ \left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z$$
$$+ \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0}$$

Mass conservation/ Continuity Eq.

Governing Equations of Fluid Flow and Heat Transfer

Mass Conservation in Three Dimensions

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0}$$

✓ **For an incompressible fluid, the density ρ is constant.**

$$\operatorname{div} \mathbf{u} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Governing Equations of Fluid Flow and Heat Transfer

- Rates of change following a fluid particle and for a fluid element
  - Lagrangian approach/ changes of properties of a fluid particle
  - **Total or substantial derivative of ϕ**
  - **ϕ: Function of the position (x,y,z), property per unit mass**

$$\frac{D\phi}{Dt}=\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}\frac{dx}{dt}+\frac{\partial\phi}{\partial y}\frac{dy}{dt}+\frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

  - **A fluid particle follows the flow, so**

$$dx/dt=u$$
$$dy/dt=v$$
$$dz/dt=w$$

  - **Hence, the substantive derivative of ϕ is given by**

$$\boxed{\frac{D\phi}{Dt}=\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial z}=\frac{\partial\phi}{\partial t}+\mathbf{u}\ .\ \mathrm{grad}\ \phi}$$

Governing Equations of Fluid Flow and Heat Transfer

- Rates of change following a fluid particle and for a fluid element

  - ✓ $\dfrac{D\phi}{Dt}$ defines the rate of change of property ϕ per unit mass.

  - ✓ $\rho\dfrac{D\phi}{Dt}$ The rate of change of property φ per unit volume

$$\boxed{\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial t} + \mathbf{u}\ .\ \mathrm{grad}\ \phi}$$

$$\boxed{\rho\frac{D\phi}{Dt} = \rho\left(\frac{\partial\phi}{\partial t} + \mathbf{u}\ .\ \mathrm{grad}\ \phi\right)}$$

  - ✓ **Eulerian approach/ changes of properties in a fluid element**
    - Far more common than Lagrangian approach
    - Develop equations for collections of fluid elements making up a region fixed in space

Governing Equations of Fluid Flow and Heat Transfer

- Rates of change following a fluid particle and for a fluid element
  - ✓ LHS of the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u})$$

  - ✓ **The generalization of these terms for an arbitrary conserved property**

$$\frac{\partial (\rho\phi)}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) \qquad \begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ per unit volume}\end{pmatrix} + \begin{pmatrix}\text{Net rate of flow of } \phi\\ \text{out of fluid element}\\ \text{per unit volume}\end{pmatrix}$$

$$\frac{\partial (\rho\phi)}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) = \rho\left[\frac{\partial \phi}{\partial t} + \mathbf{u} \,.\, \mathrm{grad}\,\phi\right] + \phi\left[\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{u})\right]$$

$$\boxed{\rho\frac{D\phi}{Dt} = \rho\left(\frac{\partial \phi}{\partial t} + \mathbf{u} \,.\, \mathrm{grad}\,\phi\right)} \qquad \boxed{\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{u}) = 0}$$

$$\frac{\partial (\rho\phi)}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) = \rho\frac{D\phi}{Dt}$$

Rate of increase of ϕ of fluid element	+	Net rate of flow of ϕ out of fluid element	=	Rate of increase of ϕ for a fluid particle

Governing Equations of Fluid Flow and Heat Transfer

- Rates of change following a fluid particle and for a fluid element
  - ✓ Relevant entries of ϕ for momentum and energy equations

x-momentum	u	$\rho \frac{Du}{Dt}$	$\frac{\partial(\rho u)}{\partial t} + \mathrm{div}(\rho u \mathbf{u})$
y-momentum	v	$\rho \frac{Dv}{Dt}$	$\frac{\partial(\rho v)}{\partial t} + \mathrm{div}(\rho v \mathbf{u})$
z-momentum	w	$\rho \frac{Dw}{Dt}$	$\frac{\partial(\rho w)}{\partial t} + \mathrm{div}(\rho w \mathbf{u})$
energy	E	$\rho \frac{DE}{Dt}$	$\frac{\partial(\rho E)}{\partial t} + \mathrm{div}(\rho E \mathbf{u})$

Governing Equations of Fluid Flow and Heat Transfer

❖ Momentum equation in three dimensions

✓ Newton's second law

Rate of increase of momentum of fluid particle	=	Sum of forces on fluid particle

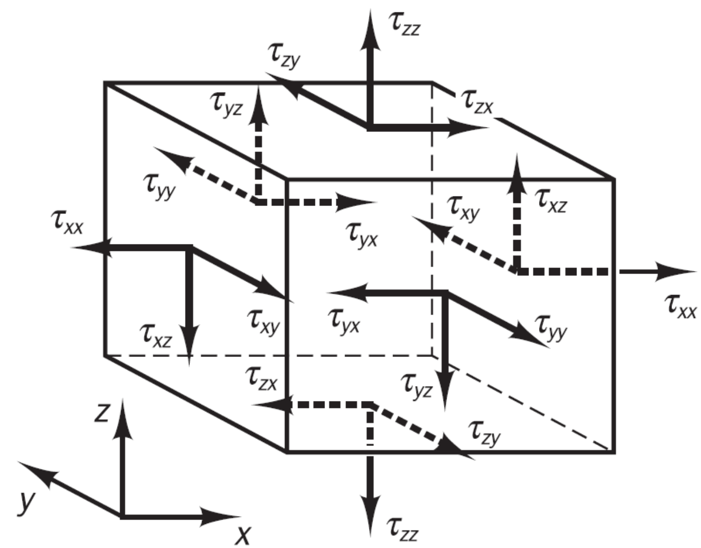
$$\rho\frac{Du}{Dt} \qquad \rho\frac{Dv}{Dt} \qquad \rho\frac{Dw}{Dt}$$

Surface forces
- Pressure force (p)
- Viscous force (τ)
- Gravity force

Body forces
- Centrifugal force
- Coriolis force
- Electromagnetic force
- Gravity force

Pressure = normal stress = p

Viscous stress = τ



τ_{ij} : stress component acts in the j-direction on a surface normal to i-direction

Momentum equation in three dimensions

- ✓ *x*-component of the forces due to pressure and viscous stress
  - On the pair of faces (E,W)

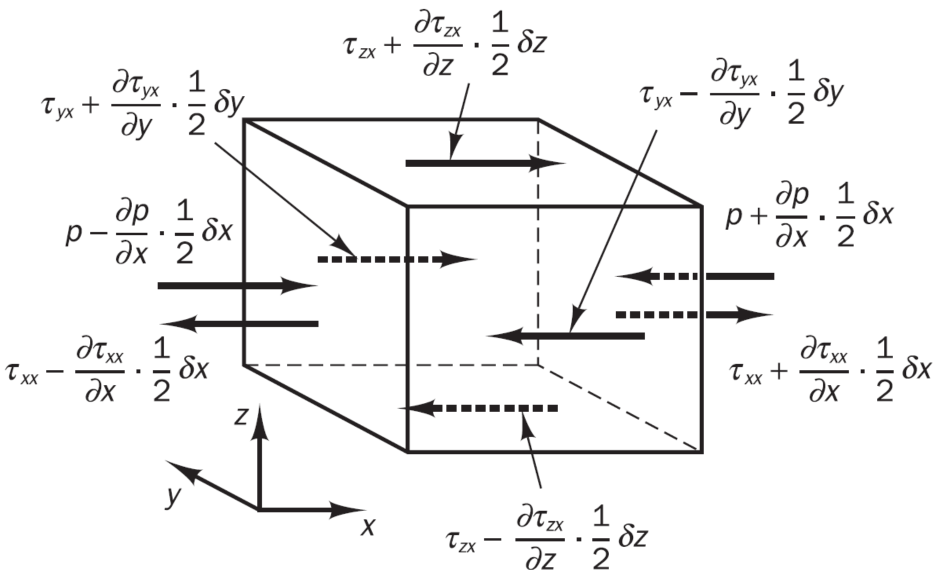
$$\left[\left(p-\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)-\left(\tau_{xx}-\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z+\left[-\left(p+\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)+\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z=\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}\right)\delta x\,\delta y\,\delta z$$

  - On the pair of faces (N,S)

$$-\left(\tau_{yx}-\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z+\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z=\frac{\partial \tau_{yx}}{\partial y}\delta x\,\delta y\,\delta z$$

  - On the pair of faces (T,B)

$$-\left(\tau_{zx}-\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y+\left(\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y=\frac{\partial \tau_{zx}}{\partial z}\delta x\,\delta y\,\delta z$$



Governing Equations of Fluid Flow and Heat Transfer

- Momentum equation in three dimensions
  - ✓ *x*-component of the forces due to pressure and viscous stress
    - Total surface force per unit volume

$$\boxed{\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}}$$

  - ✓ ***x*-component of the momentum equation**

Rate of increase of momentum of fluid particle	=	Sum of forces on fluid particle

$$\boxed{\rho\frac{Du}{Dt}=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+S_{Mx}}$$

Governing Equations of Fluid Flow and Heat Transfer

❖ Momentum equation in three dimensions

Rate of increase of momentum of fluid particle	=	Sum of forces on fluid particle

✓ *x*-**component of the momentum equation**

$$\boxed{\rho \frac{Du}{Dt} = \frac{\partial(-p+\tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx}}$$

✓ *y*-**component of the momentum equation**

$$\boxed{\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p+\tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My}}$$

✓ *z*-**component of the momentum equation**

$$\boxed{\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p+\tau_{zz})}{\partial z} + S_{Mz}}$$

Body force

$$S_{Mx} = 0$$

$$S_{My} = 0$$

$$S_{Mz} = -\rho g$$

Governing Equations of Fluid Flow and Heat Transfer

- Energy equation in three dimensions
  - **The first law of thermodynamics**

$$\rho\frac{DE}{Dt} \leftarrow \boxed{\text{Rate of increase of energy of fluid particle} = \text{Net rate of heat added to fluid particle} + \boxed{\text{Net rate of work done on fluid particle}}}$$

  - Rate of work done by surface forces

$$(F_{\text{surface forces}})(V)$$

  - **In x-direction,**

$$\boxed{\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}} \rightarrow \boxed{\left[\frac{\partial(u(-p+\tau_{xx}))}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}\right]\delta x\,\delta y\,\delta z}$$

  - **In y and z-directions,**

$$\boxed{\left[\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v(-p+\tau_{yy}))}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}\right]\delta x\,\delta y\,\delta z} \quad \boxed{\left[\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w(-p+\tau_{zz}))}{\partial z}\right]\delta x\,\delta y\,\delta z}$$

Governing Equations of Fluid Flow and Heat Transfer

- Energy equation in three dimensions
  - ✓ **Total rate of work done on the fluid particle by surface stresses**

$$\left[\frac{\partial(u(-p+\tau_{xx}))}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}\right]\delta x\,\delta y\,\delta z$$

$$+\left[\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v(-p+\tau_{yy}))}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}\right]\delta x\,\delta y\,\delta z$$

$$+\left[\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w(-p+\tau_{zz}))}{\partial z}\right]\delta x\,\delta y\,\delta z$$

$$-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}=-\mathrm{div}(p\mathbf{u})$$

$$\boxed{[-\mathrm{div}(p\mathbf{u})]+\left[\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}\right]}$$

Governing Equations of Fluid Flow and Heat Transfer

- Energy equation in three dimensions

Rate of increase of energy of fluid particle	=	Net rate of heat added to fluid particle	+	Net rate of work done on fluid particle

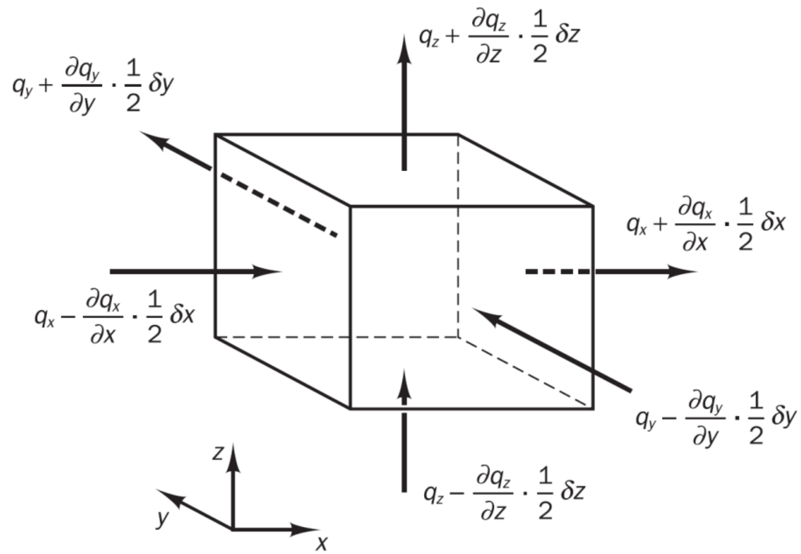
✓ Net rate of heat transfer to the fluid particle

- In x-direction,

$$\left[\left(q_x - \frac{\partial q_x}{\partial x}\frac{1}{2}\delta x\right) - \left(q_x + \frac{\partial q_x}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z = -\frac{\partial q_x}{\partial x}\,\delta x\,\delta y\,\delta z$$

- In y and z-directions,

$$-\frac{\partial q_y}{\partial y}\,\delta x\,\delta y\,\delta z \qquad -\frac{\partial q_z}{\partial z}\,\delta x\,\delta y\,\delta z$$



Governing Equations of Fluid Flow and Heat Transfer

- Energy equation in three dimensions

Rate of increase of energy of fluid particle	=	Net rate of heat added to fluid particle	+	Net rate of work done on fluid particle

✓ **Total rate of heat added to the fluid particle per unit volume**

$$-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}=-\text{div } \mathbf{q}$$

✓ **Fourier's law of heat conduction**

$$q_x=-k\frac{\partial T}{\partial x} \quad q_y=-k\frac{\partial T}{\partial y} \quad q_z=-k\frac{\partial T}{\partial z}$$

$$\mathbf{q}=-k \text{ grad } T$$ in vector form

$$\boxed{-\text{div } \mathbf{q}=\text{div}(k \text{ grad } T)}$$

Governing Equations of Fluid Flow and Heat Transfer

❖ Energy equation in three dimensions

Rate of increase of energy of fluid particle	=	Net rate of heat added to fluid particle	+	Net rate of work done on fluid particle	+ Energy Source

$$\rho\frac{DE}{Dt} \qquad \mathrm{div}(k\ \mathrm{grad}\ T) \qquad [-\mathrm{div}(p\mathbf{u})] + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right]$$

✓ **Energy equation**

$$E = i + \tfrac{1}{2}(u^2 + v^2 + w^2)$$

- E: Sum of internal energy and kinetic energy

$$\rho\frac{DE}{Dt} = -\mathrm{div}(p\mathbf{u}) + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right] + \mathrm{div}(k\ \mathrm{grad}\ T) + S_E$$

Governing Equations of Fluid Flow and Heat Transfer

- Energy equation in three dimensions

  ✓ **Kinetic energy equation**

$$\rho\frac{Du}{Dt}=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+S_{Mx} \qquad \times\ u$$

$$\rho\frac{Dv}{Dt}=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial(-p+\tau_{yy})}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+S_{My} \qquad \times\ v$$

$$\rho\frac{Dw}{Dt}=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial(-p+\tau_{zz})}{\partial z}+S_{Mz} \qquad \times\ w$$

$$\boxed{\begin{aligned}\rho\frac{D[\frac{1}{2}(u^2+v^2+w^2)]}{Dt}=&-\mathbf{u}\,.\,\mathrm{grad}\,p+u\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)\\&+v\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)\\&+w\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+\mathbf{u}\,.\,\mathbf{S}_M\end{aligned}}$$

Governing Equations of Fluid Flow and Heat Transfer

- Energy equation in three dimensions

  ✓ **Total energy equation – kinetic energy equation**

$$\rho\frac{DE}{Dt} = -\mathrm{div}(p\mathbf{u}) + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right] + \mathrm{div}(k\ \mathrm{grad}\ T) + S_E$$

$$E = i + \tfrac{1}{2}(u^2 + v^2 + w^2)$$

$$\rho\frac{D[\frac{1}{2}(u^2+v^2+w^2)]}{Dt} = -\mathbf{u}\ .\ \mathrm{grad}\ p + u\left(\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\right) + v\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}\right) + w\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z}\right) + \mathbf{u}\ .\ \mathbf{S}_M$$

  ✓ **Internal energy equation**

$$\boxed{\begin{aligned}\rho\frac{Di}{Dt} = -p\ \mathrm{div}\ \mathbf{u} + \mathrm{div}(k\ \mathrm{grad}\ T) &+ \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} \\ &+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} \\ &+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i\end{aligned}}$$

Governing Equations of Fluid Flow and Heat Transfer

- Energy equation in three dimensions
  - ✓ **For the special case of an incompressible fluid, temperature equation**

$$i = cT \qquad div\,\mathbf{u} = 0$$

$$\boxed{\begin{aligned}\rho c\frac{DT}{Dt} = \mathrm{div}(k\ \mathrm{grad}\ T) &+ \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} \\ &+ \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i\end{aligned}}$$

Governing Equations of Fluid Flow and Heat Transfer

- Energy equation in three dimensions
  - ✓ **Enthalpy equation**

$$h = i + p/\rho \qquad h_0 = h + \tfrac{1}{2}(u^2 + v^2 + w^2)$$ Total enthalpy

$$h_0 = i + p/\rho + \tfrac{1}{2}(u^2 + v^2 + w^2) = E + p/\rho$$

  - ✓ **Total enthalpy equation**

$$\rho\frac{DE}{Dt} = -\mathrm{div}(p\mathbf{u}) + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right] + \mathrm{div}(k\ \mathrm{grad}\ T) + S_E$$

$$\rho\frac{DE}{Dt} = \frac{\partial(\rho E)}{\partial t} + \mathrm{div}(\rho E\mathbf{u}) = \frac{\partial(\rho h_0)}{\partial t} - \frac{\partial p}{\partial t} + \mathrm{div}(\rho h_0\mathbf{u}) - \mathrm{div}(p\mathbf{u})$$

$$\boxed{\frac{\partial(\rho h_0)}{\partial t} + \mathrm{div}(\rho h_0\,\mathbf{u}) = \mathrm{div}(k\ \mathrm{grad}\ T) + \frac{\partial p}{\partial t} + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right] + S_h}$$

Summary

Governing Equations of Fluid Flow and Heat Transfer

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0}$$

$$\boxed{\rho \frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx}}$$

$$\boxed{\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p + \tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My}}$$

$$\boxed{\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p + \tau_{zz})}{\partial z} + S_{Mz}}$$

τ_{ij} : stress component acts in the *j*-direction on a surface normal to *i*-direction

$$\begin{aligned}\rho \frac{DE}{Dt} = &-\operatorname{div}(p\mathbf{u}) + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x}\right. \\ &\left. + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right] \\ &+ \operatorname{div}(k \operatorname{grad} T) + S_E\end{aligned}$$

$$\begin{aligned}\rho \frac{Di}{Dt} = &-p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} \\ &+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} \\ &+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i\end{aligned}$$

Equations of State

- Thermodynamic variables

$$\rho, p, i \text{ and } T$$

  - ✓ **Assumption of thermodynamic equilibrium**

- Equations of the state
  - ✓ **Relate two state variables to the other variables**

$$p = p(\rho, T) \qquad i = i(\rho, T)$$

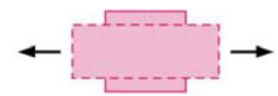
- Compressible fluids
  - ✓ **EOS provides the linkage between the energy equation and other governing equations.**

- Incompressible fluids
  - ✓ **No linkage between the energy equation and the others.**
  - ✓ **The flow field can be solved by considering mass and momentum equations.**

Navier-Stokes Equations for a Newtonian Fluid

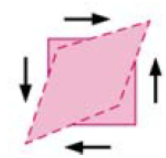
- Viscous stresses τ_{ij} in momentum and energy equations
  - ✓ Viscous stresses can be expressed as functions of the local deformation rate (or strain rate).
  - ✓ In 3D flows the local rate of deformation is composed of
    - the linear deformation rate
    - the volumetric deformation rate.
  - ✓ All gases and many liquids are **isotropic.**

- The rate of linear deformation of a fluid element
  - ✓ **Nine components in 3D**
  - ✓ **Linear elongating deformation** $s_{xx} = \frac{\partial u}{\partial x} \quad s_{yy} = \frac{\partial v}{\partial y} \quad s_{zz} = \frac{\partial w}{\partial z}$

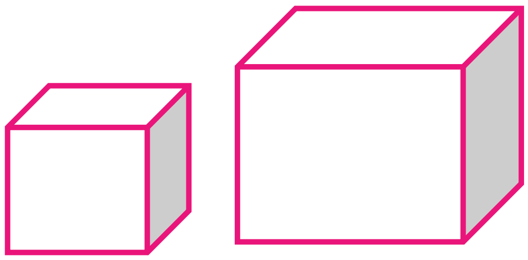


  - ✓ **Shearing linear deformation components**

$$s_{xy} = s_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad s_{xz} = s_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad s_{yz} = s_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$



- The rate of volume deformation of a fluid element



$$e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div } \mathbf{u}$$

Navier-Stokes Equations for a Newtonian Fluid

- Viscous stresses τ_{ij} in momentum and energy equations

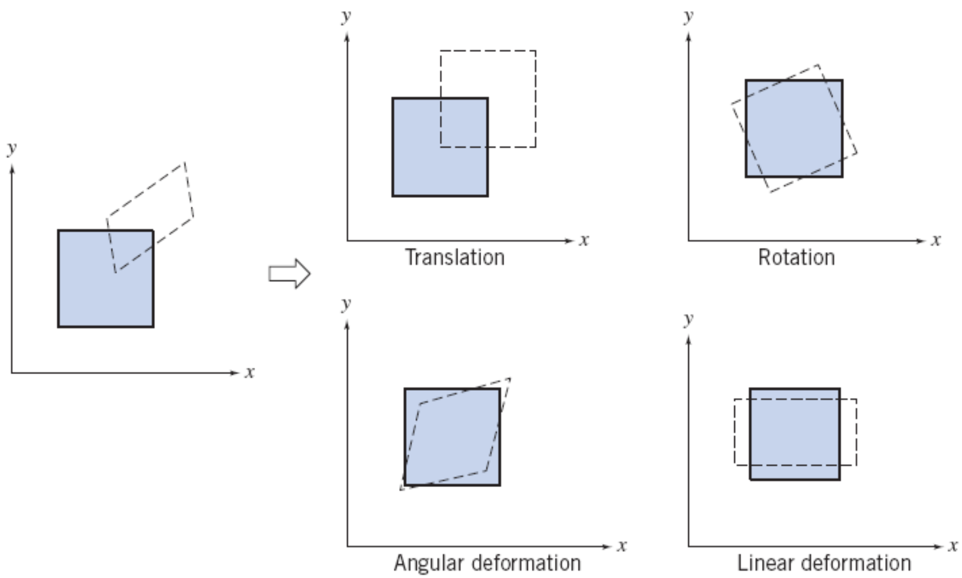


Fig. 5.5 Pictorial representation of the components of fluid motion.

All gases and many liquids are isotropic.

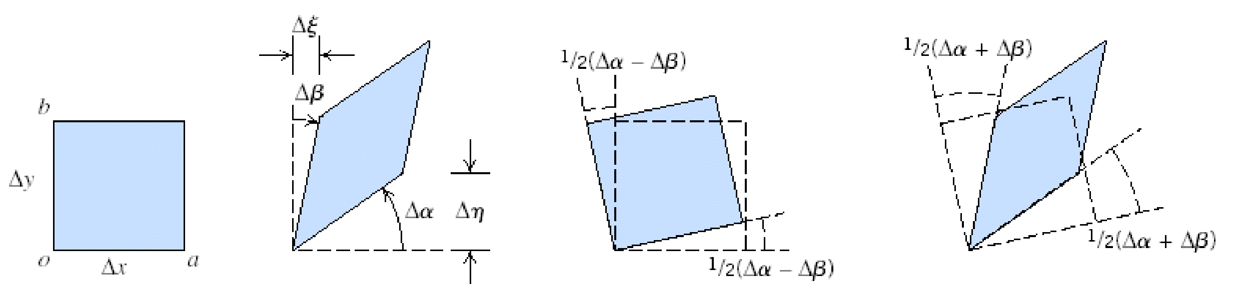


Fig. 5.7 Rotation and angular deformation of perpendicular line segments in a two-dimensional flow.

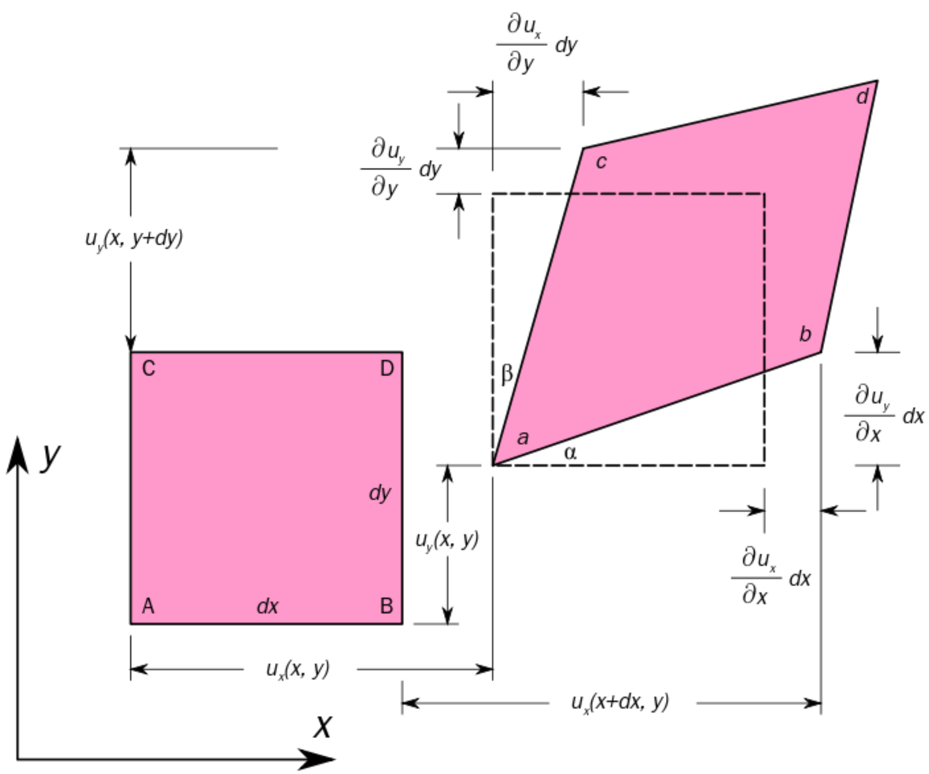
Navier-Stokes Equations for a Newtonian Fluid

❖ Viscous stresses τ_{ij} in momentum and energy equations

$$\tan\alpha = \frac{\dfrac{\partial u_y}{\partial x}dx}{dx + \dfrac{\partial u_x}{\partial x}dx} = \frac{\dfrac{\partial u_y}{\partial x}}{1 + \dfrac{\partial u_x}{\partial x}}$$

$$\tan\beta = \frac{\dfrac{\partial u_x}{\partial y}dy}{dy + \dfrac{\partial u_y}{\partial y}dy} = \frac{\dfrac{\partial u_x}{\partial y}}{1 + \dfrac{\partial u_y}{\partial y}}$$

$$\tan\alpha \approx \frac{\partial u_y}{\partial x} \approx \alpha \qquad \tan\beta \approx \frac{\partial u_x}{\partial y} \approx \beta$$



$$\gamma_{xy} = \alpha + \beta$$

The rate at which
two sides close toward each other

$$s_{xy} = \frac{1}{2}\gamma_{xy}$$

Navier-Stokes Equations for a Newtonian Fluid

- Newtonian fluid
  - ✓ Viscous stresses are proportional to the rates of deformation.
  - ✓ **Two constants of proportionality**
    - Dynamic viscosity (μ): to relate stresses to linear deformations
    - Second viscosity (λ): to relate stresses to volumetric deformation

$$(\tau_{ij})_{linear} = 2\mu s_{ij}$$

$$(\tau_{ii})_{volume} = \lambda(e_{xx} + e_{yy} + e_{zz})$$

$$e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div}\,\mathbf{u}$$

- Viscous stress components

$$s_{xx} = \frac{\partial u}{\partial x} \quad s_{yy} = \frac{\partial v}{\partial y} \quad s_{zz} = \frac{\partial w}{\partial z} \qquad s_{xy} = s_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad s_{xz} = s_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad s_{yz} = s_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\,\text{div}\,\mathbf{u} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda\,\text{div}\,\mathbf{u} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda\,\text{div}\,\mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

  - ✓ **Second viscosity**
    - For gases: $\lambda = -\frac{2}{3}\mu$
    - For liquid: $div\ \mathbf{u} = 0$

Navier-Stokes Equations for a Newtonian Fluid

- Momentum equations

$$\rho \frac{Du}{Dt} = \frac{\partial(-p+\tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx}$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + S_{Mx}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + S_{My}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} + \lambda \operatorname{div} \mathbf{u}\right] + S_{Mz}$$

Navier-Stokes Equations for a Newtonian Fluid

- Rearrangement

$$\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\,\mathrm{div}\,\mathbf{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)+\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial x}(\lambda\,\mathrm{div}\,\mathbf{u})\right]$$

$$=\mathrm{div}(\mu\,\mathrm{grad}\,u)+[s_{Mx}]$$

- N.-S. equations can be written as follows with modified source terms; $S_M = S_M + [s_M]$

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\mathrm{div}(\mu\,\mathrm{grad}\,u)+S_{Mx}$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\mathrm{div}(\mu\,\mathrm{grad}\,v)+S_{My}$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+\mathrm{div}(\mu\,\mathrm{grad}\,w)+S_{Mz}$$

Navier-Stokes Equations for a Newtonian Fluid

- For incompressible fluids with constant μ

$$[s_{Mx}] = \left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}(\lambda \operatorname{div} \mathbf{u})\right]$$

$$= \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)\right] = \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)\right]$$

$$= \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = 0$$

Navier-Stokes Equations for a Newtonian Fluid

❖ Internal energy equation

$$\rho \frac{Di}{Dt} = -p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z}$$
$$+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z}$$
$$+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u} \quad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda \operatorname{div} \mathbf{u} \quad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda \operatorname{div} \mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\rho \frac{Di}{Dt} = -p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) + \Phi + S_i$$

✓ **Dissipation function Φ**

- Always positive
- Source of internal energy due to deformation work on the fluid particle.
- Mechanical energy is converted into internal energy or heat.

$$\Phi = \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\} + \lambda(\operatorname{div} \mathbf{u})^2$$

Conservative form of the governing equations of fluid flow

Mass $$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0$$

x-momentum $$\frac{\partial (\rho u)}{\partial t} + div(\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + div(\mu\, grad\, u) + S_{Mx}$$

y-momentum $$\frac{\partial (\rho v)}{\partial t} + div(\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + div(\mu\, grad\, v) + S_{My}$$

z-momentum $$\frac{\partial (\rho w)}{\partial t} + div(\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + div(\mu\, grad\, w) + S_{Mz}$$

Internal energy $$\frac{\partial (\rho i)}{\partial t} + div(\rho i \mathbf{u}) = -p\, div\, \mathbf{u} + div(k\, grad\, T) + \Phi + S_i$$

\+ EOS $$u, v, w, p, i, \rho, T$$

This system is mathematically closed!