

## Vibrational Analysis

principle of virtual work

virtual internal work

$$\delta V = \underbrace{\int_v \delta \underline{\Sigma}^T \underline{\Omega} dV}_{\text{Virtual external work}} + \underbrace{\int_v \delta \underline{u}^T \underline{p} \ddot{\underline{u}} dV}_{\text{work by inertia force}} + \underbrace{\int_v \delta \underline{u}^T \underline{c} \dot{\underline{u}} dV}_{\text{work by damping force}}$$

$$\delta W = \delta \underline{\underline{\delta}}^T \underline{P}(t) + \int_v \delta \underline{u}^T \underline{b}(t) dV - \int_v \delta \underline{\underline{\delta}}^T \underline{\underline{f}}(t) dV$$

$$\underline{\underline{\delta}} = \underline{\underline{f}} \underline{\underline{\delta}}$$

$$\ddot{\underline{\underline{\delta}}} = \underline{\underline{f}} \ddot{\underline{\underline{\delta}}}$$

$$\underline{\underline{\Sigma}} = \underline{\underline{B}} \underline{\underline{\delta}}$$

$$\delta V = \delta W$$

$$\delta \underline{\underline{\delta}}^T \int_v \underline{p} \underline{f}^T \underline{f} dV \ddot{\underline{\underline{\delta}}} + \delta \underline{\underline{\delta}}^T \int_v \underline{f} \ddot{\underline{f}}^T \underline{f} dV \dot{\underline{\underline{\delta}}} + \delta \underline{\underline{\delta}}^T \int_v \underline{B}^T \underline{E} \underline{B} dV \dot{\underline{\underline{\delta}}}$$

$$\ddot{\underline{\underline{\delta}}} = \delta \underline{\underline{\delta}}^T \underline{P} + \delta \underline{\underline{\delta}}^T \int_v \underline{f}^T \underline{b} dV$$

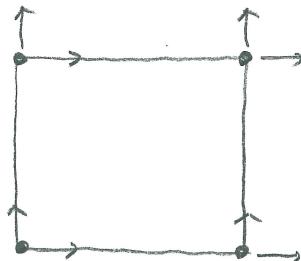
$$\ddot{\underline{\underline{\delta}}} + \underline{\underline{C}} \dot{\underline{\underline{\delta}}} + \underline{\underline{K}} \underline{\underline{\delta}} = \underline{P} + \underline{P}_b$$

$$\ddot{\underline{\underline{\delta}}} = \int_v \underline{P} \underline{f}^T \underline{f} dV = \text{consistent mass matrix}$$

$\leftrightarrow$  Lumped mass matrix

$$\ddot{\underline{\underline{\delta}}} = \int_v \underline{C} \underline{f}^T \underline{f} dV$$

$$\ddot{\underline{\underline{\delta}}} = \int_v \underline{B}^T \underline{E} \underline{B} dV$$



consistent  
mass matrix

$$\tilde{M} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$\tilde{M} (8 \times 8)$

Lumped mass

$$\text{matrix} = \begin{bmatrix} \frac{1}{4}M & & & \\ & \frac{1}{4}M & & \\ & & \ddots & \\ & & & \frac{1}{4}M \end{bmatrix}$$

beneficial for matrix calculation

If the displacements is defined in local axes,

$$\underline{\delta}' = R \underline{\delta}$$

$$\int \delta \underline{u}^T \rho \ddot{\underline{u}} dV = \underline{\delta}^T R^T \int \rho f^T f dV R \underline{\delta}$$

$$\tilde{M} = R^T \int \rho f^T f dV R$$

Assemble all the mass matrix to construct structural mass matrix

$$\tilde{M} \ddot{\underline{Q}} + \underline{K} \underline{Q} = \underline{P} + \underline{P}_b$$

$$\underline{M} \ddot{\underline{Q}} + \underline{C} \dot{\underline{Q}} + \underline{K} \underline{Q} = \underline{P} + \underline{P}_b$$

Usually,  $\underline{C}$  is not defined directly, but, is defined by using Rayleigh damping method  $\underline{C} = \alpha_0 \underline{K} + \alpha_1 \underline{M}$

Forced vibration - Modal superposition method

Step by step integration method

Fourier Transform

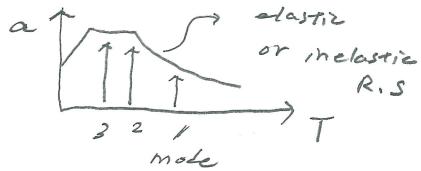
## Forced Vibration

### Modal Superposition Method

combination of independent 1 DOF modal responses

response spectrum analysis

SRSS required



Time history analysis - rarely used

simple sum of modal responses

### Step by step Integration Method

Acceleration distribution is assumed  $\Rightarrow$  Velocity, Displacement

$$\tilde{K} \tilde{U} = \tilde{P} \quad \tilde{U} \rightarrow \dot{\tilde{U}} \text{ and } \ddot{\tilde{U}} \text{ at each time step}$$

Acceptable for both linear and nonlinear analysis

time history analysis

### Fourier Transform

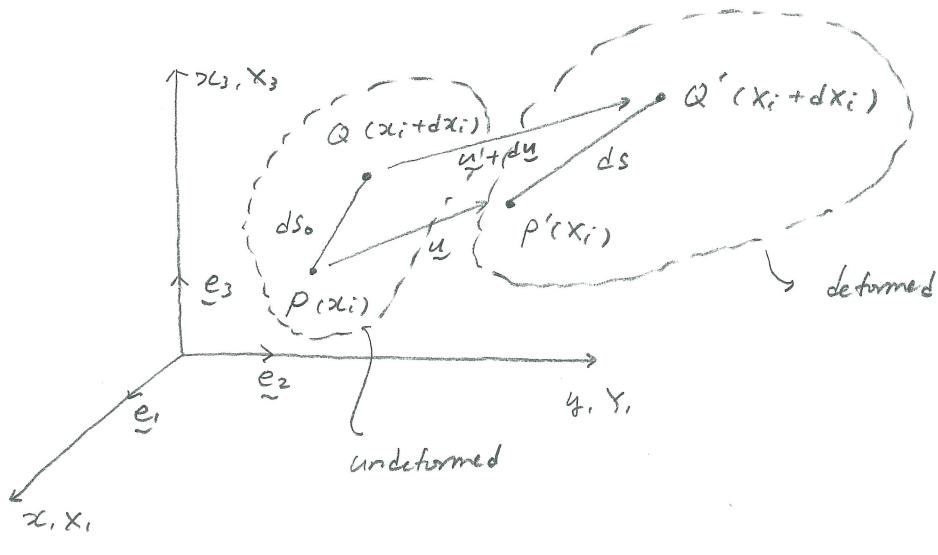
applied force is defined as sum of sinusoidal forms

combination of regular sinusoidal equations

Time history analysis

Acceptable only for linear analysis

## Analysis of Strain



$x_i$  - initial coordinate (Lagrangian description)

$X_i$  - current (final) coordinate (Eulerian)

Consider Reference points  $P$  and  $P'$  on the undeformed and deformed parts and neighbors  $Q$  and  $Q'$

$P$  and  $Q$  separated by a distance  $ds_0$

$P'$  and  $Q'$  separated by a distance  $ds$

$$\text{Then, } (ds_0)^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i dx_i \quad (1)$$

$$(ds)^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i dx_i$$

The displacement vector  $\underline{u} = P \rightarrow P'$   
 $\underline{u} + d\underline{u} = Q \rightarrow Q'$

$$\text{Then, } d\underline{s}_0 + \underline{u} + d\underline{u} = d\underline{s} + \underline{u}$$

$$\text{or } d\underline{u} = d\underline{s} - d\underline{s}_0 \quad (2)$$

from Eq. (1)

$$ds^2 - ds_0^2 = dx_i dx_i - dx_i dx_i \quad (3)$$

Consider referencing the initial coordinates

Then the final coordinates are functions of the initial ones, i.e

$$X_i = X_i(x_1, x_2, x_3) \text{ and}$$

$$\begin{aligned} dx_i &= \frac{\partial X_i}{\partial x_1} dx_1 + \frac{\partial X_i}{\partial x_2} dx_2 + \frac{\partial X_i}{\partial x_3} dx_3 \\ &= X_{ij} dx_j \end{aligned} \quad (4)$$

Furthermore  $\underline{u}$  is given by  $\underline{u} = u_i e_i$

$$\text{and } u_i = X_i - x_i \quad (5)$$

from (3) and (4)

$$\begin{aligned}
 (ds)^2 - (ds_0)^2 &= X_{ij} dx_i X_{ik} dx_k - dx_i dx_i \\
 &= (X_{ij} X_{ik} - \delta_{ij} \delta_{ik}) dx_i dx_k \\
 &\quad \delta = \text{kronnecker delta} \quad \delta_{ij} = 1 \\
 &\quad \text{from Eq. (5)} \quad \text{only when } i=j \\
 &= [(x_i + u_i)_{,j} (x_i + u_i)_{,k} - \delta_{jk}] dx_i dx_k \\
 &= [(\delta_{ij} + u_{ij}) (\delta_{ik} + u_{ik}) - \delta_{jk}] dx_i dx_k \\
 &= [\underline{\delta_{ij} u_{ik}} + \underline{\delta_{ik} u_{ij}} + u_{ij} u_{ik}] dx_i dx_k
 \end{aligned}$$

Rearranging indices

$$ds^2 - ds_0^2 = [u_{ij} + u_{ji} + u_{ki} u_{kj}] dx_i dx_j \quad (6)$$

If we define  $\Sigma_{ij}^L$  as the components of the Lagrangian

strain tensor (Green), and

$$\underline{\Sigma_{ij}^L} = \frac{1}{2} [u_{ij} + u_{ji} + u_{ki} u_{kj}] \quad (7)$$

$(u_{ii} = \frac{\partial u_i}{\partial x_i})$

Then

$$ds^2 - ds_0^2 = 2 \Sigma_{ij}^L dx_i dx_j \quad (8)$$

Now consider the Eulerian description

$$x_i = x_i(x_1, x_2, x_3)$$

$$dx_i = x_{i;j} dx_j \quad (9)$$

$$\underline{e}_i = e_i E_i \quad u_i = x_i - x_i \quad (10)$$

$\underline{E}_i$  = base vectors of  $x_i$  system

from Eq. (3) and (9)

$$(ds)^2 - (ds_0)^2 = dx_i dx_j - x_{i;j} x_{j;k} dx_k$$

$$= (\delta_{ij} \delta_{ik} - x_{i;j} x_{i;k}) dx_j dx_k$$

from (10)

$$= [\delta_{jk} - (x_i - u_i)_j (x_i - u_i)_k] dx_j dx_k$$

$$= [\delta_{jk} - (\delta_{ij} - u_{ij}) (\delta_{ik} - u_{ik})] dx_j dx_k$$

$$= [\delta_{jk} - \delta_{jk} + u_{j;k} + u_{k;j} - u_{i;j} u_{i;k}] dx_j dx_k$$

$$= [u_{j;k} + u_{k;j} - u_{i;j} u_{i;k}] dx_j dx_k$$

or

$$ds^2 - ds_0^2 = [u_{i;j} + u_{j;i} - u_{k;i} u_{k;j}] dx_i dx_j \quad (11)$$

Introducing the Eulerian strain Tensor (Alamansi)

$$\varepsilon_{ij}^E = \frac{1}{2} [u_{i,j} + u_{j,i} - u_{k,i} u_{k,j}] \quad u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

Then

$$ds^2 - ds_0^2 = 2 \varepsilon_{ij}^E dx_i dx_j$$

physical interpretation

$$\text{If } dx_1 = dx_3 = 0 \quad \text{and} \quad dx_2 = ds_0$$

$$ds^2 - ds_0^2 = (ds - ds_0)(ds + ds_0)$$

$$\frac{ds - ds_0}{ds_0} = \frac{(ds)^2 - (ds_0)^2}{ds_0(ds + ds_0)}$$

$$= \frac{2 \varepsilon_{22} ds_0^2}{ds_0(ds + ds_0)}$$

$$\approx \frac{2 \varepsilon_{22} ds_0^2}{2 ds_0^2}$$

$$= \varepsilon_{22}$$

$$\text{Thus, } \varepsilon_{22} = \frac{ds - ds_0}{ds_0}$$

### Linearization

If displacement gradients are small, ie if  $U_{k,i} \ll 1$

$U_{k,i} \cdot U_{k,j}$  can be neglected.

Also, the initial and final coordinate systems are identical

$$\text{Thus } \frac{\partial}{\partial x_i} = \frac{\partial}{\partial X_i}$$

and  $\Sigma_{ij}^L = \Sigma_{ij}^E = \Sigma_{ij} = \frac{1}{2} [U_{i;j} + U_{j;i}]$

$\Sigma_{ij}$  = small strain tensor (Cauchy)

$$\Sigma_{11} = \frac{\partial U_1}{\partial x_1}, \quad \Sigma_{22} = \frac{\partial U_2}{\partial x_2}, \quad \Sigma_{33} = \frac{\partial U_3}{\partial x_3}$$

$$\Sigma_{12} = \frac{1}{2} \left[ \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} \right] = \Sigma_{21}$$

$$\Sigma_{13} = \frac{1}{2} \left[ \frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1} \right] = \Sigma_{31}$$

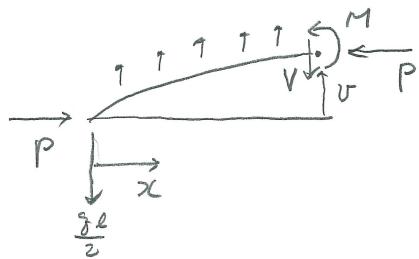
$$\Sigma_{23} = \frac{1}{2} \left[ \frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2} \right] = \Sigma_{32}$$

## Geometric Nonlinear Analysis

Beam - column element



Taking M-equilibrium at x,



$$+M + Pv = \frac{8x^2}{2} - \frac{8l}{2}x$$

$$EI v'' + Pv = \frac{8x^2}{2} - \frac{8l}{2}x$$

$$EI v'' + Pv'' = -g \Rightarrow \text{governing Eq.}$$

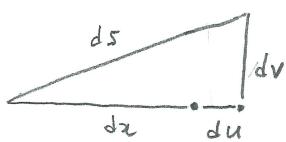
For large displacement,

$$\varepsilon_{ij} = \frac{1}{2} [ u_{ij} + u_{j,i} + u_{k,i} u_{k,j} ]$$

$$\Rightarrow \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2$$

Interpretation

change in length due to change in geometry



$$ds^2 = (dx + du)^2 + dv^2$$

$$= dx^2 \left[ 1 + 2 \frac{du}{dx} + \left( \frac{du}{dx} \right)^2 + \left( \frac{dv}{dx} \right)^2 \right]$$

$$ds = dx \sqrt{1 + 2 \frac{du}{dx} + \left( \frac{du}{dx} \right)^2 + \left( \frac{dv}{dx} \right)^2}$$

$$\approx dx \left( 1 + \frac{du}{dx} + \frac{1}{2} \left( \frac{du}{dx} \right)^2 + \frac{1}{2} \left( \frac{dv}{dx} \right)^2 \right)$$

$$\varepsilon_x = \frac{ds - dx}{dx}$$

$$= \frac{ds - dx}{dx} = \underbrace{\frac{du}{dx}}_{\text{strain}} + \frac{1}{2} \left( \frac{du}{dx} \right)^2 + \frac{1}{2} \left( \frac{dv}{dx} \right)^2$$

additional strain due to change in geometry

Total  $\varepsilon_x$  including flexural action

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2} \left( \frac{dv}{dx} \right)^2 \quad \text{neglecting } \frac{1}{2} \left( \frac{du}{dx} \right)^2$$

$$= \underbrace{\frac{du_0}{dx}}_{\text{Axial}} - \gamma \underbrace{\frac{d^2 v}{dx^2}}_{\text{flexural}} + \underbrace{\frac{1}{2} \left( \frac{dv}{dx} \right)^2}_{\text{large deformation}}$$



$$\delta u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \delta v = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

$$\underline{f}_1 = [f_{u1} \ f_{u2}] \quad \underline{f}_2 = [f_{v1} \ f_{v2} \ f_{v3} \ f_{v4}]$$

$$u_0 = \underline{f}_1 \ \underline{\delta u}$$

$$v = \underline{f}_2 \ \underline{\delta v}$$

$$\varepsilon_x = \underline{f}_1' \ \underline{\delta u} - \gamma \underline{f}_2'' \ \underline{\delta v} + \frac{1}{2} \underline{\delta v}^T \underline{f}_2'^T \underline{f}_2' \ \underline{\delta v}$$

Approximation : the shape function is the same after large deformation.  
 $\Rightarrow$  This is not true true shape function should satisfy the force equilibrium eq.

$$\varepsilon_x = u'_0 - \gamma v'' + \frac{1}{2}(v')^2$$

$$\delta \varepsilon_x = \delta u'_0 - \gamma \delta v'' + v' \delta v'$$

$$\delta \varepsilon_x \sigma = (\delta u'_0 - \gamma \delta v'' + v' \delta v') \in (u'_0 - \gamma v'' + \frac{1}{2}(v')^2)$$

$$= E(u'_0 - \gamma v'' + \frac{1}{2}v'^2) \delta u'_0 - E\gamma \delta v'' (u'_0 - \gamma v'' + \frac{1}{2}v'^2) + E v' \delta v' (u'_0 - \gamma v'' + \frac{1}{2}v'^2)$$

$$\int y \, dA = 0 \quad , \quad \int y^2 \, dA = I$$

$$\int \delta \varepsilon_x \sigma \, dA = EA(u'_0 + \frac{1}{2}v'^2) \delta u'_0 + EI v'' \delta v'' + EA v' \delta v' (u'_0 + \frac{1}{2}v'^2)$$

$$\approx EA u'_0 \delta u'_0 + EI v'' \delta v'' + EA v' \delta v' u'_0$$

$$u_0 = \underline{f}_1 \underline{\delta}_u \quad \text{and} \quad v = \underline{f}_2 \underline{\delta}_v$$

$$\int \delta \epsilon \sigma dA = \delta \underline{\delta}_u^T EA \underline{f}_1' \underline{f}_1' \underline{\delta}_u + \delta \underline{\delta}_v^T EI \underline{f}_2'' \underline{f}_2'' \underline{\delta}_v \\ + \underbrace{\delta \underline{\delta}_v^T (EA u_0') \underline{f}_2'^T \underline{f}_2'}_{\text{current axial force } P_0}$$

$$\delta U = \int \delta \epsilon \sigma dv \\ = \delta \underline{\delta}_u^T [EA \int \underline{f}_1'^T \underline{f}_1' dx] \underline{\delta}_u + \delta \underline{\delta}_v^T [EI \int \underline{f}_2''^T \underline{f}_2'' dx] \underline{\delta}_v \\ + P_0 \int \underline{f}_2'^T \underline{f}_2 dx \underline{\delta}_v$$

$$\delta U = \delta W$$

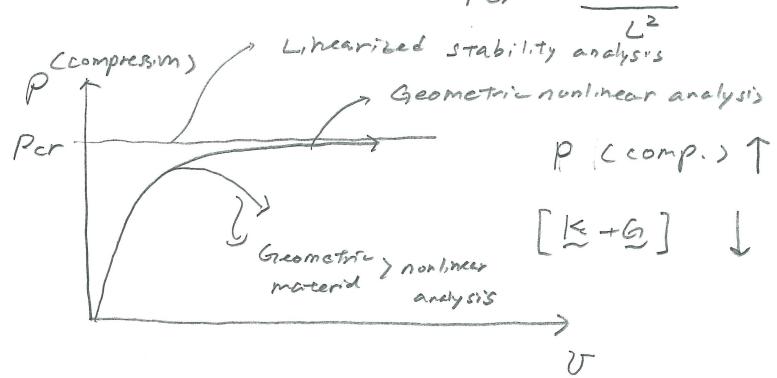
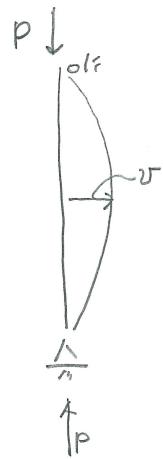
$$\underbrace{[EA \int \underline{f}_1'^T \underline{f}_1' dx]}_{P K_A} \underline{\delta}_u + \underbrace{[EI \int \underline{f}_2''^T \underline{f}_2'' dx]}_{K_B} + \underbrace{[P_0 \int \underline{f}_2'^T \underline{f}_2' dx]}_{G} \underline{\delta}_v = P$$

$K_A$  = elastic stiffness for axial action

$K_B$  = elastic stiffness for bending action

$G$  = geometric stiffness ( $< 0$  if  $P_0 < 0$ )

$$[K_A + G] \underline{\delta} = P$$



$$[\underline{K} + \underline{G}] \underline{\delta} = \underline{P}$$

↓ linearization (assuming the geometry  
is the same before buckling)

$$\Rightarrow [\underline{K} + \underline{G}] d\underline{\delta} = d\underline{P}$$

$$[\underline{K} + \underline{G}] d\underline{\delta} = \underline{0} \quad \text{Linearized Stability Analysis}$$

$$[\underline{K} + P \hat{\underline{G}}] d\underline{\delta} = \underline{0}$$

⇒ Eigenvalue problem ⇒ solve  $P_r$  (buckling force)

$d\underline{\delta}$  for  $P$  = buckling mode

$$\underline{G} = P_0 \int f_2'^T f_2' dx$$

$$= P_0 \begin{bmatrix} \int f_{v1}' f_{v1}' dx & \int f_{v1}' f_{v2}' dx & \int f_{v1}' f_{v3}' dx & \int f_{v1}' f_{v4}' dx \\ \int f_{v2}' f_{v1}' dx & \int f_{v2}' f_{v2}' dx & \int f_{v2}' f_{v3}' dx & \int f_{v2}' f_{v4}' dx \\ \text{sym.} & \int f_{v3}' f_{v1}' dx & \int f_{v3}' f_{v2}' dx & \int f_{v3}' f_{v4}' dx \\ & & & \int f_{v4}' f_{v1}' dx \end{bmatrix}$$

## Geometric Nonlinear Analysis for plate bending

$$\varepsilon_{ij} = \frac{1}{2} [u_{i,j} + u_{j,i} + u_{k,i} u_{kj}]$$

$$\left\{ \begin{array}{l} \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\ \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{array} \right. \quad \left( = \frac{1}{2} [u_{xx} + u_{yy} + u_{x,y} u_{y,x} + v_{x,y} v_{y,x} + w_{xx} w_{yy}] \right)$$

for plate bending,

(membrane)  
plane stress

$$u = u_0 - z \frac{\partial w}{\partial x}$$

$$v = v_0 - z \frac{\partial w}{\partial y}$$

$$\left\{ \begin{array}{l} \varepsilon_x = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\ \varepsilon_y = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\ \gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \end{array} \right. \quad \text{additional strain due to large deformation}$$

$$\bar{\sigma} = \begin{bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\gamma}_{xy} \end{bmatrix}$$

$$\bar{\epsilon} = \begin{bmatrix} \frac{E}{1-v^2} & \frac{vE}{1-v^2} & 0 \\ \frac{vE}{1-v^2} & \frac{E}{1-v^2} & 0 \\ 0 & 0 & G \end{bmatrix}$$

$$\bar{\sigma}_x = \frac{E}{(1-v^2)} \left[ u_{0,x} - z w_{,xx} + v v_{0,y} - v z w_{,yy} + \frac{1}{2} w_{,x}^2 + \frac{1}{2} v w_{,y}^2 \right]$$

$$\bar{\sigma}_y = \frac{E}{(1-v^2)} \left[ v u_{0,x} - v z w_{,xx} + v v_{0,y} - z w_{,yy} + \frac{1}{2} v w_{,x}^2 + \frac{1}{2} w_{,y}^2 \right]$$

$$\bar{\gamma}_{xy} = G \left[ u_{0,y} + v_{0,x} - 2z w_{,xy} + w_{,x} w_{,y} \right]$$

$$\int \delta \Sigma^T \sigma dV = \int (\delta \varepsilon_x \sigma_x + \delta \varepsilon_y \sigma_y + \delta \gamma_{xy} \tau_{xy}) dV$$

$$\delta \varepsilon_x \sigma_x = \frac{E}{(1-v^2)} \left[ \delta u_{0,x} - z \delta w_{,xx} + \delta w_{,x} w_{,x} \right] \times \\ \left[ u_{0,x} - z w_{,xx} + v w_{,yy} - v z w_{,yy} + \left( \frac{1}{2} w_{,x}^2 + \frac{1}{2} v w_{,y}^2 \right) \right]$$

$\sigma$  small

considering  $\int z dA = 0$  and  $\int z^2 dA = I$

$$\delta \varepsilon_x \sigma_x = \frac{E}{(1-v^2)} \delta u_{0,x} [u_{0,x} + v w_{,y}] \quad \textcircled{1}$$

$$+ \frac{E z^2}{(1-v^2)} [\delta w_{,xx} w_{,xx} + v \delta w_{,x} w_{,yy}] \quad \textcircled{2}$$

$$+ \frac{E}{(1-v^2)} \delta w_{,x} w_{,x} [u_{0,x} + v w_{,y}] \quad \textcircled{3}$$

$\textcircled{1}$  = membrane action

$\textcircled{2}$  = plate bending action

$\textcircled{3}$  = effect of axial force on bending action

$$\textcircled{3} \approx \sigma_{x_0} \delta w_{,x} w_{,x} \quad (\sigma_{x_0} = \frac{E}{(1-v^2)} [u_{0,x} + v w_{,y}])$$

Additional terms due to P-D effect.

$$\left\{ \begin{array}{l} \delta \varepsilon_x \sigma_x \Rightarrow \sigma_{x_0} \delta w_{,x} w_{,x} \\ \delta \varepsilon_y \sigma_y \Rightarrow \sigma_{y_0} \delta w_{,y} w_{,y} \\ \delta \gamma_{xy} \tau_{xy} \Rightarrow \tau_{xy_0} (\delta w_{,x} w_{,y} + \delta w_{,y} w_{,x}) \end{array} \right.$$

$$\delta U = \delta W$$

$$\delta \underline{\underline{\delta}}_1^T \underline{\underline{K}}_1 + \delta \underline{\underline{\delta}}_2^T \underline{\underline{K}}_2 + \delta \underline{\underline{\delta}}_2^T \underline{\underline{G}} \underline{\underline{\delta}}_2 = \delta \underline{\underline{\delta}}_1^T \underline{\underline{P}}_1 + \delta \underline{\underline{\delta}}_2^T \underline{\underline{P}}_2$$

$$\underline{\underline{\delta}}_1 = \begin{bmatrix} u_{01} \\ v_{01} \\ u_{02} \\ v_{02} \\ \vdots \end{bmatrix} \quad \underline{\underline{\delta}}_2 = \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \vdots \end{bmatrix}$$

$$[\underline{\underline{K}}_1 + \underline{\underline{K}}_2 + \underline{\underline{G}}] \underline{\underline{\delta}} = \underline{\underline{P}}$$

$\underline{\underline{K}}_1$  = membrane stiffness

$\underline{\underline{K}}_2$  = plate bending stiffness

$\underline{\underline{G}}$  = Geometric stiffness

$$\delta U_G = \int \sigma_{x0} \delta w_{,x} w_{,x} dV + \int \sigma_{y0} \delta w_{,y} w_{,y} dV \\ + \int z_{xy0} \delta w_{,x} w_{,y} dV + \int z_{xy0} \delta w_{,y} w_{,x} dV$$

$$\underline{\underline{G}} = \int \sigma_{x0} \underline{\underline{f}}_{2,x}^T \underline{\underline{f}}_{2,x} dV + \int \sigma_{y0} \underline{\underline{f}}_{2,y}^T \underline{\underline{f}}_{2,y} dV \\ + \int z_{xy0} \underline{\underline{f}}_{2,x}^T \underline{\underline{f}}_{2,y} dV + \int z_{xy0} \underline{\underline{f}}_{2,y}^T \underline{\underline{f}}_{2,x} dV$$

$$w = \underline{\underline{f}}_2 \cdot \underline{\underline{\delta}}_2 \quad \underline{\underline{f}}_2 = \text{shape function for plate bending}$$

$$[\underline{K}_1 + \underline{K}_2 + \underline{G}] \underline{\delta} = \underline{P}$$

$$\Rightarrow [\underline{K}_1 + \underline{K}_2 + \underline{G}] d\underline{\delta} = d\underline{P} \quad \text{linearization}$$

$$[\underline{K}_1 + \underline{K}_2 + \underline{G}] d\underline{\delta} = 0 \quad \text{linearized stability analysis}$$

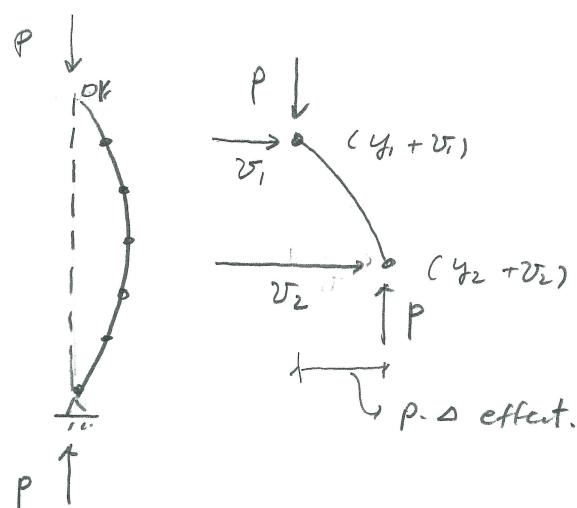
$$|\underline{K}_1 + \underline{K}_2 + \lambda \hat{\underline{G}}| = 0 \quad \text{solve } \lambda \quad \lambda P = \text{buckling force}$$

$d\underline{\delta}$  = buckling mode

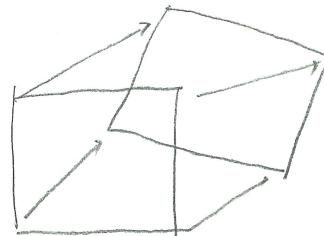
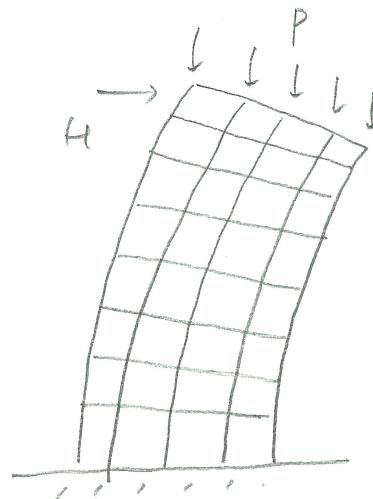
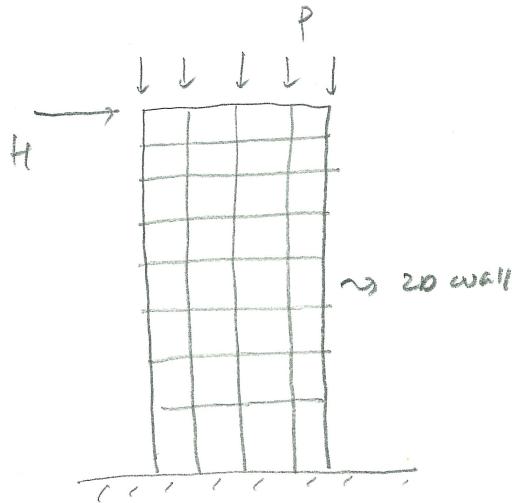
## Geometric Nonlinear Analysis using updated coordinates

Though nonlinear analysis can be performed using FEM analysis, still several assumptions are used. For example, the shape function used for small deformation is still used for large deformation. Therefore, the geometric stiffness approach can be acceptable if the deflection is limited to a moderate value.

Further, the P-Δ effect can be directly addressed when the coordinates of the structure is updated with the deflections.



### Plane stress Element



$$\begin{aligned} \underline{\underline{x}}^t &= \underline{\underline{x}}^0 + \Delta \underline{\underline{u}} \\ \underline{\underline{x}}^0 &= \underline{\underline{x}}^t \\ \text{new } \underline{\underline{x}}^t &= \underline{\underline{x}}^0 + \text{new } \Delta \underline{\underline{u}} \end{aligned}$$

until  $\Delta \underline{\underline{u}} \rightarrow 0$ , the coordinates  $\underline{\underline{x}}^t$  is updated.

# Nonlinear Analysis

## ① General Input

- ② Input
  - Material properties
  - Geometry and node numbering
  - Element numbering and properties
  - Load and combinations
  - Nonlinear Analysis
    - control method
    - load step size  $\Delta P$
    - target force or displacement  $P_{target}$

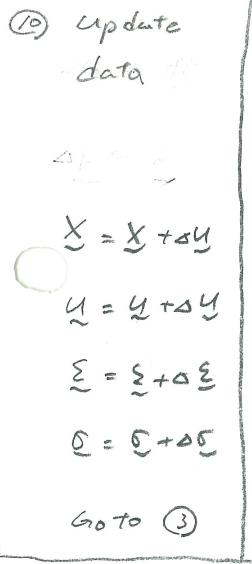
→ ③ Construct element stiffness  $\underline{k}_e$  and

Assemble  $\underline{k}_e$ 's to make structural stiffness  $\underline{K}$

(in  $\underline{K}$ , geometric stiffness  $\underline{G}$  may be included)

$$\underline{K} \Delta \underline{U} = \Delta \underline{P}$$

Solve incremental displacement  $\Delta \underline{U}$



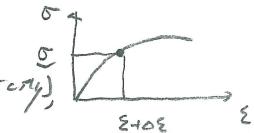
⑤ Calculate incremental and/or total strains and stresses at Gauss Integration points

$$\underline{\epsilon} \underline{\Sigma} = \underline{B} \Delta \underline{\delta}$$

Elastic material  $\underline{\Sigma} = \underline{\sigma} + \Delta \underline{\Sigma} (= E \underline{B} \Delta \underline{\delta})$

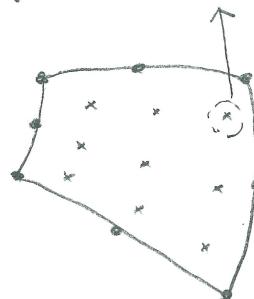
material nonlinearity  $\underline{\Sigma} = f(\underline{\epsilon} + \Delta \underline{\epsilon})$

if advanced material rule is used (plasticity),  
 $\underline{\Sigma} - \underline{\epsilon}$  relationship is more complicated.



⑥ Calculate internal forces

$$\underline{Q} = \sum \int \underline{B}^T \underline{\Sigma} dV$$



⑦ Calculate unbalance force

$$\underline{P} (= \underline{P} + \Delta \underline{P}) - \underline{Q} = \underline{R}$$

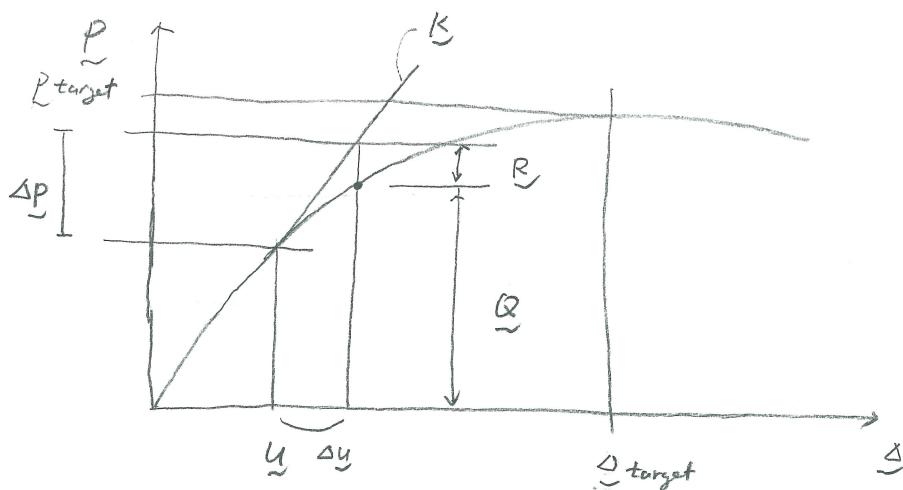
⑧ If  $\underline{R}$  is small enough, Go to ⑨

otherwise  $\Delta \underline{P} = \underline{R}$  Go to 10

⑨ CHECK  $\underline{P} \rightarrow \underline{P}_{\text{target}}$

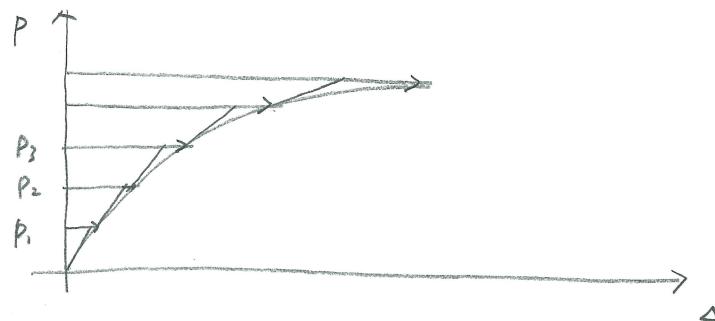
If yes  $\Rightarrow$  finish

If not  $\Delta \underline{P} = \underline{P}$  Go to 10.

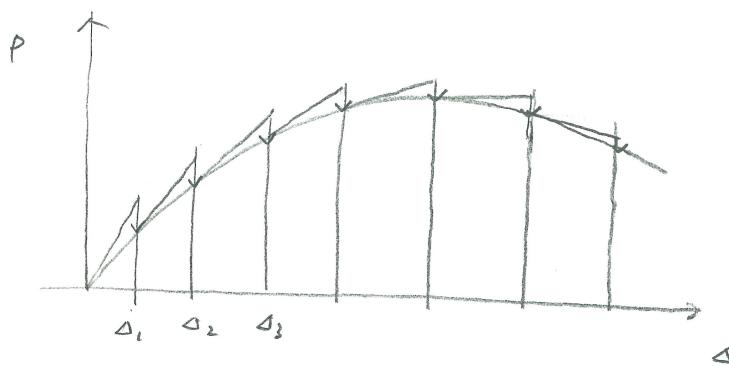


## Solution scheme for nonlinear analysis

### Force - control



### Displacement - control



### Arc-length method

