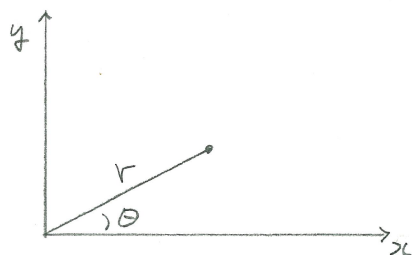


Polar coordinate

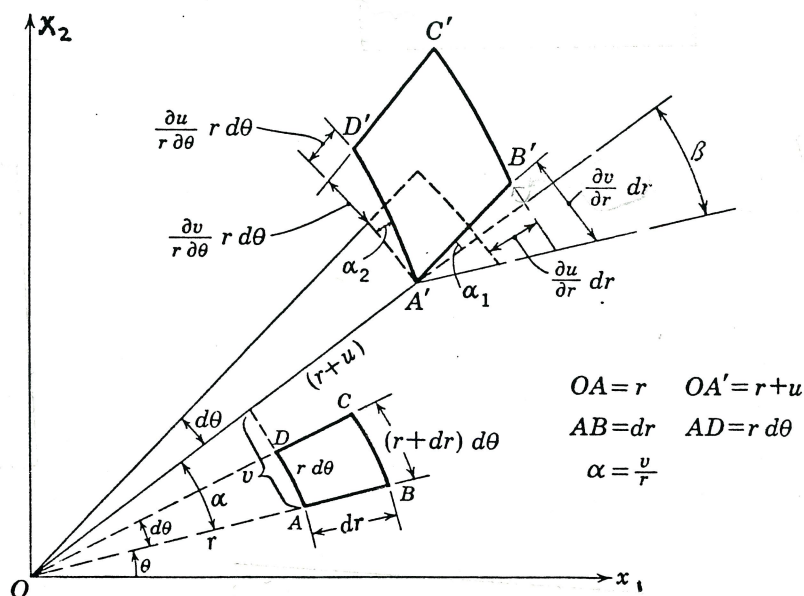


$$x = r \cos \theta$$

$$y = r \sin \theta$$

u = displacement in r direction

v = displacement in θ direction



Radial Normal strain

$$\epsilon_r = \frac{A'B' - AB}{AB}$$

$$A'B' = (1 + \epsilon_r) AB = (1 + \epsilon_r) dr$$

$$(A'B')^2 = (1 + \epsilon_r)^2 dr^2$$

$$= \left(dr + \frac{\partial u}{\partial r} dr \right)^2 + \left(\alpha dr \right)^2$$

Eliminating relatively small terms,

$$\epsilon_r = \frac{\partial u}{\partial r}$$

Hoop strain

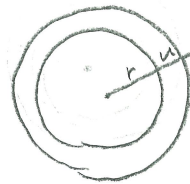
$$\epsilon_\theta = \frac{A'D' - AD}{AD}$$

$$(AD')^2 = (1 + \epsilon_\theta)^2 (r d\theta)^2$$

$$= \left[(r+u) d\theta + \frac{1}{r} \frac{\partial v}{\partial \theta} r d\theta \right]^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} r d\theta \right)^2$$

$$2\epsilon_\theta + \epsilon_\theta^2 = 2 \left[\frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \left(\frac{u}{r} \right) \left(\frac{1}{r} \frac{\partial v}{\partial \theta} \right) \right] + \left(\frac{u}{r} \right)^2 + \left(\frac{1}{r} \frac{\partial v}{\partial \theta} \right)^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2$$

$$\epsilon_\theta = \left[\frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right]$$



$$\epsilon_\theta = \frac{2\pi u}{2\pi r} = \frac{u}{r}$$

Shear strain

$$\gamma_{r\theta} = (\alpha_1 + \alpha_2)$$

$$\text{Since } \alpha_1, \alpha_2 \text{ are small, } \alpha_1 + \beta \approx \tan(\alpha_1 + \beta) = \frac{\frac{\partial v}{\partial r} dr}{(1 + \frac{\partial u}{\partial r}) dr} \approx \frac{\partial v}{\partial r}$$

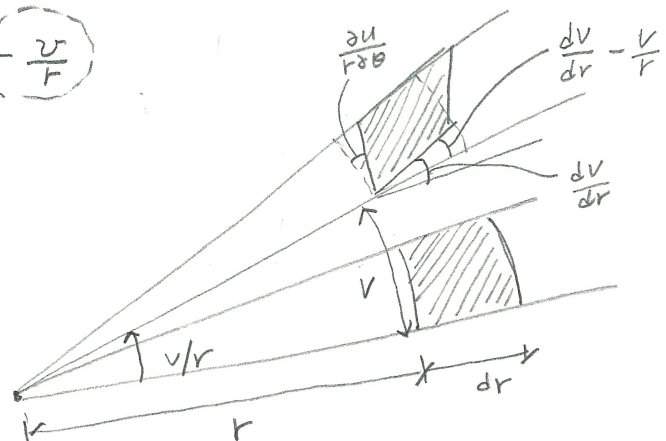
$$\text{Thus } \alpha_1 = \frac{\partial v}{\partial r} - \beta = \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$\begin{aligned} \alpha_2 = \tan \alpha_2 &= \frac{\left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right) r d\theta}{\left[(r+u) d\theta + \frac{1}{r} \frac{\partial v}{\partial \theta} r d\theta \right]} \\ &= \frac{\left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right) r d\theta}{\left(1 + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) r d\theta} \approx \frac{1}{r} \frac{\partial u}{\partial \theta} \end{aligned}$$

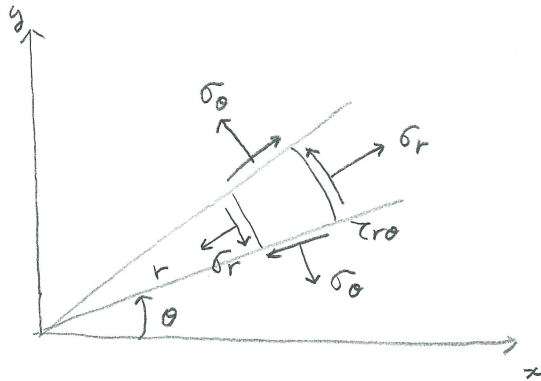
$$\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial u}{\partial \theta}$$

$$\begin{cases} \epsilon_r = \frac{\partial u}{\partial r} \\ \epsilon_\theta = \frac{\partial v}{r \partial \theta} \left(+ \frac{u}{r} \right) \\ \gamma_{r\theta} = \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} \left(- \frac{v}{r} \right) \end{cases} \quad \begin{pmatrix} dx \rightarrow dr \\ dy \rightarrow r d\theta \end{pmatrix}$$

() $\Rightarrow$ extra consideration

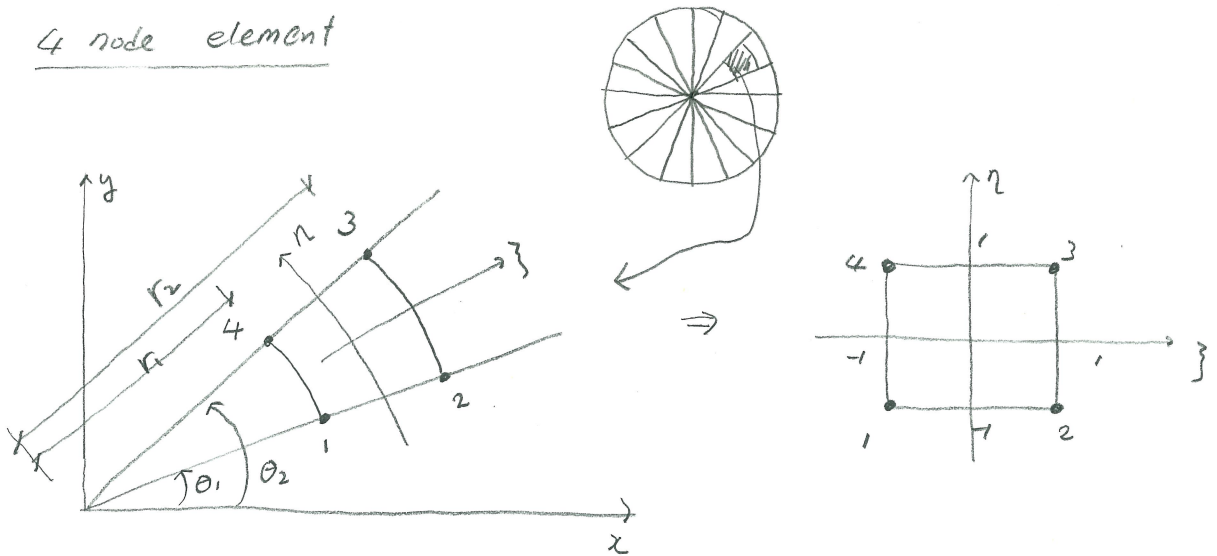


Stress - Strain relationship



$$\underline{\sigma} = \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{bmatrix} = \underline{D} \underline{\epsilon} = \underline{D} \begin{bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{bmatrix}$$

4 node element



$$u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 = \sum f_i u_i$$

$$v = \sum v_i f_i$$

$$f_1 = \frac{1}{4} (1-\zeta)(1-\eta)$$

$$f_2 = \frac{1}{4} (1+\zeta)(1-\eta)$$

$$f_3 = \frac{1}{4} (1+\zeta)(1+\eta)$$

$$f_4 = \frac{1}{4} (1-\zeta)(1+\eta)$$

$$\begin{cases} r = \frac{1}{2} r_1 (1-\zeta) + \frac{1}{2} r_2 (1+\zeta) & r = \text{function of } \zeta \\ \theta = \frac{1}{2} \theta_1 (1-\eta) + \frac{1}{2} \theta_2 (1+\eta) & \theta = \text{function of } \eta \end{cases}$$

$$\begin{cases} r = \frac{1}{2} (r_1 + r_2) + \frac{1}{2} (r_2 - r_1) \zeta \\ \theta = \frac{1}{2} (\theta_1 + \theta_2) + \frac{1}{2} (\theta_2 - \theta_1) \eta \end{cases}$$

$$\frac{dr}{d\zeta} = \frac{1}{2} (r_2 - r_1) = \frac{1}{2} \Delta r$$

$$\frac{d\theta}{d\eta} = \frac{1}{2} (\theta_2 - \theta_1) = \frac{1}{2} \Delta \theta$$

Strain - generic displacement

$$\begin{bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & \frac{\partial}{r \partial \theta} \\ \frac{\partial}{r \partial \theta} & (\frac{\partial}{\partial r} - \frac{1}{r}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\underline{\underline{\epsilon}} = \underline{\underline{d}} \underline{\underline{u}}$$

Strain - nodal displacement

$$\underline{\underline{\epsilon}} = \underline{\underline{d}} \underline{\underline{f}} \underline{\underline{\delta}}$$

$$= \underline{\underline{B}} \underline{\underline{\delta}}$$

$$\underline{B} = \begin{bmatrix} \frac{\partial f_1}{\partial r} & 0 & \frac{\partial f_4}{\partial r} & 0 \\ \frac{f_1}{r} & \frac{\partial f_1}{r \partial \theta} & \frac{f_4}{r} & \frac{\partial f_4}{r \partial \theta} \\ \frac{\partial f_1}{r \partial \theta} & \left(\frac{\partial f_1}{\partial r} - \frac{f_1}{r} \right) & \frac{\partial f_4}{r \partial \theta} & \left(\frac{\partial f_4}{\partial r} - \frac{f_4}{r} \right) \end{bmatrix}$$

$$\underline{K} = \int \underline{B}^T \underline{E} \underline{B} dV$$

$$= \pm \int \underline{B}^T \underline{E} \underline{B} dA$$

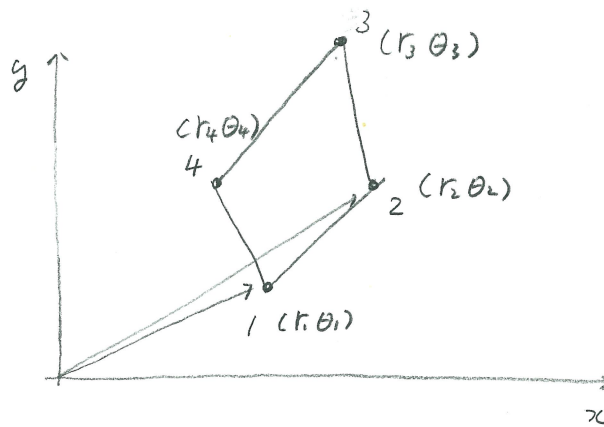
$$dA = r dr d\theta$$

$$dr = \frac{1}{2} \Delta r d\zeta, \quad d\theta = \frac{1}{2} \Delta \theta d\eta$$

$$\underline{K} = \pm \frac{\Delta r \Delta \theta}{4} \int_{-1}^1 \int_{-1}^1 \underline{B}^T \underline{E} \underline{B} r d\zeta d\eta$$

$$\left\{ \begin{aligned} \frac{\partial f_i}{\partial r} &= \frac{\partial f_i}{\partial \zeta} \frac{\partial \zeta}{\partial r} + \frac{\partial f_i}{\partial \eta} \frac{\partial \eta}{\partial r} \\ &= \frac{\partial f_i}{\partial \zeta} \frac{2}{\Delta r} \\ \frac{\partial f_i}{\partial \theta} &= \frac{\partial f_i}{\partial \zeta} \frac{\partial \zeta}{\partial \theta} + \frac{\partial f_i}{\partial \eta} \frac{\partial \eta}{\partial \theta} \\ &= \frac{\partial f_i}{\partial \eta} \frac{2}{\Delta \theta} \end{aligned} \right.$$

Isoparametric 4 node Element



$$\begin{cases} r = \sum_i^4 f_i r_i \\ \theta = \sum_i f_i \theta_i \end{cases} \quad \begin{cases} u = \sum f_i u_i \\ v = \sum f_i v_i \end{cases}$$

$$\begin{bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & \frac{\partial}{r \partial \theta} \\ \frac{\partial}{r \partial \theta} & (\frac{\partial}{\partial r} - \frac{1}{r}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_i}{\partial r} & 0 & \dots & \dots \\ \frac{f_i}{r} & \frac{\partial f_i}{r \partial \theta} & \dots & \dots \\ \frac{\partial f_i}{r \partial \theta} & (\frac{\partial f_i}{\partial r} - \frac{f_i}{r}) & \dots & \dots \end{bmatrix}$$

$$\frac{\partial f_i}{\partial r} = \frac{\partial f_i}{\partial \xi} \frac{\partial \xi}{\partial r} + \frac{\partial f_i}{\partial \eta} \frac{\partial \eta}{\partial r}$$

$$\frac{\partial f_i}{\partial \theta} = \frac{\partial f_i}{\partial \xi} \frac{\partial \xi}{\partial \theta} + \frac{\partial f_i}{\partial \eta} \frac{\partial \eta}{\partial \theta}$$

$$\begin{bmatrix} \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial r}{\partial \zeta} & \frac{\partial \theta}{\partial \zeta} \\ \frac{\partial r}{\partial \eta} & \frac{\partial \theta}{\partial \eta} \end{bmatrix}}_{\underline{J}} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix}$$

$$\frac{\partial r}{\partial \zeta} = \sum f_{i,\zeta} r_i \quad \frac{\partial \theta}{\partial \zeta} = \sum f_{i,\zeta} \theta_i$$

$$\frac{\partial r}{\partial \eta} = \sum f_{i,\eta} r_i \quad \frac{\partial \theta}{\partial \eta} = \sum f_{i,\eta} \theta_i$$

$$\frac{\partial f_i}{\partial r} = J_{11}^* \frac{\partial f_i}{\partial \zeta} + J_{12}^* \frac{\partial f_i}{\partial \eta}$$

$$\frac{\partial f_i}{\partial \theta} = J_{21}^* \frac{\partial f_i}{\partial \zeta} + J_{22}^* \frac{\partial f_i}{\partial \eta}$$

$$\underline{K} = \int \underline{B}^T \underline{E} \underline{B} dV$$

$$= \pm \int \underline{B}^T \underline{E} \underline{B} dA = \pm \int \underline{B}^T \underline{E} \underline{B} r dr d\theta$$

$$= \pm \int \underline{B}^T \underline{E} \underline{B} r |J| d\zeta d\eta$$

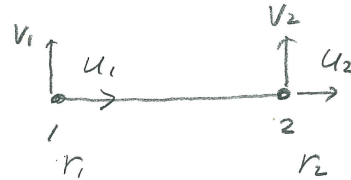
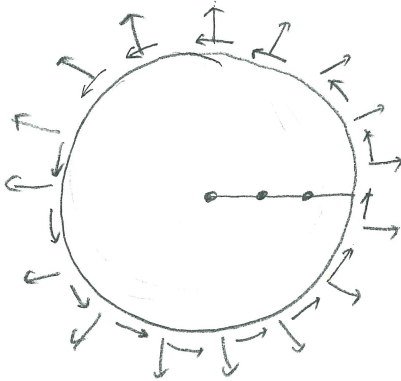
$$\text{if } r_4 = r_1 \quad \theta_1 = \theta_2$$

$$r_3 = r_2 \quad \theta_4 = \theta_3$$

$$\underline{J} = \begin{bmatrix} \frac{r_2 - r_1}{2} & 0 \\ 0 & \frac{\theta_2 - \theta_1}{2} \end{bmatrix}$$

$$|\underline{J}| = \frac{1}{4} r \Delta \theta$$

Axisymmetric Geometry and loadings



$$u = f_1 u_1 + f_2 u_2 \quad v = f_1 v_1 + f_2 v_2$$

$$r = f_1 r_1 + f_2 r_2$$

$$f_1 = \frac{1}{2}(1-\xi) \quad f_2 = \frac{1}{2}(1+\xi)$$

$$r = \frac{1}{2}(1-\xi)r_1 + \frac{1}{2}(1+\xi)r_2$$

$$= \frac{1}{2}(r_1 + r_2) + \frac{1}{2}(r_2 - r_1)\xi$$

$$\frac{dr}{d\xi} = \frac{1}{2}(r_2 - r_1) = \frac{1}{2}\Delta r$$

Strain - Green's displ.

$$\begin{bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & \frac{\partial}{r\partial\theta} \\ \frac{\partial}{r\partial\theta} & (\frac{\partial}{\partial r} - \frac{1}{r}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

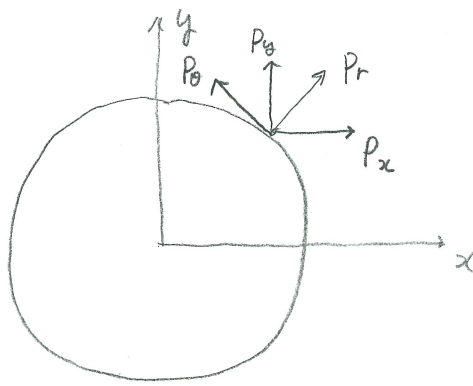
$$\underline{B} = \left[\begin{array}{cc|cc} \frac{\partial f_1}{\partial r} & 0 & \frac{\partial f_2}{\partial r} & 0 \\ f_1/r & \frac{\partial f_1}{r\partial\theta} & f_2/r & \frac{\partial f_2}{r\partial\theta} \\ \frac{\partial f_1}{r\partial\theta} & (\frac{\partial f_1}{\partial r} - \frac{f_1}{r}) & \frac{\partial f_2}{r\partial\theta} & (\frac{\partial f_2}{\partial r} - \frac{f_2}{r}) \end{array} \right]$$

$$\frac{\partial f_1}{\partial r} = \frac{\partial f_1}{\partial \beta} \frac{\partial \beta}{\partial r} = \frac{\partial f_1}{\partial \beta} \cdot \frac{2}{\Delta r}$$

$$K = \pm \int \beta^T \underline{\underline{E}} \beta \, dr = \pm \int_{-1}^1 \beta^T \underline{\underline{E}} \beta \, \frac{1}{2} \Delta r \, d\beta$$

(plane stress)

Axisymmetric Geometry subject to Nonaxisymmetric loadings



$$P_r = P_x \cos \theta + P_y \sin \theta$$

$$P_\theta = -P_x \sin \theta + P_y \cos \theta$$

★ Any load combinations can be described by Fourier Series

$$P_r = \sum_{m=0}^{\infty} (P_{rm} \cos m\theta + \overline{P_{rm}} \sin m\theta)$$

$$P_\theta = \sum_{m=0}^{\infty} (-P_{\theta m} \sin m\theta + \overline{P_{\theta m}} \cos m\theta)$$

$$\left\{ \begin{array}{l} P_{r0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r \, d\theta \\ P_{rm} = \frac{1}{\pi} \int_{-\pi}^{\pi} P_r \cos m\theta \, d\theta \\ \overline{P_{rm}} = \frac{1}{\pi} \int_{-\pi}^{\pi} P_r \sin m\theta \, d\theta \\ \overline{P_{r0}} = 0 \end{array} \right.$$

$$\int_{-\pi}^{\pi} \cos m\theta \cos n\theta \, d\theta = 0 \quad m \neq n$$

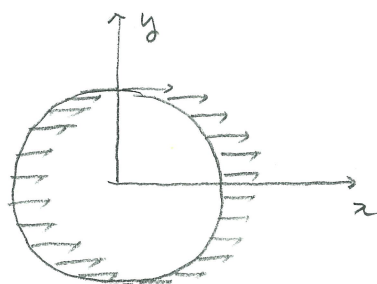
$$\int_{-\pi}^{\pi} \cos^2 m\theta \, d\theta = \begin{cases} \pi & m > 0 \\ 2\pi & m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin m\theta \sin n\theta \, d\theta = 0 \quad m \neq n$$

$$\int_{-\pi}^{\pi} \sin^2 m\theta \, d\theta = \begin{cases} \pi & m > 0 \\ 0 & m = 0 \end{cases}$$

$$\left\{ \begin{array}{l} \overline{P_{\theta 0}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{\theta} d\theta \\ \overline{P_{\theta m}} = \frac{1}{\pi} \int_{-\pi}^{\pi} P_{\theta} \cos m\theta d\theta \\ P_{\theta m} = -\frac{1}{\pi} \int_{-\pi}^{\pi} P_{\theta} \sin m\theta d\theta \\ P_{\theta 0} = 0 \end{array} \right.$$

For example, $P_x = \delta$ $P_y = 0$

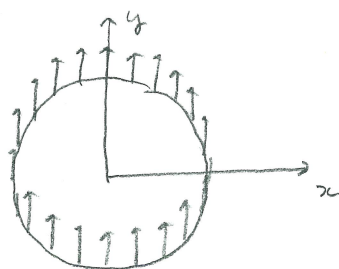


$$\left\{ \begin{array}{l} P_r = \delta \cos \theta \\ P_{\theta} = -\delta \sin \theta \end{array} \right.$$

$$P_{r1} = \delta, \quad \overline{P_{rm}} = 0, \quad P_{r0} = 0, \quad (P_{rm} = 0 \text{ } m \neq 1)$$

$$P_{\theta 1} = \delta, \quad \overline{P_{\theta m}} = 0, \quad (P_{\theta m} = 0 \text{ } m \neq 1)$$

$$P_x = 0, \quad P_y = \delta$$



$$\left\{ \begin{array}{l} P_r = \delta \sin \theta \\ P_{\theta} = \delta \cos \theta \end{array} \right.$$

$$\left. \begin{array}{l} \overline{P_{r1}} = \delta \\ \overline{P_{\theta 1}} = \delta \end{array} \right\} \text{others} = 0$$

★ For linear system,

$$\text{INPUT} \quad \begin{cases} P_r = \sum_{m=0}^{\infty} (P_{rm} \cos m\theta + \overline{P_{rm}} \sin m\theta) \\ P_\theta = \sum (-P_{\theta m} \sin m\theta + \overline{P_{\theta m}} \cos m\theta) \end{cases}$$

$$\text{OUTPUT} \quad \begin{cases} U = \sum (U_m \cos m\theta + \overline{U_m} \sin m\theta) \\ V = \sum (-V_m \sin m\theta + \overline{V_m} \cos m\theta) \end{cases}$$

Also

$$\begin{array}{l} \text{INPUT} \quad \begin{cases} P_{rm} \cos m\theta \\ -P_{\theta m} \sin m\theta \end{cases} \Rightarrow \text{symmetric w.r.t } x\text{-axis} \\ \downarrow \\ \text{output} \quad \begin{cases} U_m \cos m\theta \\ -V_m \sin m\theta \end{cases} \Rightarrow \text{''} \end{array}$$

$$\begin{array}{l} \text{INPUT} \quad \begin{cases} \overline{P_{rm}} \sin m\theta \\ \overline{P_{\theta m}} \cos m\theta \end{cases} \Rightarrow \text{symmetric w.r.t } y\text{-axis} \\ \downarrow \\ \text{output} \quad \begin{cases} \overline{U_m} \sin m\theta \\ \overline{V_m} \cos m\theta \end{cases} \Rightarrow \text{''} \end{array}$$

For response which is symmetric w.r.t x-axis

$$\underline{U_m} = \begin{bmatrix} U_m \cos m\theta \\ -V_m \sin m\theta \end{bmatrix}$$

★ Strain - General Displacement

$$\varepsilon_r = \frac{\partial u}{\partial r} = \frac{\partial u_m}{\partial r} \cos m\theta$$

$$\varepsilon_\theta = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} = \frac{u_m - m v_m}{r} \cos m\theta$$

$$\gamma_{r\theta} = \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \left[-\frac{\partial v_m}{\partial r} + \frac{v_m - m u_m}{r} \right] \sin m\theta$$

★ Using Principle of Virtual Displacement

$$\int \delta \underline{\varepsilon}^T \underline{\sigma} dV = \int \delta \underline{\varepsilon}^T \underline{E} \underline{\varepsilon} dV$$

$$\underline{E} \underline{\varepsilon} = \begin{bmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} E_1 \varepsilon_r + E_2 \varepsilon_\theta \\ E_2 \varepsilon_r + E_1 \varepsilon_\theta \\ E_3 \gamma_{r\theta} \end{bmatrix}$$

$$\delta \underline{\varepsilon}^T \underline{E} \underline{\varepsilon} = \begin{bmatrix} \delta \varepsilon_r & \delta \varepsilon_\theta & \delta \gamma_{r\theta} \end{bmatrix} \begin{bmatrix} E_1 \varepsilon_r + E_2 \varepsilon_\theta \\ E_2 \varepsilon_r + E_1 \varepsilon_\theta \\ E_3 \gamma_{r\theta} \end{bmatrix}$$

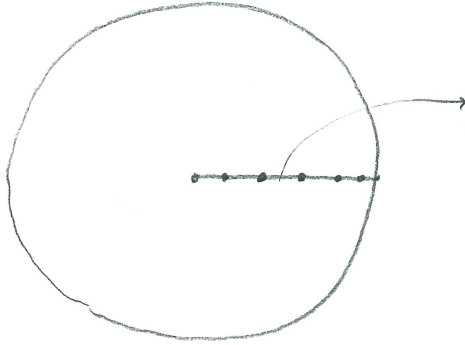
$$= E_1 \underbrace{\delta \varepsilon_r \varepsilon_r}_{\cos^2 m\theta} + E_2 \underbrace{\delta \varepsilon_\theta \varepsilon_r}_{\cos^2 m\theta} + E_2 \underbrace{\delta \varepsilon_r \varepsilon_\theta}_{\cos^2 m\theta} + E_1 \underbrace{\delta \varepsilon_\theta \varepsilon_\theta}_{\cos^2 m\theta} + E_3 \underbrace{\delta \gamma_{r\theta} \gamma_{r\theta}}_{\sin^2 m\theta}$$

$$\star \int \delta \underline{\varepsilon}^T \underline{\sigma} dV = \pm \int_0^{2\pi} \int \delta \underline{\varepsilon}^T \underline{\sigma} r dr d\theta$$

$$\int_0^{2\pi} \cos^2 m\theta d\theta = \begin{cases} \pi & m \neq 0 \\ 2\pi & m = 0 \end{cases} \quad \int_0^{2\pi} \sin^2 m\theta d\theta = \begin{cases} \pi & m \neq 0 \\ 0 & m = 0 \end{cases}$$

$$\delta \underline{\varepsilon}^T \underline{P} \Rightarrow \begin{cases} \int \cos^2 m\theta \\ \int \sin^2 m\theta \end{cases}$$

★ $\Rightarrow \int \delta \underline{\underline{\epsilon}}^T \underline{\underline{E}} \underline{\underline{\epsilon}} dV = \text{function of only "r" which is independent of } \theta$



$$r = \underbrace{\frac{1}{2}(1-\xi)}_{f_1} r_1 + \underbrace{\frac{1}{2}(1+\xi)}_{f_2} r_2$$

$$r = \frac{1}{2}(r_1 + r_2) + \frac{1}{2}(r_2 - r_1)\xi$$

$$u = f_1 u_1 + f_2 u_2$$

$$f_1 = \frac{1}{2}(1-\xi)$$

$$v = f_1 v_1 + f_2 v_2$$

$$f_2 = \frac{1}{2}(1+\xi)$$

for m th displacement coefficient

$$u_m = f_1 u_{1m} + f_2 u_{2m}$$

$$v_m = f_1 v_{1m} + f_2 v_{2m}$$

strain - nodal displacement

$$\underline{\underline{\epsilon}}_m = \underline{\underline{B}}_m \underline{\underline{q}}_m$$

$$\underline{\underline{B}}_m = \begin{bmatrix} \frac{\partial f_1}{\partial r} & 0 & \frac{\partial f_2}{\partial r} & 0 \\ \frac{f_1}{r} & -\frac{m}{r} f_1 & \frac{1}{r} f_2 & -\frac{m}{r} f_2 \\ -\frac{m}{r} f_1 & (\frac{1}{r} f_1 - \frac{df_1}{dr}) & -\frac{m}{r} f_2 & (\frac{1}{r} f_2 - \frac{df_2}{dr}) \end{bmatrix}$$

$$\frac{\partial f_i}{\partial r} = \frac{\partial f_i}{\partial \xi} \frac{d\xi}{dr}$$

$$K_m = \pi \int_{r_1}^{r_2} \underline{B}_m^T \underline{D} \underline{B}_m r dr \quad m > 0$$

Actually

$$\begin{cases} \underline{K}_m \begin{bmatrix} U_m \\ -V_m \end{bmatrix} = \begin{bmatrix} P_{rm} \\ -P_{\theta m} \end{bmatrix} \\ \underline{\bar{K}}_m \begin{bmatrix} \bar{U}_m \\ \bar{V}_m \end{bmatrix} = \begin{bmatrix} \bar{P}_{rm} \\ \bar{P}_{\theta m} \end{bmatrix} \end{cases} \quad \text{Solve } U_m, -V_m, \bar{U}_m, \bar{V}_m$$

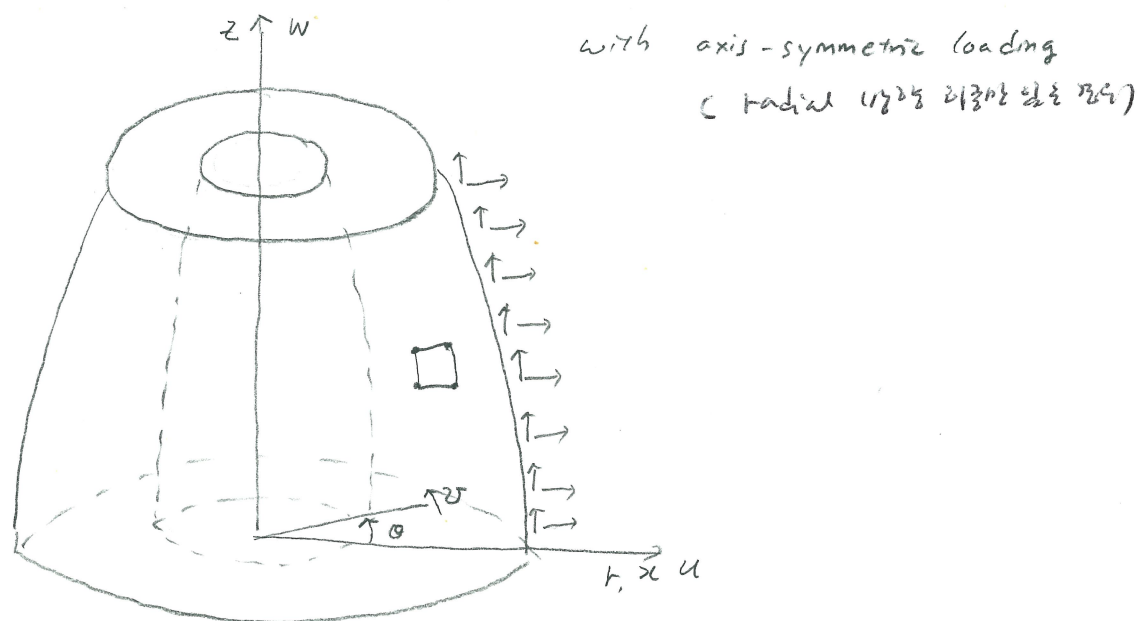
$$\star \left\{ \begin{aligned} u &= \sum_{m=0}^{\infty} (U_m \cos m\theta + \bar{U}_m \sin m\theta) \\ v &= \sum (-V_m \sin m\theta + \bar{V}_m \cos m\theta) \end{aligned} \right\} \Rightarrow \text{solution}$$

$$\underline{B}_m = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial r} & 0 & \frac{\partial f_2}{\partial r} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial f_1}{\partial r} & 0 & -\frac{\partial f_2}{\partial r} \end{bmatrix}}_{\underline{B}_1} + \frac{1}{r} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{bmatrix}}_{\underline{B}_2} - \frac{m}{r} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & f_1 & 0 & f_2 \\ f_1 & 0 & f_2 & 0 \end{bmatrix}}_{\underline{B}_3}$$

$$\underline{B}_m = \underline{B}_1 + \frac{1}{r} \underline{B}_2 - \frac{m}{r} \underline{B}_3$$

$$\begin{aligned} \underline{B}_m^T \underline{E} \underline{B}_m &= (\underline{B}_1^T + \frac{1}{r} \underline{B}_2^T - \frac{m}{r} \underline{B}_3^T) \underline{E} (\underline{B}_1 + \frac{1}{r} \underline{B}_2 - \frac{m}{r} \underline{B}_3) \\ &= \underline{B}_1^T \underline{E} \underline{B}_1 + \frac{1}{r} (\underline{B}_1^T \underline{E} \underline{B}_2 + \underline{B}_2^T \underline{E} \underline{B}_1) + \frac{1}{r^2} (\underline{B}_2^T \underline{E} \underline{B}_2) \\ &\quad - \frac{m}{r} (\underline{B}_3^T \underline{E} \underline{B}_1 + \underline{B}_1^T \underline{E} \underline{B}_3) - \frac{m}{r^2} (\underline{B}_3^T \underline{E} \underline{B}_2 + \underline{B}_2^T \underline{E} \underline{B}_3) + \frac{m^2}{r^2} \underline{B}_3^T \underline{E} \underline{B}_3 \\ &= \underline{A} + m \underline{B} + m^2 \underline{C} \end{aligned}$$

Chapter 5. Axisymmetric Solids (Solid of revolution)



$$\underline{u} = \begin{bmatrix} u \\ w \end{bmatrix}$$

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{bmatrix}$$

If circumference force is applied,

$$\underline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\epsilon_r = \frac{\partial u}{\partial r}$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\epsilon_\theta = \frac{1}{2\pi r} [2\pi(r+u) - 2\pi r] = \frac{u}{r}$$

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \\ \gamma_{z\theta} \\ \gamma_{r\theta} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}$$

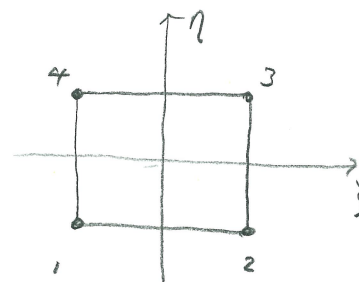
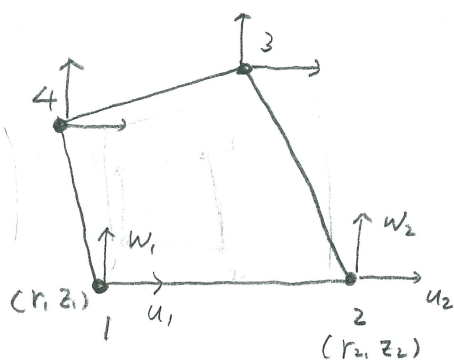
$$\underline{\epsilon} = \underline{D} \underline{u}$$

For isotropic material

$$\underline{\sigma} = \underline{E} \underline{\epsilon}$$

$$\underline{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 \\ \nu & (1-\nu) & \nu & 0 \\ \nu & \nu & (1-\nu) & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$$

Isoparametric Quadrilateral Element



$$\underline{\delta} = \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{bmatrix}$$

$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 \\ w = f_1 w_1 + f_2 w_2 + f_3 w_3 + f_4 w_4 \end{cases}$$

$$f_i = \frac{1}{4} (1 + \xi_i \xi) (1 + \eta_i \eta)$$

$$\begin{cases} r = f_1 r_1 + f_2 r_2 + f_3 r_3 + f_4 r_4 \\ z = f_1 z_1 + f_2 z_2 + f_3 z_3 + f_4 z_4 \end{cases}$$

$$\underline{\Sigma} = \underline{d} \underline{f} \underline{z}$$

$$= \underline{\beta} \underline{z}$$

$$\underline{\beta} = \begin{bmatrix} f_{1,r} & 0 \\ 0 & f_{1,z} \\ \frac{f_1}{r} & 0 \\ f_{1,z} & f_{1,r} \end{bmatrix} \quad \dots \quad \begin{bmatrix} f_{4,r} & 0 \\ 0 & f_{4,z} \\ \frac{f_4}{r} & 0 \\ f_{4,z} & f_{4,r} \end{bmatrix}$$

$$\frac{\partial f_i}{\partial r} = \frac{\partial f_i}{\partial \beta} \frac{\partial \beta}{\partial r} + \frac{\partial f_i}{\partial \eta} \frac{\partial \eta}{\partial r} = J_{11}^* \frac{\partial f_i}{\partial \beta} + J_{12}^* \frac{\partial f_i}{\partial \eta}$$

$$\frac{\partial f_i}{\partial z} = \frac{\partial f_i}{\partial \beta} \frac{\partial \beta}{\partial z} + \frac{\partial f_i}{\partial \eta} \frac{\partial \eta}{\partial z} = J_{21}^* \frac{\partial f_i}{\partial \beta} + J_{22}^* \frac{\partial f_i}{\partial \eta}$$

$$\underline{J} = \begin{bmatrix} \frac{\partial r}{\partial \beta} & \frac{\partial z}{\partial \beta} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix}$$

$$\underline{J}^{-1} = \underline{J}^* = \begin{bmatrix} \frac{\partial \beta}{\partial r} & \frac{\partial \eta}{\partial r} \\ \frac{\partial \beta}{\partial z} & \frac{\partial \eta}{\partial z} \end{bmatrix}$$

$$\frac{\partial r}{\partial \beta} = \sum_{i=1}^4 f_{i,\beta} r_i$$

$$\frac{\partial z}{\partial \beta} = \sum f_{i,\beta} z_i$$

$$\frac{\partial r}{\partial \eta} = \sum f_{i,\eta} r_i$$

$$\frac{\partial z}{\partial \eta} = \sum f_{i,\eta} z_i$$

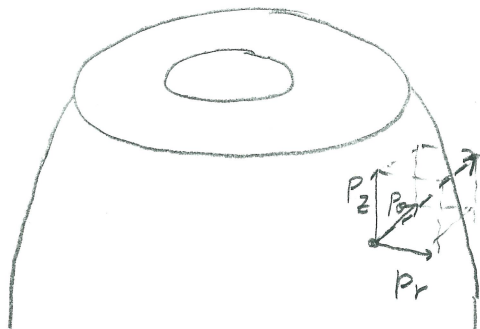
$$K = \iint B^T E B \, r \, dr \, dz$$

$$= \iint B^T E B \, r |J| \, d\zeta \, d\eta$$

$$P_b = \iint f^T b \, r |J| \, d\zeta \, d\eta$$

$$P_o = \iint B^T E \underline{\epsilon}_o \, r |J| \, d\zeta \, d\eta$$

Axisymmetric Solid with non-axisymmetric loading



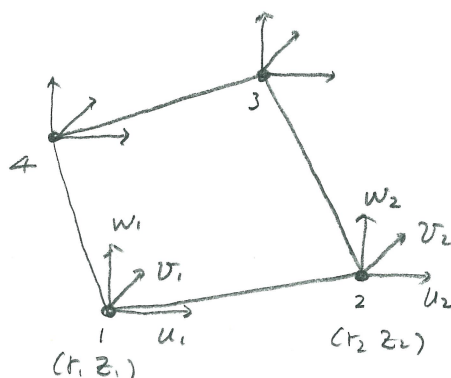
$$P_r = \sum_{m=0}^{\infty} \left[P_{rm} \cos m\theta + \overline{P_{rm}} \sin m\theta \right]$$

$$P_{\theta} = \sum \left[P_{\theta m} \sin m\theta + \overline{P_{\theta m}} \cos m\theta \right]$$

$$P_z = \sum \left[P_{zm} \cos m\theta + \overline{P_{zm}} \sin m\theta \right]$$

Symmetric
w.r.t x-axis

Symmetric
w.r.t y-axis



only cross-sectional
analysis is required

$$r = \sum f_i r_i$$

$$z = \sum f_i z_i$$

$$\underline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\underline{\Sigma} = \begin{bmatrix} \Sigma_r \\ \Sigma_{\theta} \\ \Sigma_z \\ \gamma_{r\theta} \\ \delta_{\theta z} \\ \delta_{zr} \end{bmatrix}$$

$$u = \sum f_i u_i$$

$$v = \sum f_i v_i$$

$$w = \sum f_i w_i$$

$$\Sigma_r = \frac{\partial u}{\partial r}$$

$$\Sigma_\theta = \frac{\partial v}{r \partial \theta} + \frac{u}{r}$$

$$\Sigma_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$\gamma_{\theta z} = \frac{\partial w}{r \partial \theta} + \frac{\partial v}{\partial z}$$

$$\gamma_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\begin{bmatrix} \Sigma_r \\ \Sigma_\theta \\ \Sigma_z \\ \gamma_{r\theta} \\ \gamma_{\theta z} \\ \gamma_{zr} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ \frac{1}{r} & \frac{\partial}{r \partial \theta} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{r \partial \theta} & (\frac{\partial}{\partial r} - \frac{1}{r}) & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{r \partial \theta} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

INPUT

output

$$\left. \begin{array}{l} P_{rm} \cos m\theta \\ P_{\theta m} \sin m\theta \\ P_{zm} \cos m\theta \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} u_m \cos m\theta \\ v_m \sin m\theta \\ w_m \cos m\theta \end{array} \right.$$

$$\left. \begin{array}{l} \overline{P}_{rm} \sin m\theta \\ \overline{P}_{\theta m} \cos m\theta \\ \overline{P}_{zm} \sin m\theta \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \overline{u}_m \sin m\theta \\ \overline{v}_m \cos m\theta \\ \overline{w}_m \sin m\theta \end{array} \right.$$

$$\underline{\Sigma}_m = \begin{bmatrix} \frac{\partial U_m}{\partial r} \cos m\theta & 0 & 0 \\ \frac{U_m \cos m\theta}{r} & \frac{m V_m \cos m\theta}{r} & 0 \\ 0 & 0 & \frac{\partial W_m}{\partial z} \cos m\theta \\ \frac{m U_m \sin m\theta}{r} & \left(\frac{\partial V_m}{\partial r} - \frac{V_m}{r} \right) \sin m\theta & 0 \\ 0 & \frac{\partial V_m}{\partial z} \sin m\theta & -\frac{m W_m}{r} \sin m\theta \\ \frac{\partial U_m}{\partial z} \cos m\theta & 0 & \frac{\partial W_m}{\partial r} \cos m\theta \end{bmatrix}$$

$$\underline{\Sigma}_m = \underline{B}_m \underline{\beta}_m$$

$$\begin{matrix} \underline{B}_m = \\ 6 \times 12 \end{matrix} \begin{bmatrix} f_{I,r} \cos m\theta & 0 & 0 \\ \frac{f_I}{r} \cos m\theta & \frac{m f_I}{r} \cos m\theta & 0 \\ 0 & 0 & f_{I,z} \cos m\theta \\ \frac{m f_I}{r} \sin m\theta & \left(f_{I,r} - \frac{f_I}{r} \right) \sin m\theta & 0 \\ 0 & f_{I,z} \sin m\theta & -\frac{m f_I}{r} \sin m\theta \\ f_{I,z} \cos m\theta & 0 & f_{I,r} \cos m\theta \end{bmatrix}$$

$$\underline{\underline{\bar{B}}}_m = \begin{bmatrix} f_{1,r} & 0 & 0 & \vdots \\ \frac{f_1}{r} & \frac{mf_1}{r} & 0 & \vdots \\ \frac{mf_1}{r} & (f_{1,r} - \frac{f_1}{r}) & 0 & \vdots \\ 0 & f_{1,z} & -\frac{mf_1}{r} & \vdots \\ f_{1,z} & 0 & f_{1,r} & \vdots \end{bmatrix}$$

6x12

$$\int \delta \underline{\underline{\xi}}^T \underline{\underline{Q}} dV = \delta \underline{\underline{\xi}}^T \underline{\underline{P}}$$

$$\Rightarrow \delta \underline{\underline{\xi}}_m^T \int \underline{\underline{B}}_m^T \underline{\underline{E}} \underline{\underline{B}}_m dA \underline{\underline{\xi}}_m = \delta \underline{\underline{\xi}}_m^T \underline{\underline{P}}_m \cdot k\pi$$

$$\underline{\underline{k}}/\pi \int \underline{\underline{B}}_m^T \underline{\underline{E}} \underline{\underline{B}}_m dA \underline{\underline{\xi}}_m = \underline{\underline{P}}_m \cdot \underline{\underline{k}}/\pi$$

$1 \leq m$

$$\begin{cases} m=0 & k=2 \\ m=1, 2, \dots, m & k=1 \end{cases}$$

$$\int_0^{2\pi} \cos^2 m\theta d\theta = \int_0^{2\pi} \sin^2 m\theta d\theta = \pi$$

$$\left\{ \begin{array}{l} \sum_m \underline{\underline{U}}_m = \underline{\underline{P}}_m \\ \sum_m \underline{\underline{U}}_m = \underline{\underline{P}}_m \end{array} \right\} \Rightarrow \text{solve } \begin{bmatrix} U_{mi} \\ V_{mi} \\ W_{mi} \end{bmatrix} \quad \begin{bmatrix} \bar{U}_{mi} \\ \bar{V}_{mi} \\ \bar{W}_{mi} \end{bmatrix}$$

$$\left\{ \begin{array}{l} u_i = \sum (u_{mi} \cos m\theta + \overline{u_{mi}} \sin m\theta) \\ v_i = \sum (v_{mi} \sin m\theta + \overline{v_{mi}} \cos m\theta) \\ w_i = \sum (w_{mi} \cos m\theta + \overline{w_{mi}} \sin m\theta) \end{array} \right.$$