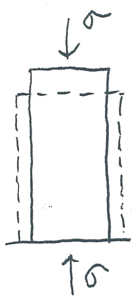
Chapter 4. Stiffness Analysis of Frames - I

4.1 Stress - strain relationships

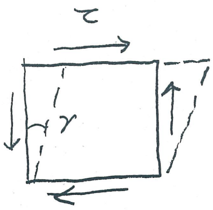
under uniaxial stress



$$\varepsilon = \frac{\sigma}{E} \qquad E = \text{elastic modulus}$$

$$\varepsilon_\ell = -\nu\,\varepsilon = -\nu \frac{\sigma}{E} \qquad \nu = \text{poisson's ratio}$$

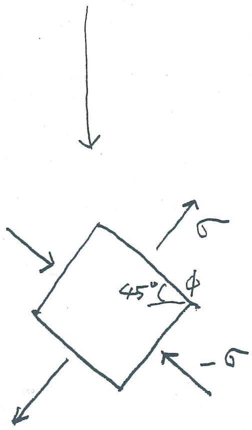
under pure shear

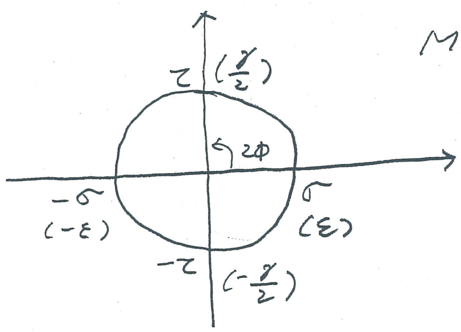


$$\gamma = \frac{\tau}{G} \qquad \gamma = 2\,\varepsilon_{xy}$$

(engineering shear strain)

$$G = \frac{E}{2(1+\nu)}$$





Mohr circle

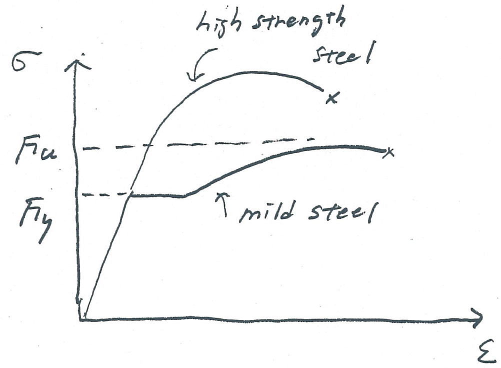
$$\varepsilon = \frac{\sigma}{E} - \nu \frac{(-\sigma)}{E} = \frac{\sigma}{E}(1+\nu) = \frac{\gamma}{2}$$

under pure shear $\tau = \sigma$

$$\tau = \frac{E}{2(1+\nu)}\,\gamma = G\gamma$$

Material properties

steel



F_y = 250 ~ 700 MPa

ν = 0.3

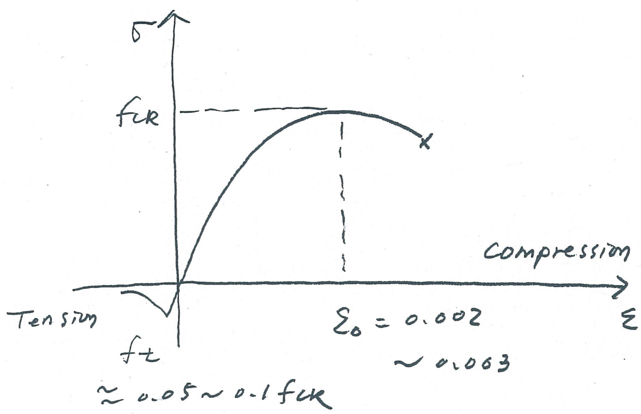
E = 200,000 MPa

Relatively expensive

high strength and stiffness
in both tension and compression

low fire resistance

concrete



f_{ck} = 21 ~ 80 MPa
(ordinary cement)

ν = 0.15

E = $4700\sqrt{f_{ck}}$ MPa

economical

high strength only in compression

high fire resistance

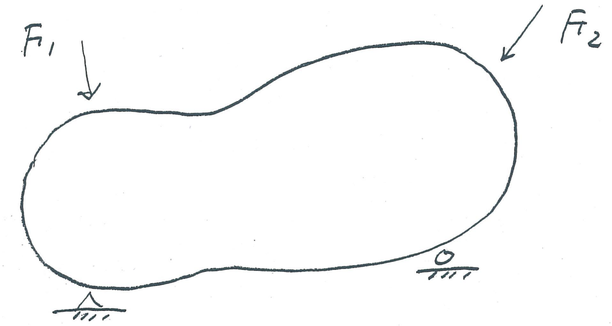
high durability

4.2 Work and energy

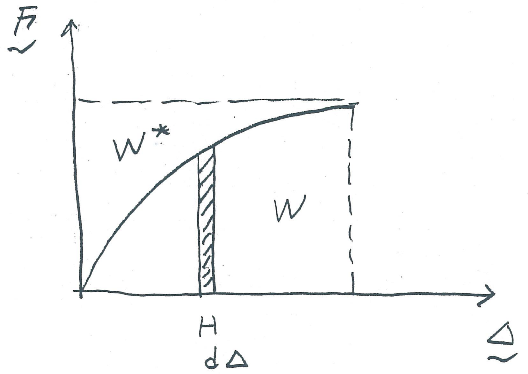
Force ⇒ vector with magnitude and orientation

Work ⇒ scalor with only magnitude (= Σ force × displacement)

Work principle can be conveniently used to derive $\underset{\sim}{K}$ matrix.



External work



W = work $= \int F \, d\Delta$

W^* = complimentary work

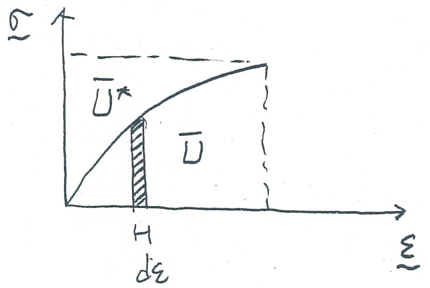
$= \int \Delta \, dF$

For linear elastic material

$$W = \frac{1}{2} \underset{\sim}{F} \cdot \underset{\sim}{\Delta} = \frac{1}{2} \underset{\sim}{\Delta}^T \underset{\sim}{K} \underset{\sim}{\Delta}$$

$$W^* = \frac{1}{2} \underset{\sim}{F} \cdot \underset{\sim}{\Delta} = \frac{1}{2} \underset{\sim}{F_f}^T \underset{\sim}{d} \underset{\sim}{F_f}$$

Internal Work



$\bar{U}$ = strain energy density

$$= \int \underset{\sim}{\sigma}\, d\underset{\sim}{\varepsilon}$$

U = strain energy (internal energy)

$$= \int \bar{U}\, dV$$

$\bar{U}^*$ = complimentary strain energy density

$$= \int \underset{\sim}{\varepsilon}\, d\underset{\sim}{\sigma}$$

$$U^* = \int \bar{U}^*\, dV$$

For linear elastic material,

$$\bar{U} = \frac{1}{2}\underset{\sim}{\sigma} \cdot \underset{\sim}{\varepsilon} = \bar{U}^*$$

Energy Conservation

External Work = Internal Work

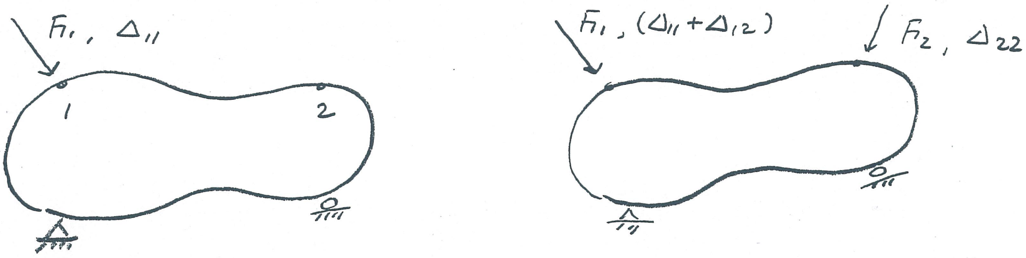
$$W = U \quad \text{and} \quad W^* = U^*$$

Example 4.1

4.3 Reciprocity

$$\left.\begin{array}{lll} \underset{\sim}{K} : & k_{ij} = k_{ji} \\ \underset{\sim}{d} : & d_{ij} = d_{ji} \end{array}\right] \Rightarrow \text{symmetry} \Rightarrow \text{computational Efficiency}$$

Maxwell's Reciprocal Theorem



Apply F_1 and Then F_2 $\Rightarrow$ work W_I

$$W_{I1} \ (\text{By } F_1) = \frac{1}{2}\Delta_{11}F_1 = \frac{1}{2}d_{11}F_1F_1$$

$$W_{I2} \ (\text{By } F_2) = \frac{1}{2}\Delta_{22}F_2 + \Delta_{12}F_1$$

$$= \frac{1}{2}d_{22}F_2F_2 + d_{12}F_2F_1$$

$$W_I = W_{I1} + W_{I2} = \frac{1}{2}d_{11}F_1^2 + d_{12}F_1F_2 + \frac{1}{2}d_{22}F_2^2$$

Apply F_2 and then F_1 $\Rightarrow$ work W_{II}

$$W_{II} = W_{II1} + W_{II2} = \frac{1}{2}d_{11}F_1^2 + d_{21}F_1F_2 + \frac{1}{2}d_{22}F_2^2$$

For linear elastic system,

$$W_I \equiv W_{II}$$

$$\Rightarrow \quad d_{12} \equiv d_{21} \quad \Rightarrow k_{12} \equiv k_{21}$$

4.4 Flexibility – Stiffness Transformations

4.4.1 Stiffness – to – flexibility transformation

$$\begin{bmatrix} \underset{\sim}{F}_f \\ \underset{\sim}{F}_s \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{k}_{ff} & \underset{\sim}{k}_{fs} \\ \hline \underset{\sim}{k}_{sf} & \underset{\sim}{k}_{ss} \end{array}\right] \begin{bmatrix} \underset{\sim}{\Delta}_f \\ \underset{\sim}{\Delta}_s \end{bmatrix}$$

if $\underset{\sim}{\Delta}_s = 0$

$$\begin{bmatrix} \underset{\sim}{F}_f \\ \underset{\sim}{F}_s \end{bmatrix} = \begin{bmatrix} \underset{\sim}{k}_{ff} \\ \underset{\sim}{k}_{sf} \end{bmatrix} \underset{\sim}{\Delta}_f$$

$$\underset{\sim}{F}_f = \underset{\sim}{k}_{ff} \underset{\sim}{\Delta}_f \Rightarrow \underset{\sim}{\Delta}_f = \underset{\sim}{d} \underset{\sim}{F}_f$$

$$\underset{\sim}{d} = \underset{\sim}{k}_{ff}^{-1}$$

4.4.2 flexibility – to – stiffness transformation

$$\underset{\sim}{F}_f = \underset{\sim}{d}^{-1} \underset{\sim}{\Delta}_f$$

$$= \underset{\sim}{K}_{ff} \underset{\sim}{\Delta}_f$$

$$\underset{\sim}{K}_{ff} = \underset{\sim}{d}^{-1}$$

other terms? $\underset{\sim}{k}_{fs}$, $\underset{\sim}{k}_{sf}$, $\underset{\sim}{k}_{ss}$?

$$\underset{\sim}{F}_s = \underset{\sim}{\Phi}\, \underset{\sim}{F}_f \qquad \text{by force-equilibrium}$$

$$\underset{\sim}{\Phi} = \text{equilibrium matrix}$$

$$= \underset{\sim}{\Phi}\, \underset{\sim}{d}^{-1} \underset{\sim}{\Delta}_f = \underset{\sim}{k}_{sf} \underset{\sim}{\Delta}_f$$

$$\underset{\sim}{k}_{sf} = \underset{\sim}{\Phi}\, \underset{\sim}{d}^{-1}$$

From reciprocal theorem,

$$\underset{\sim}{k}_{fs} = \underset{\sim}{k}_{sf}^T = (\underset{\sim}{d}^{-1})^T \underset{\sim}{\Phi}^T = \underset{\sim}{d}^{-1} \underset{\sim}{\Phi}^T$$

$$\underset{\sim}{d}^{-1} = \text{symmetric}$$

$$\underset{\sim}{F}_s = \underset{\sim}{\Phi}\, \underset{\sim}{F}_f = \underset{\sim}{\Phi} \left[\underset{\sim}{k}_{ff} \underset{\sim}{\Delta}_f + \underset{\sim}{k}_{fs} \underset{\sim}{\Delta}_s \right]$$

$$= \underset{\sim}{\Phi} \left[\underset{\sim}{d}^{-1} \underset{\sim}{\Delta}_f + \underset{\sim}{d}^{-1} \underset{\sim}{\Phi}^T \underset{\sim}{\Delta}_s \right]$$

$$\underset{\sim}{F}_s = \underset{\sim}{k}_{sf} \underset{\sim}{\Delta}_f + \underset{\sim}{k}_{ss} \underset{\sim}{\Delta}_s$$

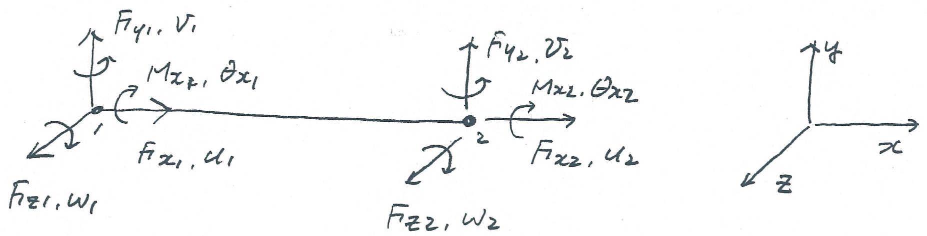
$$\underset{\sim}{k}_{ss} = \underset{\sim}{\Phi}\, \underset{\sim}{d}^{-1} \underset{\sim}{\Phi}^T$$

$$\underset{\sim}{k} = \left[\begin{array}{c|c} \underset{\sim}{k}_{ff} & \underset{\sim}{k}_{fs} \\ \hline \underset{\sim}{k}_{sf} & \underset{\sim}{k}_{ss} \end{array} \right]$$

$$= \left[\begin{array}{c|c} \underset{\sim}{d}^{-1} & \underset{\sim}{d}^{-1} \underset{\sim}{\Phi}^T \\ \hline \underset{\sim}{\Phi}\, \underset{\sim}{d}^{-1} & \underset{\sim}{\Phi}\, \underset{\sim}{d}^{-1} \underset{\sim}{\Phi}^T \end{array} \right]$$

Example 4.2

4.5 Framework Element Stiffness Matrix



6 Degree of freedom per each node

For 3 dimensional frame element, 12 dof

$\underset{\sim}{u} = \langle u_1\ v_1\ w_1\ \theta_{x1}\ \theta_{y1}\ \theta_{z1}\ \ u_2\ v_2\ w_2\ \theta_{x2}\ \theta_{y2}\ \theta_{z2} \rangle$

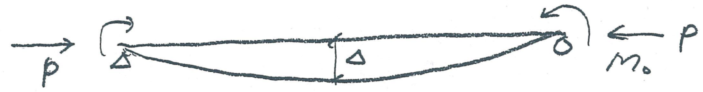
$\underset{\sim}{F} = \langle F_{x1}\ F_{y1}\ F_{z1}\ M_{x1}\ M_{y1}\ M_{z1}\ \ F_{x2}\ F_{y2}\ F_{z2}\ M_{x2}\ M_{y2}\ M_{z2} \rangle$

Assuming
1) small deformation
2) cross-section is bisymmetric
3) no warping (distortion of cross-section)

The behavior of a 3-D frame element can be uncoupled into four actions, which is independent

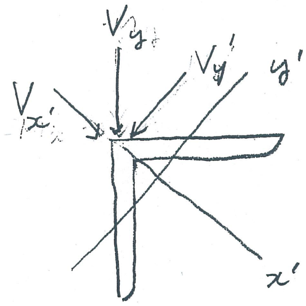
1) axial force $\underset{\sim}{u} = \langle u_1\ \ u_2 \rangle$
2) pure torsion $\underset{\sim}{u} = \langle \theta_{x1},\ \theta_{x2} \rangle$
3) major axis bending $\underset{\sim}{u} = \langle v_1,\ \theta_{z1},\ v_2\ \theta_{z2} \rangle$
4) minor axis bending $\underset{\sim}{u} = \langle w_1\ \theta_{y1}\ w_2\ \theta_{y2} \rangle$

large deformation - coupled actions of flexure and axial compression



M at center $= M_o + P \cdot \Delta$

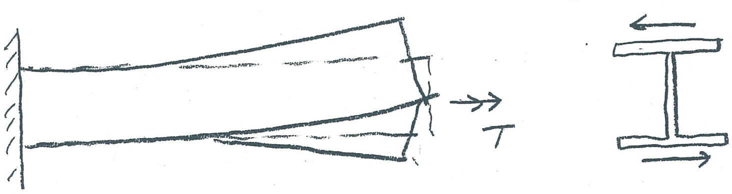
Asymmetric cross-section



V_y induces $M_{x'}$ and $M_{y'}$ in local axis.

coupled action of major axis bending and minor axis bending

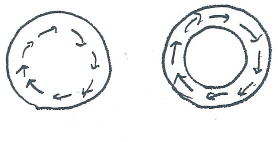
warping (warping torsion versus pure torsion)



coupled action of torsion and minor axis bending.

when a torsional moment is applied to a asymmetric section or open section, the cross-section is distorted.

As a result, normal stresses are developed by the torsion



$\Rightarrow$ pure torsion causing shear stress only

no warping

4.5.1 Axial Force Member

$$u_2 = \int_0^L \varepsilon \, dx = \int_0^L \frac{\sigma}{E} dx = \int_0^L \frac{F_{x2}}{EA} dx = \frac{F_{x2} L}{EA}$$

$$\underset{\sim}{d} = \frac{L}{EA} \qquad \underset{\sim}{k}_{ff} = \underset{\sim}{d}^{-1} = \frac{EA}{L}$$

$$F_{x1} = \underset{\sim}{\phi} F_{x2} \qquad \underset{\sim}{\phi} = -1$$

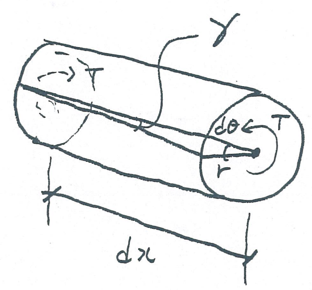
$$\underset{\sim}{k}_{sf} = \underset{\sim}{\phi}\, \underset{\sim}{d}^{-1} = -\frac{EA}{L}$$

$$\underset{\sim}{k}_{fs} = \underset{\sim}{d}^{-1} \underset{\sim}{\phi}^T = -\frac{EA}{L}$$

$$\underset{\sim}{k}_{ss} = \underset{\sim}{\phi}\, \underset{\sim}{d}^{-1} \underset{\sim}{\phi}^T = \frac{EA}{L}$$

$$\underset{\sim}{k} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

4.5.2 Pure Torsional Member



$$\gamma \cdot dx = r \, d\theta$$

$$\gamma = r \frac{d\theta}{dx} = r\beta$$

$$\tau = G\gamma = Gr\beta \qquad dA = r\, d\phi\, dr$$

$$T = \int \tau \cdot r \, dA = \iint G r^3 \beta \, d\phi \, dr$$

$$= GJ\beta \qquad J = \text{torsional constant}$$

$$= \iint r^3 dr\, d\phi$$

rate of twist $\beta \left(= \frac{d\theta_x}{dx}\right) = \frac{M_{x2}}{GJ}$

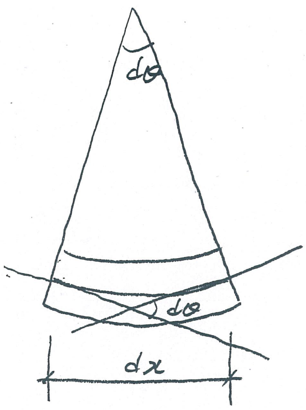
$$\theta_{x2} = \int_0^L \beta \, dx = \frac{L}{GJ} M_{x2}$$

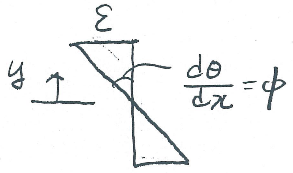
$$\underset{\sim}{d} = \frac{L}{GJ} \qquad \underset{\sim}{k}_{ff} = \underset{\sim}{d}^{-1} = \frac{GJ}{L}$$

$$M_{x1} = -M_{x2} \quad \Rightarrow \quad \underset{\sim}{\phi} = -1$$

$$\underset{\sim}{k} = \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

4.5.3 Beam bent about its z axis





$$\varepsilon = -y \frac{d\theta}{dx}$$
$$= -y \frac{d^2 v}{dx^2}$$

$$\varepsilon_x = -y \frac{d^2 v}{dx^2}$$

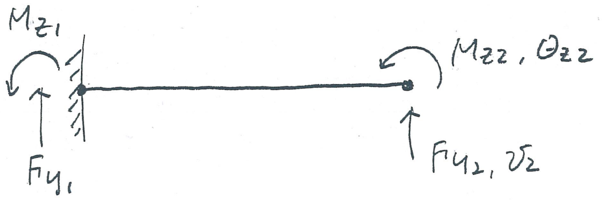
$$\sigma_x = -E y \frac{d^2 v}{dx^2}$$

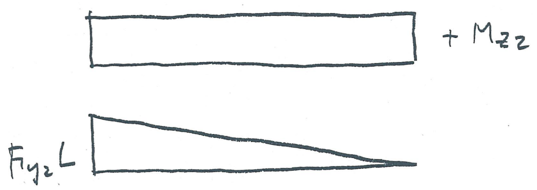
$$M_x = -\int \sigma_x \, y \, dA = E I_z \frac{d^2 v}{dx^2} = E I_z \phi$$

M_x = Generalized stress

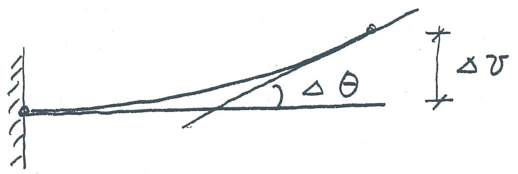
$\frac{d^2 v}{dx^2}$ = Generalized strain

EI_z = Generalized stiffness





moment - area method (curvature area method)



$$\Delta\theta = \int \phi \, dx = \int \frac{M}{EI} dx \quad \left(= \int d\theta\right)$$

$$\Delta v = \int d\theta \cdot x = \int \frac{M}{EI} \cdot x \, dx$$

$$\theta_{z2} = \frac{M_{z2}}{EI} \cdot L + \frac{F_{y2}L}{EI} \cdot \frac{L}{2}$$

$$v_2 = \frac{M_{z2}}{EI} L \cdot \frac{L}{2} + \frac{F_{y2}L}{EI} \cdot \frac{L}{2} \cdot \frac{2}{3}L$$

$$\begin{bmatrix} v_2 \\ \theta_{z2} \end{bmatrix} = \underbrace{\frac{L}{EI_z} \begin{bmatrix} \frac{L^2}{3} & \frac{L}{2} \\ \frac{L}{2} & 1 \end{bmatrix}}_{\underset{\sim}{d}} \begin{bmatrix} F_{y2} \\ M_{z2} \end{bmatrix}$$

$$\underset{\sim}{\Delta_f} = \underset{\sim}{d} \; \underset{\sim}{F_f}$$

By force - equilibrium

$$F_{y1} = -F_{y2}$$

$$M_{z1} = -F_{y2} \cdot L - M_{z2}$$

$$\begin{bmatrix} F_{y1} \\ M_{z1} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ -L & -1 \end{bmatrix}}_{\phi} \begin{bmatrix} F_{y2} \\ M_{z2} \end{bmatrix}$$

$$\underset{\sim}{F_s} = \underset{\sim}{\phi} \; \underset{\sim}{F_f}$$

$$\underset{\sim}{k} = \left[\begin{array}{c|c} \underset{\sim}{d}^{-1} & \underset{\sim}{d}^{-1}\underset{\sim}{\phi}^T \\ \hline \underset{\sim}{\phi}\,\underset{\sim}{d}^{-1} & \underset{\sim}{\phi}\,\underset{\sim}{d}^{-1}\underset{\sim}{\phi}^T \end{array}\right]$$

rearrange $\underset{\sim}{k}$

$$= \frac{EI_z}{L} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{c|c|c|c} 12/L^2 & 6/L & -12/L^2 & 6/L \\ \hline 6/L & 4 & -6/L & 2 \\ \hline -12/L^2 & -6/L & 12/L^2 & -6/L \\ \hline 6/L & 2 & -6/L & 4 \end{array}\right] \end{array} \qquad \text{Eq. (4.32)}$$

see examples 4.3 and 4.4

4.5.4 Beam bent about its y axis

$\underset{\sim}{k}$ for this action can be independently drived in the same manner

see eq. (4.33)

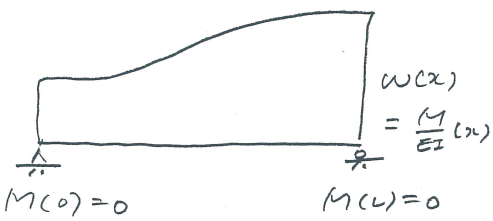
Example 4.5

conjugate beam method

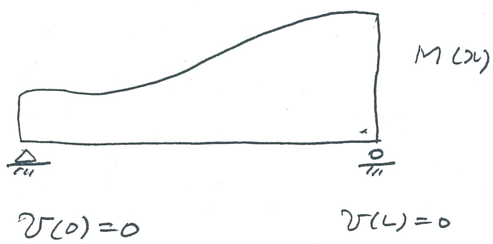
$$\frac{dM}{dx} = V \qquad\qquad \frac{dv}{dx} = \theta$$

$$\Longleftarrow$$

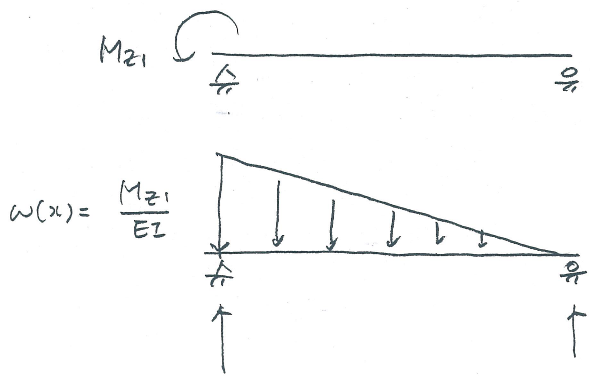
$$\frac{dV}{dx} = w \qquad\qquad \frac{d\theta}{dx} = \phi \quad \left(= \frac{M}{EI}\right)$$



conjugate beam



actual beam



$$\underbrace{V_1 = \frac{M}{EI}\cdot\frac{L}{3}}_{\Downarrow \atop \theta_{z1}} \qquad\qquad \underbrace{V_2 = \frac{M}{EI}\cdot\frac{L}{6}}_{\Downarrow \atop \theta_{z2}}$$

4.5.5 Complete Element Stiffness matrix

$$\underset{\sim}{k} = \underset{\sim}{k}_{axial} + \underset{\sim}{k}_{torsion} + \underline{k}_{z\text{-bending}} + \underline{k}_{y\text{-bending}}$$

All actions can be superimposed because they are assumed to be independent $\Rightarrow$ Eq. (4.34)

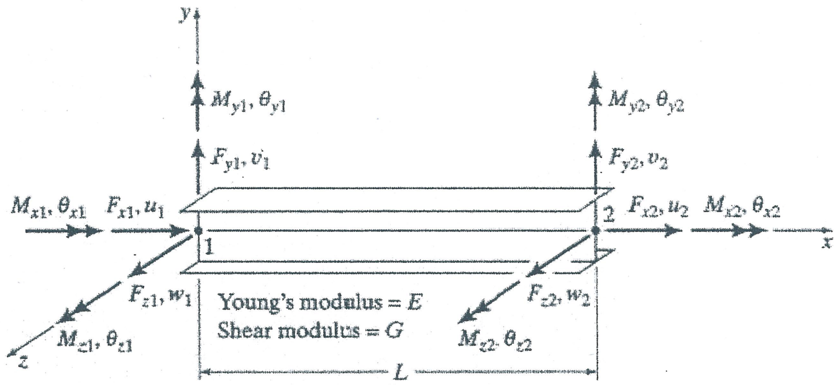


Figure 4.6 Bisymmetrical framework element.

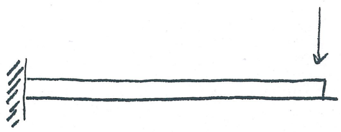
$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \\ M_{x1} \\ M_{y1} \\ M_{z1} \\ \hdashline F_{x2} \\ F_{y2} \\ F_{z2} \\ M_{x2} \\ M_{y2} \\ M_{z2} \end{Bmatrix} = E \left[\begin{array}{cccccc:cccccc}
\frac{A}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{A}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12I_z}{L^3} & 0 & 0 & 0 & \frac{6I_z}{L^2} & 0 & -\frac{12I_z}{L^3} & 0 & 0 & 0 & \frac{6I_z}{L^2} \\
0 & 0 & \frac{12I_y}{L^3} & 0 & -\frac{6I_y}{L^2} & 0 & 0 & 0 & -\frac{12I_y}{L^3} & 0 & -\frac{6I_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{J}{2(1+\nu)L} & 0 & 0 & 0 & 0 & 0 & -\frac{J}{2(1+\nu)L} & 0 & 0 \\
0 & 0 & -\frac{6I_y}{L^2} & 0 & \frac{4I_y}{L} & 0 & 0 & 0 & \frac{6I_y}{L^2} & 0 & \frac{2I_y}{L} & 0 \\
0 & \frac{6I_z}{L^2} & 0 & 0 & 0 & \frac{4I_z}{L} & 0 & -\frac{6I_z}{L^2} & 0 & 0 & 0 & \frac{2I_z}{L} \\ \hdashline
-\frac{A}{L} & 0 & 0 & 0 & 0 & 0 & \frac{A}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12I_z}{L^3} & 0 & 0 & 0 & -\frac{6I_z}{L^2} & 0 & \frac{12I_z}{L^3} & 0 & 0 & 0 & -\frac{6I_z}{L^2} \\
0 & 0 & -\frac{12I_y}{L^3} & 0 & \frac{6I_y}{L^2} & 0 & 0 & 0 & \frac{12I_y}{L^3} & 0 & \frac{6I_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{J}{2(1+\nu)L} & 0 & 0 & 0 & 0 & 0 & \frac{J}{2(1+\nu)L} & 0 & 0 \\
0 & 0 & -\frac{6I_y}{L^2} & 0 & \frac{2I_y}{L} & 0 & 0 & 0 & \frac{6I_y}{L^2} & 0 & \frac{4I_y}{L} & 0 \\
0 & \frac{6I_z}{L^2} & 0 & 0 & 0 & \frac{2I_z}{L} & 0 & -\frac{6I_z}{L^2} & 0 & 0 & 0 & \frac{4I_z}{L}
\end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_{x1} \\ \theta_{y1} \\ \theta_{z1} \\ \hdashline u_2 \\ v_2 \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ \theta_{z2} \end{Bmatrix}$$

4.6 A commentary on deformations and displacement variables

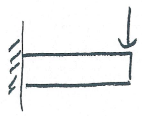
4.6.1 Neglected deformations

- Transverse shear

$\Delta = \Delta_{flexure} + \Delta_{shear}$

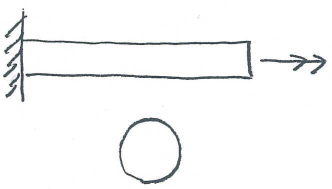


$\Delta \approx \Delta_{flexure}$



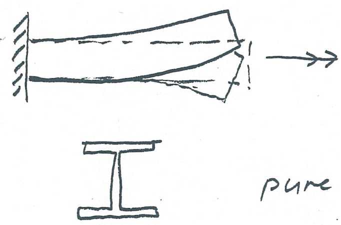
$\Delta = \Delta_{flexure} + \Delta_{shear}$

- Warping torsion



pure torsion

$T \rightarrow \tau$



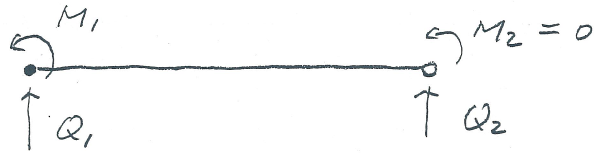
pure torsion
+ warping torsion

$T \rightarrow \tau$ & σ

Torsion + minor axis bending
(⇐ warping torsion)

Example 4.14

Stiffness of element with a hinge

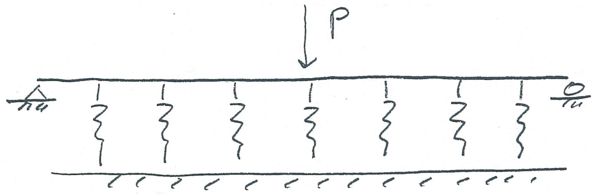


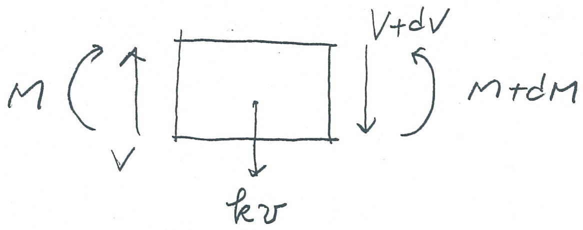
$$EI\begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix} = 0 \quad (4)$$

Eliminate Eq. (4) $\quad \theta_2 = -\frac{3}{2L} v_1 - \frac{1}{2}\theta_1 + \frac{3}{2L} v_2$

$$EI\begin{bmatrix} \frac{3}{L^3} & \frac{3}{L^2} & -\frac{3}{L^3} \\ \frac{3}{L^2} & \frac{3}{L} & -\frac{3}{L^2} \\ -\frac{3}{L^3} & -\frac{3}{L^2} & \frac{3}{L^3} \end{bmatrix}\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \end{bmatrix}$$

Example 4.15





$$\begin{cases} \dfrac{dV}{dx} = -kv \\ \dfrac{dM}{dx} = V \end{cases}$$

$$(\phi =)\ \frac{d^2 v}{dx^2} = \frac{M}{EI}, \qquad \frac{d^2 M}{dx^2} = -kv$$

$$\Rightarrow \underline{EI \frac{d^4 v}{dx^4} = -kv}$$

B/C, $\quad x = 0 \quad v(0) = 0$

$$M(0) = \frac{d^2 v}{dx^2}(0) = 0$$

$x = l \quad v(l) = 0$

$$M(l) = \frac{d^2 v}{dx^2}(l) = 0$$