

## Basic Assumptions on flexural Elements (Beams, Plate, Shell)

### 1) Kirchhoff theory

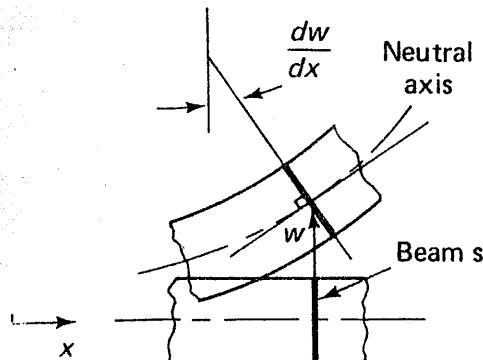
- Shear deformation is neglected
- The straight line remains perpendicular to the mid-surface during deformation
- It is difficult to satisfy interelement continuity on displacements and edge rotations because the plate (or shell) rotations are calculated from the transverse displacements. (higher-order shape function is used compared to the number of nodes)
- Using an assemblage of flat elements to represent a shell element, a relatively large no. of elements may be required to represent the shell geometry.

### 2) Mindlin Theory

- The displacements and rotations of the mid-surface normals are independent.
- Shear deformations are included.
- The line originally normal to the mid-surface does in general not remain perpendicular to the mid-surface during the deformation
- The interelement continuity conditions on these quantities can be satisfied directly as in the analysis of continua.

of the beam mid-surface. This kinematic assumption, illustrated in Fig. 5.29(a), leads to the well-known beam-bending governing differential equation in which the transverse displacement  $w$  is the only variable (see Example 3.20). Therefore, using beam elements formulated with this theory, displacement continuity between elements requires that  $w$  and  $dw/dx$  be continuous.

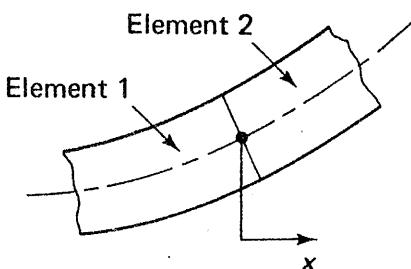
Considering now beam-bending analysis with the effect of shear deformations, we retain the assumption that a plane section originally normal to



Deformation of cross-section

$$\frac{dw}{dx} = -\frac{du}{dz}$$

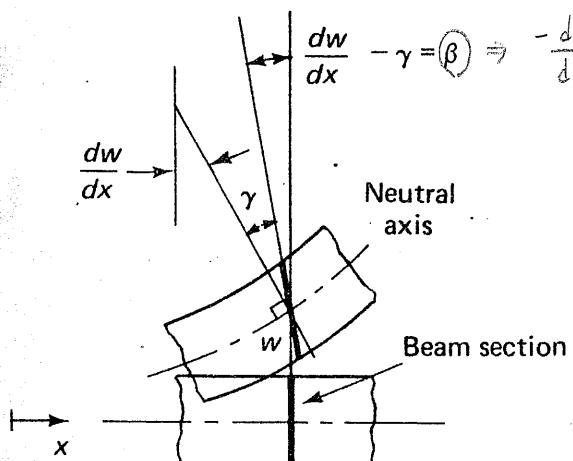
$$\gamma_{zz} = 0$$



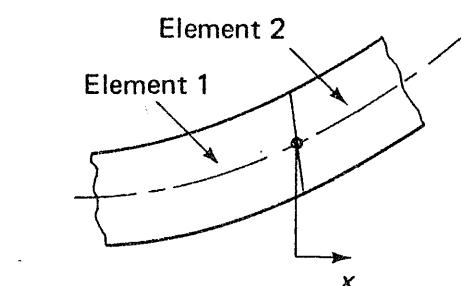
Boundary conditions between beam elements

$$w \Big|_{x=0} = w \Big|_{x=0}; \quad \frac{dw}{dx} \Big|_{x=0} = \frac{dw}{dx} \Big|_{x=0}$$

(a) Beam deformations excluding shear effect



Deformation of cross-section



Boundary conditions between beam elements

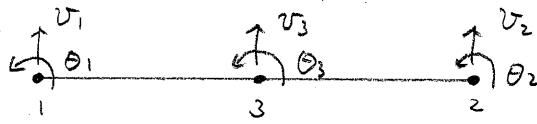
$$w \Big|_{x=0} = w \Big|_{x=0}$$

$$\beta \Big|_{x=0} = \beta \Big|_{x=0}$$

(b) Beam deformations including shear effect

**FIGURE 5.29** Beam deformation assumptions.

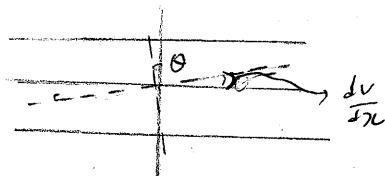
### Beam Element (Mindlin Theory)



$$\underline{\xi} = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} \quad \begin{aligned} u &= \sum_{i=1}^3 f_i v_i \\ \theta &= \sum f_i \theta_i \end{aligned} \quad \left\{ \begin{array}{l} f_1 = -\frac{1}{2} \} (1-3) \\ f_2 = \frac{1}{2} \} (1+3) \\ f_3 = \frac{1}{2} (1-3)(1+3) \end{array} \right.$$



$$\left\{ \begin{array}{l} u = -y \theta = -y \sum f_i \theta_i \quad \theta = \text{rotation w.r.t. the vertical axis} = -\frac{du}{dy} \\ v = \sum f_i v_i \end{array} \right.$$



$\frac{dv}{dx}$  = rotation w.r.t. the horizontal axis

$$\underline{\xi} = \begin{bmatrix} \epsilon_x \\ \gamma_{xy} \end{bmatrix} \quad \begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = -y \frac{\partial \theta}{\partial x} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\theta + \frac{\partial v}{\partial x} \quad (\text{shear deformation is included}) \end{aligned}$$

$$\epsilon_x = \frac{\partial u}{\partial x} = -y \sum \frac{\partial f_i}{\partial x} \theta_i$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\sum f_i \theta_i + \sum \frac{\partial f_i}{\partial x} v_i$$

$$\underline{\Sigma} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 0 & -y f_{1,x} & 0 & -y f_{2,x} & 0 & -y f_{3,x} \\ f_{1,x} & -f_1 & f_{2,x} & -f_2 & f_{3,x} & -f_3 \end{bmatrix}$$

$$\underline{K} = \iint \underline{B}^T \underline{\Sigma} \underline{B} b dy dx$$

### Discussion

$$\begin{aligned}
 \delta V &= \int \delta \underline{\epsilon}^T E \underline{\epsilon} dV \quad \underline{\epsilon} = \begin{bmatrix} \epsilon_x \\ \gamma_{xy} \end{bmatrix} \\
 &= \int \delta \epsilon_x E \epsilon_x dV + \int \delta \gamma_{xy} G \gamma_{xy} dV \\
 &= \int g^2 \delta \left( \frac{\partial \theta}{\partial x} \right) E \left( \frac{\partial \theta}{\partial x} \right) dV + \int \delta \left( \frac{\partial v}{\partial x} - \theta \right) G \left( \frac{\partial v}{\partial x} - \theta \right) dV \\
 &= EI \underbrace{\int \delta \left( \frac{\partial \theta}{\partial x} \right) \left( \frac{\partial \theta}{\partial x} \right) dx}_{\substack{\text{virtual strain energy} \\ \text{due to axial strain/stress}}} + G A \underbrace{\int \delta \left( \frac{\partial v}{\partial x} - \theta \right) \left( \frac{\partial v}{\partial x} - \theta \right) dx}_{\substack{\text{virtual strain energy} \\ \text{due to shear strain/stress}}}
 \end{aligned}$$

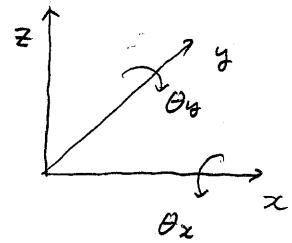
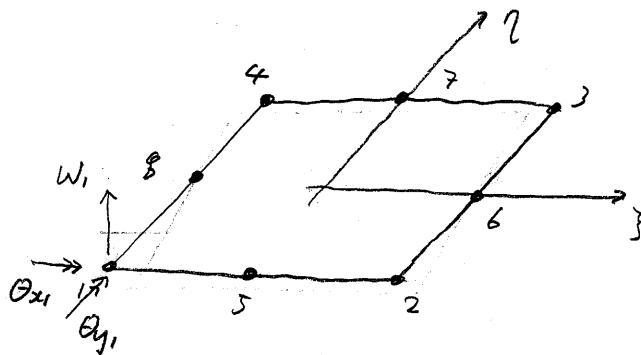
For Kirchhoff theory

$$\theta = \frac{dv}{dx}$$

$$\delta V = EI \int \delta \left( \frac{\partial \theta}{\partial x} \right) \left( \frac{d\theta}{dx} \right) dx$$

$$= EI \int \delta(v'') (v'') dx$$

## Plate Bending (Mindline Theory)



General displacements

$$\left\{ \begin{array}{l} w = f_1 w_1 + f_2 w_2 + \dots + f_8 w_8 = \sum_{i=1}^8 f_i w_i \\ \theta_x = f_1 \theta_{x1} + f_2 \theta_{x2} + \dots + f_8 \theta_{x8} = \sum f_i \theta_{xi} \\ \theta_y = \sum f_i \theta_{yi} \end{array} \right. \quad \text{f}_i \text{'s are the same} \quad ] \text{ independent of } w$$

$$\left\{ \begin{array}{l} u = z \theta_y = z \sum f_i \theta_{yi} \\ v = -z \theta_x = -z \sum f_i \theta_{xi} \\ w = \sum f_i w_i \end{array} \right.$$

$$\underline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 0 & zf_1 \\ 0 & -zf_1 & 0 \\ f_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & -zf_8 & 0 \\ f_8 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ \vdots \\ w_8 \\ \theta_{x8} \\ \theta_{y8} \end{bmatrix}$$

$$\underline{u} = \underline{f} \underline{\theta}$$

## Strain - General Displacement

$$\underline{\xi} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ \tilde{u} \end{bmatrix}$$

d

Shear deformation

## Strain - nodal displacement

$$\underline{\xi} = \underline{d} \underline{A} = \underline{d} \underline{E} \underline{B} = \underline{B} \underline{d}$$

$$\underline{B} = \begin{bmatrix} 0 & 0 & z f_{1,x} & | & 1 & 0 & 0 & z f_{8,x} \\ 0 & -z f_{1,y} & 0 & | & 1 & 0 & -z f_{8,y} & 0 \\ 0 & -z f_{1,x} & z f_{1,y} & | & - & 0 & -z f_{8,x} & z f_{8,y} \\ f_{1,y} & -f_1 & 0 & | & & f_{8,y} & -f_8 & 0 \\ f_{1,x} & 0 & f_1 & | & & f_{8,x} & 0 & f_8 \end{bmatrix}$$

$$\left\{ \begin{array}{l} f_{1,x} = f_{i,j} J_{1,x} + f_{i,j} n_{1,x} = f_{i,j} J_{11}^* + f_{i,j} J_{12}^* \\ f_{1,y} = f_{i,j} J_{1,y} + f_{i,j} n_{1,y} = f_{i,j} J_{21}^* + f_{i,j} J_{22}^* \end{array} \right.$$

$$\underline{J}^{-1} = \begin{bmatrix} J_{1,x} & n_{1,x} \\ J_{1,y} & n_{1,y} \end{bmatrix} \quad \underline{J} = \begin{bmatrix} x_{i,j} & y_{i,j} \\ x_{i,j} & y_{i,j} \end{bmatrix}$$

$$x_{i,j} = \sum f_{i,j} x_i \text{ and so on.}$$

plane stress

$$\underline{\sigma} = \underline{\epsilon} \underline{\epsilon} \quad \underline{\epsilon} = \begin{bmatrix} \underline{\epsilon}_{11} & \underline{\epsilon}_{12} \\ \underline{\epsilon}_{21} & \underline{\epsilon}_{22} \end{bmatrix} \quad \begin{array}{c} G \\ G \\ G \end{array}$$

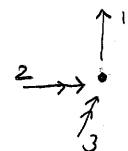
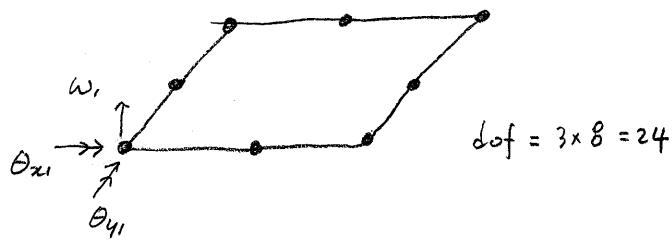
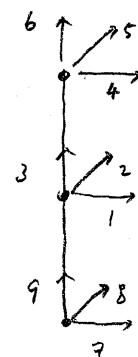
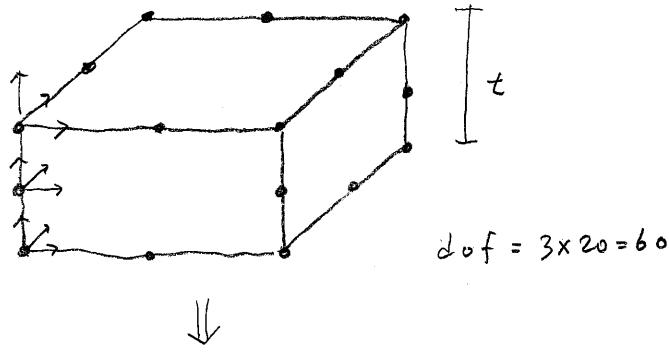
$$\begin{aligned} K &= \int \underline{\beta}^T \underline{\epsilon} \underline{\beta} dV \\ &= \int \underline{\beta}^T \underline{\epsilon} \underline{\beta} |J| d\zeta d\eta dz \\ &= \iiint \underline{\beta}^T \underline{\epsilon} \underline{\beta} |J| \frac{1}{2} d\zeta d\eta d\zeta \end{aligned}$$

### Discussion

$$\underline{\beta} = \begin{bmatrix} z \underline{\beta}_A \\ \underline{\beta}_B \end{bmatrix} \quad \underline{\epsilon} = \begin{bmatrix} \underline{\epsilon}_A & 0 \\ 0 & \underline{\epsilon}_B \end{bmatrix} \quad \underline{\epsilon}_A = 3 \times 3 \quad \underline{\epsilon}_B = 2 \times 2$$

$$\begin{aligned} \underline{\beta}^T \underline{\epsilon} \underline{\beta} &= [z \underline{\beta}_A^T \underline{\beta}_B^T] \begin{bmatrix} \underline{\epsilon}_A & 0 \\ 0 & \underline{\epsilon}_B \end{bmatrix} \begin{bmatrix} z \underline{\beta}_A \\ \underline{\beta}_B \end{bmatrix} \\ &= z^2 \underline{\beta}_A^T \underline{\epsilon}_A \underline{\beta}_A + \underline{\beta}_B^T \underline{\epsilon}_B \underline{\beta}_B \end{aligned}$$

$$\begin{aligned} K &= \int z^2 \underline{\beta}_A^T \underline{\epsilon}_A \underline{\beta}_A dV + \int \underline{\beta}_B^T \underline{\epsilon}_B \underline{\beta}_B dV \\ &= \frac{t^3}{12} \int \underline{\beta}_A^T \underline{\epsilon}_A \underline{\beta}_A dA + t \int \underline{\beta}_B^T \underline{\epsilon}_B \underline{\beta}_B dA \\ &= \underbrace{\frac{t^3}{12} \iint \underline{\beta}_A^T \underline{\epsilon}_A \underline{\beta}_A |J| d\zeta d\eta}_{\text{Bending related stiffness}} + \underbrace{t \iint \underline{\beta}_B^T \underline{\epsilon}_B \underline{\beta}_B |J| d\zeta d\eta}_{\text{shear-related stiffness}} \end{aligned}$$



$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{t}{2} \\ 0 & -\frac{t}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{t}{2} \\ 0 & \frac{t}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{bmatrix}$$

$$\underline{\delta}_s = \underline{G} \underline{\delta}_p$$

$60 \times 1 \quad 60 \times 24 \quad 24 \times 1$

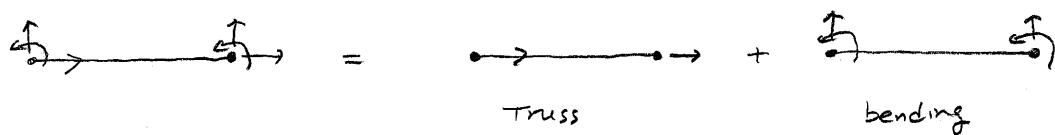
$$\underline{K}_{\text{Solid}} = \int \underline{B}_s^T \underline{\underline{E}} \underline{B}_s dV$$

$$\begin{aligned} \delta V &= \delta \underline{\underline{\delta}}_s^T \int \underline{B}_s^T \underline{\underline{E}} \underline{B}_s dV \underline{\underline{\delta}}_s \\ &= \delta \underline{\underline{\delta}}_p^T \underline{\underline{G}}^T \underbrace{\left[ \int \underline{B}_s^T \underline{\underline{E}} \underline{B}_s dV \right]}_{\underline{K}_s} \underline{\underline{G}} \underline{\underline{\delta}}_p \end{aligned}$$

$$\underline{K}_p = \underline{\underline{G}}^T \underline{K}_s \underline{\underline{G}} \Rightarrow \text{same as plate bending 8-node element (midline theory)}$$

General Shells ( $=$  membrane action + bending action)

beam - column element

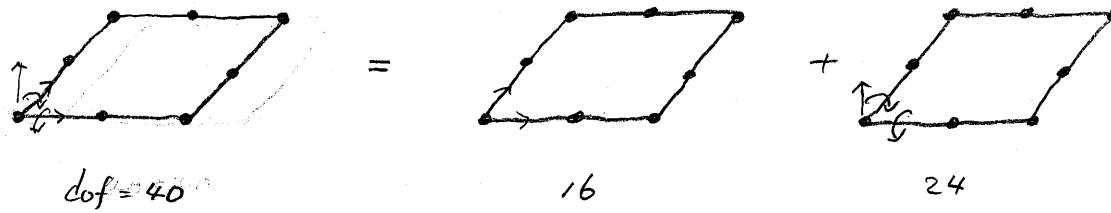


$$\begin{matrix} K \\ \sim \\ 6 \times 6 \end{matrix}$$

$$K = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \square & \square & 0 & \square & \square \\ 0 & \square & \square & 0 & \square & \square \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \square & \square & 0 & \square & \square \\ 0 & \square & \square & 0 & \square & \square \end{bmatrix}$$

bending stiffness

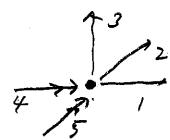
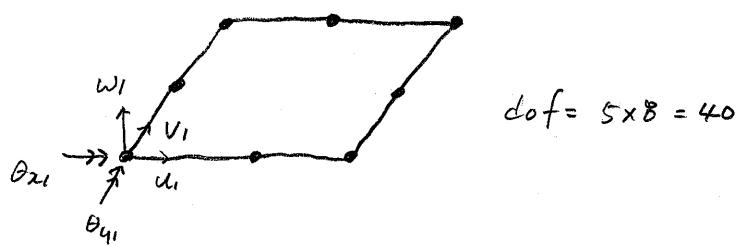
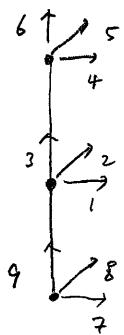
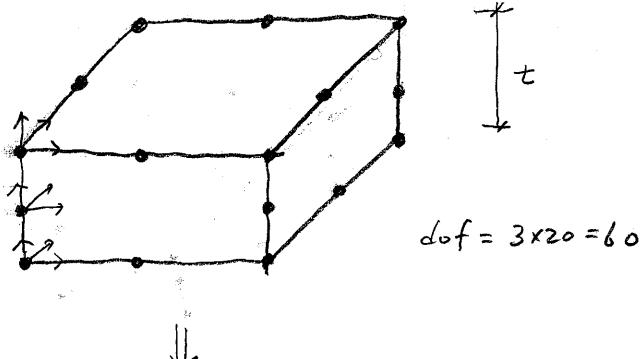
plane stress + plate bending



$$K =$$

$$K = \left[ \begin{array}{c|c} \text{plane stress} & \text{plate bending} \\ \hline \begin{matrix} \square & \square \\ \square & \square \end{matrix} & \begin{matrix} \square & \square \\ \square & \square \end{matrix} \\ \vdots & \vdots \end{array} \right]$$

Finite Element Method

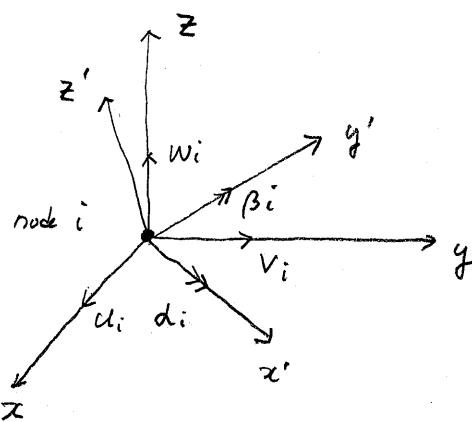
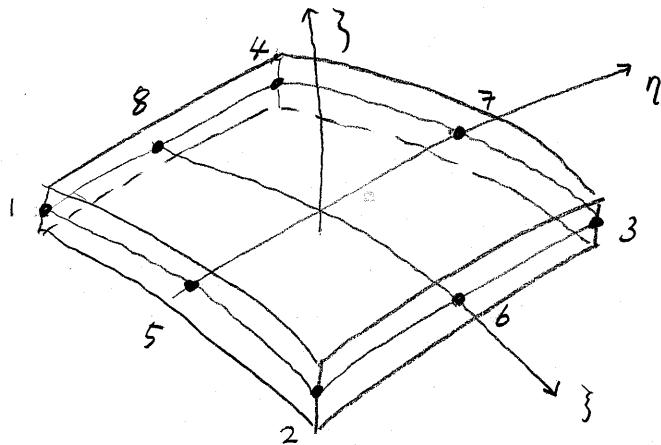


$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & \frac{t}{2} \\ 0 & 1 & 0 & -\frac{t}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -\frac{t}{2} \\ 0 & 1 & 0 & \frac{t}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_{x1} \\ \theta_{y1} \end{bmatrix}$$

$$\tilde{\delta}_S = \underbrace{G}_{60 \times 1} \underbrace{\delta_{Sh}}_{60 \times 40} \underbrace{40 \times 1}$$

$$K_{\text{shell}} = \underbrace{G^T K_S G}_{40 \times 40} \quad \underbrace{40 \times 60}_{60 \times 60} \quad \underbrace{60 \times 40}_{60 \times 40}$$

## General Shell



$x, y, z \rightarrow$  Global coordinates

$x', y', z' \rightarrow$  Local coordinates

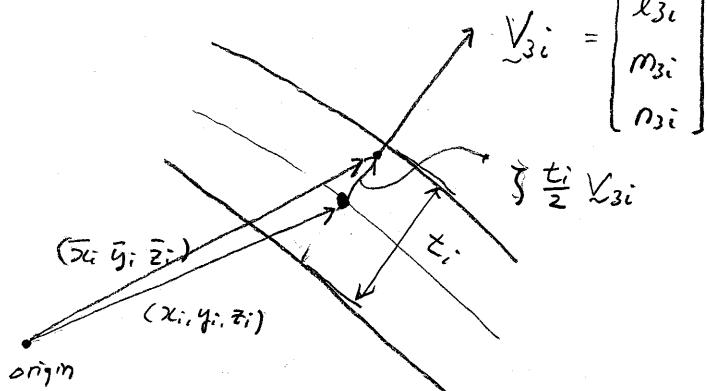
at node i

$$\text{nodal displacements} = \begin{bmatrix} u_i \\ v_i \\ w_i \\ d_i \\ d_i \\ \beta_i \end{bmatrix}$$

$\rightarrow$  Global

$\rightarrow$  local

directional vector  
of the tangential plane  
at node i

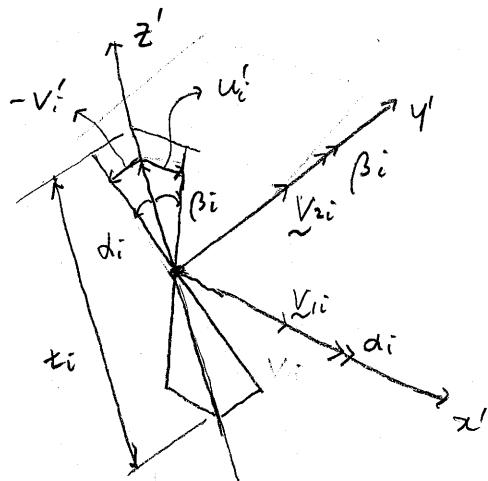
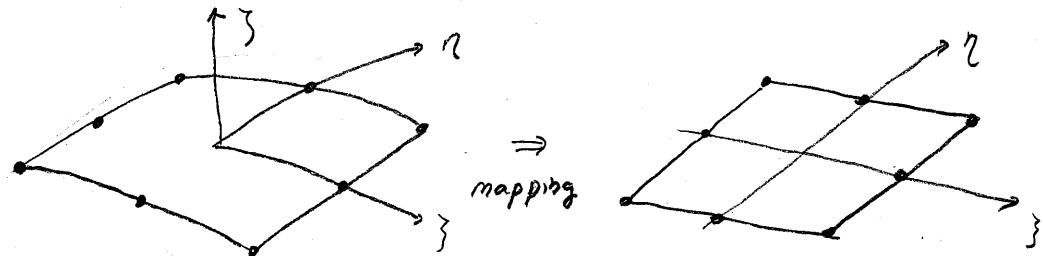


$$\begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \sqrt{\frac{t_i}{2}} \begin{bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sum_{i=1}^8 f_i \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sum_{i=1}^8 f_i \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \sum_{i=1}^8 f_i \int \frac{t_i}{2} \begin{bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{bmatrix}$$

$f_i$  = shape functions for 8-node 2-D element



$$u'_i = \int \frac{t_i}{2} \beta_i \quad , \quad v'_i = -\int \frac{t_i}{2} \alpha_i$$

$$\begin{bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{w}_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + u'_i \underline{V}_{1i} + v'_i \underline{V}_{2i}$$

$$= \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \int \frac{t_i}{2} d_i \underline{V}_{2i} + \int \frac{t_i}{2} \beta_i \underline{V}_{1i}$$

$$\begin{bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{w}_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \sum \frac{t_i}{2} \begin{bmatrix} -l_{2i} & l_{1i} \\ -m_{2i} & m_{1i} \\ -n_{2i} & n_{1i} \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^8 f_i \begin{bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{w}_i \end{bmatrix}$$

$$= \sum f_i \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \sum f_i \frac{t_i}{2} \begin{bmatrix} -l_{2i} & l_{1i} \\ -m_{2i} & m_{1i} \\ -n_{2i} & n_{1i} \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$$

$\underline{V}_{1i}$  and  $\underline{V}_{2i}$  can be defined arbitrarily.

for example,  $\underline{V}_{1i} = e_y \times \underline{V}_{3i}$  and  $\underline{V}_{2i} = \underline{V}_{3i} \times \underline{V}_{1i}$

if  $\underline{V}_{3i} \parallel e_y$ , then  $\underline{V}_{1i} = e_x \times \underline{V}_{3i}$  and  $\underline{V}_{2i} = \underline{V}_{3i} \times \underline{V}_{1i}$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_1 & 0 & 0 & -f_1 \frac{t_1}{2} l_{21} & f_1 \frac{t_1}{2} l_{11} \\ 0 & f_1 & 0 & -f_1 \frac{t_1}{2} m_{21} & f_1 \frac{t_1}{2} m_{11} \\ 0 & 0 & f_1 & -f_1 \frac{t_1}{2} n_{21} & f_1 \frac{t_1}{2} n_{11} \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_8 \\ \beta_8 \end{bmatrix}}_{f(3 \times 40)}$$

Strain - Displacement

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}}_d \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

strain - nodal Displacements

$$\underline{\epsilon} = \underline{d} \underline{u} = \underbrace{\underline{d} \underline{f} \underline{u}}_{B}$$

strain - nodal displacement relationship for 8-node shell element

$\Sigma_x$	$f_{i,x}$	$0$	$-f_{i,x} \int \frac{t_i}{2} \lambda_{2i} - f_i \int_{x_2} \frac{t_i}{2} \lambda_{2i}$	$f_{i,x} \int \frac{t_i}{2} \lambda_{ii} + f_i \int_{x_2} \frac{t_i}{2} \lambda_{ii}$
$\Sigma_y$	$= \sum_{i=1}^8$	$0$	$-f_{i,y} \int \frac{t_i}{2} m_{2i} - f_i \int_{y_2} \frac{t_i}{2} m_{2i}$	$f_{i,y} \int \frac{t_i}{2} m_{ii} + f_i \int_{y_2} \frac{t_i}{2} m_{ii}$
$\Sigma_z$	$0$	$f_{i,y}$	$-f_{i,z} \int \frac{t_i}{2} \eta_{2i} - f_i \int_{z_2} \frac{t_i}{2} \eta_{2i}$	$f_{i,z} \int \frac{t_i}{2} \eta_{ii} + f_i \int_{z_2} \frac{t_i}{2} \eta_{ii}$
$\gamma_{xz}$	$f_{i,y}$	$0$	$-f_{i,z} \int \frac{t_i}{2} \lambda_{2i} - f_i \int_{y_2} \frac{t_i}{2} \lambda_{2i}$ $-f_{i,x} \int \frac{t_i}{2} m_{2i} - f_i \int_{x_2} \frac{t_i}{2} m_{2i}$	$f_{i,y} \int \frac{t_i}{2} \lambda_{ii} + f_i \int_{y_2} \frac{t_i}{2} \lambda_{ii}$ $+ f_{i,x} \int \frac{t_i}{2} m_{ii} + f_i \int_{x_2} \frac{t_i}{2} m_{ii}$
$\gamma_{yz}$	$f_{i,z}$	$0$	$-f_{i,x} \int \frac{t_i}{2} \lambda_{2i} - f_i \int_{z_2} \frac{t_i}{2} \lambda_{2i}$ $-f_{i,y} \int \frac{t_i}{2} m_{2i} - f_i \int_{y_2} \frac{t_i}{2} m_{2i}$	$f_{i,z} \int \frac{t_i}{2} \lambda_{ii} + f_i \int_{z_2} \frac{t_i}{2} \lambda_{ii}$ $+ f_{i,y} \int \frac{t_i}{2} m_{ii} + f_i \int_{y_2} \frac{t_i}{2} m_{ii}$
$\gamma_{zx}$	$f_{i,x}$	$0$	$-f_{i,z} \int \frac{t_i}{2} \lambda_{2i} - f_i \int_{x_2} \frac{t_i}{2} \lambda_{2i}$ $-f_{i,y} \int \frac{t_i}{2} m_{2i} - f_i \int_{y_2} \frac{t_i}{2} m_{2i}$	$f_{i,x} \int \frac{t_i}{2} \lambda_{ii} + f_i \int_{x_2} \frac{t_i}{2} \lambda_{ii}$ $+ f_{i,z} \int \frac{t_i}{2} m_{ii} + f_i \int_{z_2} \frac{t_i}{2} m_{ii}$

$$\left[ \begin{array}{l} u_i \\ v_i \\ w_i \\ \alpha_i \\ \beta_i \end{array} \right]$$

$$f_{i,x} = f_{i,j} \beta_{j,x} + f_{i,\eta} \gamma_{i,x} + f_{i,j} \beta_{j,x}^*$$

$$f_{i,y} = f_{i,j} \beta_{j,y} + f_{i,\eta} \gamma_{i,y} + f_{i,j} \beta_{j,y}^*$$

$$f_{i,z} = f_{i,j} \beta_{j,z} + f_{i,\eta} \gamma_{i,z} + f_{i,j} \beta_{j,z}^*$$

$$\underline{J} = \begin{bmatrix} x_{,j} & y_{,j} & z_{,j} \\ x_{,\eta} & y_{,\eta} & z_{,\eta} \\ x_{,j} & y_{,j} & z_{,j} \end{bmatrix}$$

$$x_{,j} = \sum f_{i,j} x_i + \sum f_{i,j} \beta_{j,\frac{t_i}{2}} \ell_{3,i}$$

$$x_{,\eta} = \sum f_{i,\eta} x_i + \sum f_{i,\eta} \beta_{j,\frac{t_i}{2}} \ell_{3,i}$$

$$x_{,j} = \sum f_{i,j} \beta_{j,\frac{t_i}{2}} \ell_{3,i}$$

$$\vdots$$

$$\underline{J}^{-1} = \underline{J}^* = \begin{bmatrix} \beta_{j,x} & \gamma_{i,x} & \beta_{j,x}^* \\ \beta_{j,y} & \gamma_{i,y} & \beta_{j,y}^* \\ \beta_{j,z} & \gamma_{i,z} & \beta_{j,z}^* \end{bmatrix}$$

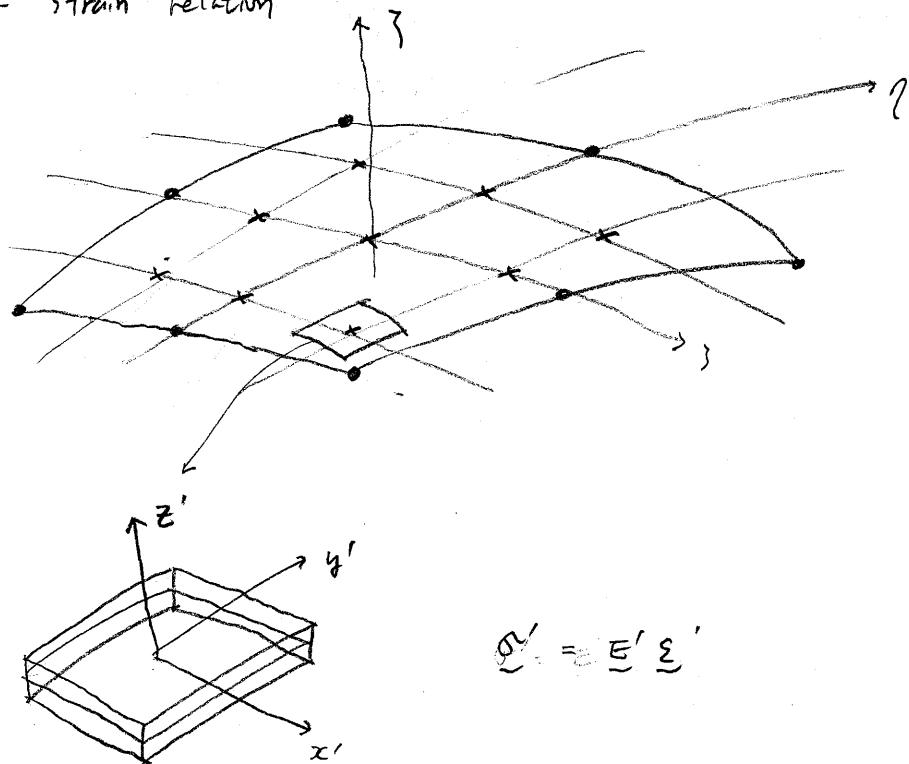
$$f_{i,x} = f_{i,j} J_{11}^* + f_{i,\eta} J_{12}^*$$

$$f_{i,y} = f_{i,j} J_{21}^* + f_{i,\eta} J_{22}^*$$

$$f_{i,z} = f_{i,j} J_{31}^* + f_{i,\eta} J_{32}^*$$

$$\beta_{j,x} = J_{13}^*, \quad \beta_{j,y} = J_{23}^*, \quad \beta_{j,z} = J_{33}^*$$

stress-strain relation



$x$  : Gauss  
Integration  
points

$$\underline{\epsilon}' = \underline{E}' \underline{\epsilon}'$$

$$\begin{bmatrix} \sigma_x' \\ \sigma_y' \\ \sigma_z' \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{31} \end{bmatrix} \begin{bmatrix} \epsilon_x' \\ \epsilon_y' \\ \epsilon_z' \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

Constitutive matrix  $\underline{E}'$  defined in Global coordinate system

$$\underline{E} = \underline{T}_\epsilon^T \underline{E}' \underline{T}_\epsilon$$

To calculate  $\underline{T}_\epsilon^T$ , we need to know the directional vectors  $\underline{V}_1'$ ,  $\underline{V}_2'$ , and  $\underline{V}_3'$  at Gaussian integration points.

At Gaussian Points

$$\underline{V}_3 = \sum f_i \underline{V}_{3i}$$

$$\underline{V}_1 = \underline{e}_y \times \underline{V}_3 \quad \text{and} \quad \underline{V}_2 = \underline{V}_1 \times \underline{V}_3$$

Stiffness matrix

$$\begin{aligned} K &= \int \underline{B}^T \underline{E} \underline{B} dV \\ &= \iiint \underline{B}^T \underline{E} \underline{B} |J| d\gamma d\eta d\zeta \end{aligned}$$

Numerical Integration

Equivalent nodal forces due to body forces

$$\underline{P}_b = \int_V \underline{f}^T \underline{b} dV = \iiint \underline{f}^T \underline{b} |J| d\gamma d\eta d\zeta$$

$$\underline{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Equivalent nodal loads due to initial strain

$$\underline{P}_0 = \int_V \underline{B}^T \underline{E} \underline{\varepsilon}_0 dV$$

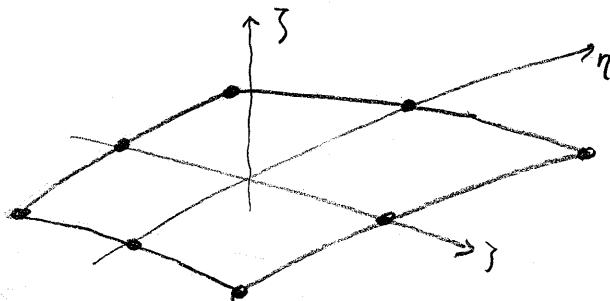
$$= \int_V \underline{B}^T \underline{I}_S^T \underline{E}' \underline{I}_S \underline{\varepsilon}_0 dV$$

$$= \int_V \underline{B}^T \underline{I}_S^T \underline{E}' \underline{\varepsilon}'_0 dV$$

### calculation of stresses

$$\begin{aligned}\Sigma' &= E' (\varepsilon' - \varepsilon'_0) \\ &= E' (T_\varepsilon \varepsilon - \varepsilon'_0) \\ &= E' (T_\varepsilon B \bar{\varepsilon} - \varepsilon'_0)\end{aligned}$$

### Membrane Element ( $\sigma_t, \tau_{\varepsilon}$ )



Rotational dof's are eliminated from the general shell element.

only membrane forces and stresses exist,

bending moments and out-of-plane stresses do not exist.

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum f_i \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sum f_i \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \frac{3}{2} t_i \begin{bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{bmatrix}$$

In the local coordinates at the Gaussian Integration points.

$$\begin{bmatrix} \sigma_x' \\ \sigma_y' \\ \tau_{xy}' \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_{11}' & \varepsilon_{12}' & 0 \\ \varepsilon_{21}' & \varepsilon_{22}' & 0 \\ 0 & 0 & \varepsilon_{33}' \end{bmatrix}}_{E'} \begin{bmatrix} \varepsilon_x' \\ \varepsilon_y' \\ \gamma_{xy}' \end{bmatrix}$$

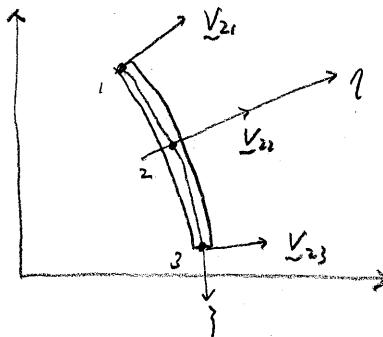
$$\underline{\underline{E}} = \underline{T}_{\epsilon}^T \underline{\underline{E}}' \underline{T}_{\epsilon}$$

(6x6)      (6x3) (3x3) (3x6)

$$\underline{B} = \left[ \begin{array}{ccc|ccc} f_{1,x} & 0 & 0 & f_{8,x} & 0 & 0 \\ 0 & f_{1,y} & 0 & 0 & f_{8,y} & 0 \\ 0 & 0 & f_{1,z} & 0 & 0 & f_{8,z} \\ f_{1,y} & f_{1,x} & 0 & f_{8,y} & f_{8,x} & 0 \\ 0 & f_{1,z} & f_{1,y} & 0 & f_{8,z} & f_{8,y} \\ f_{1,z} & 0 & f_{1,x} & f_{8,z} & 0 & f_{8,x} \end{array} \right]$$

### curved shell Element

⇒ see text.



### curved Beam Element

