Lecture 06

Dynamic Reliability of Series & Parallel Systems

Where We Are – Reliability

Reading Articles: Reliability of Complex Systems & Networks

((Handout)) Yen & Tung, "Dynamic Reliability"

((Handout)) Hiller & Lieberman, "Ch 14 Reliability"

Material has two lessons for us:

One is methodology of calculating reliability of parallel, series and other systems and networks.

The second is an appreciation of impact on system reliability of components in parallel, series and other configurations.

Supporting Book

Kottegoda & Rosso, "Ch 9 Risk & Reliability Analysis", *Statistics, Probability, & Reliability for Civil and Environmental Engineers*

Taylor Series

For continuously differentiable functions, ***near*** a point x_0, the value of f(x) should be well approximated by the first few terms of the Taylor series:

$$f(x) = f(x_0) + (x - x_0)\, f'(x_0) + (1/2)\,(x - x_0)^2 f''(x_0) + (1/3!)\,(x - x_0)^3 f'''(x_0) + \dots$$

Beware that Taylor series may converge only for small neighborhood of x_0.

Taylor Series Examples $[x_0 = 0]$

$$\exp(x) = 1 + x + x^2/2 + x^3/6 + x^4/4! + \dots$$

$$\sin(x) = x - x^3/3! + x^5/5! - \dots$$

$$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$$

Integrals for Reliability Calculations

Constant c is introduced in indefinite integrals because they do not specify upper and lower bounds on the interval over which the integral is to be evaluated.

$\int a\, dx = a x + c$

$\int a\, f(x)\, dx = a \int f(x)\, dx$

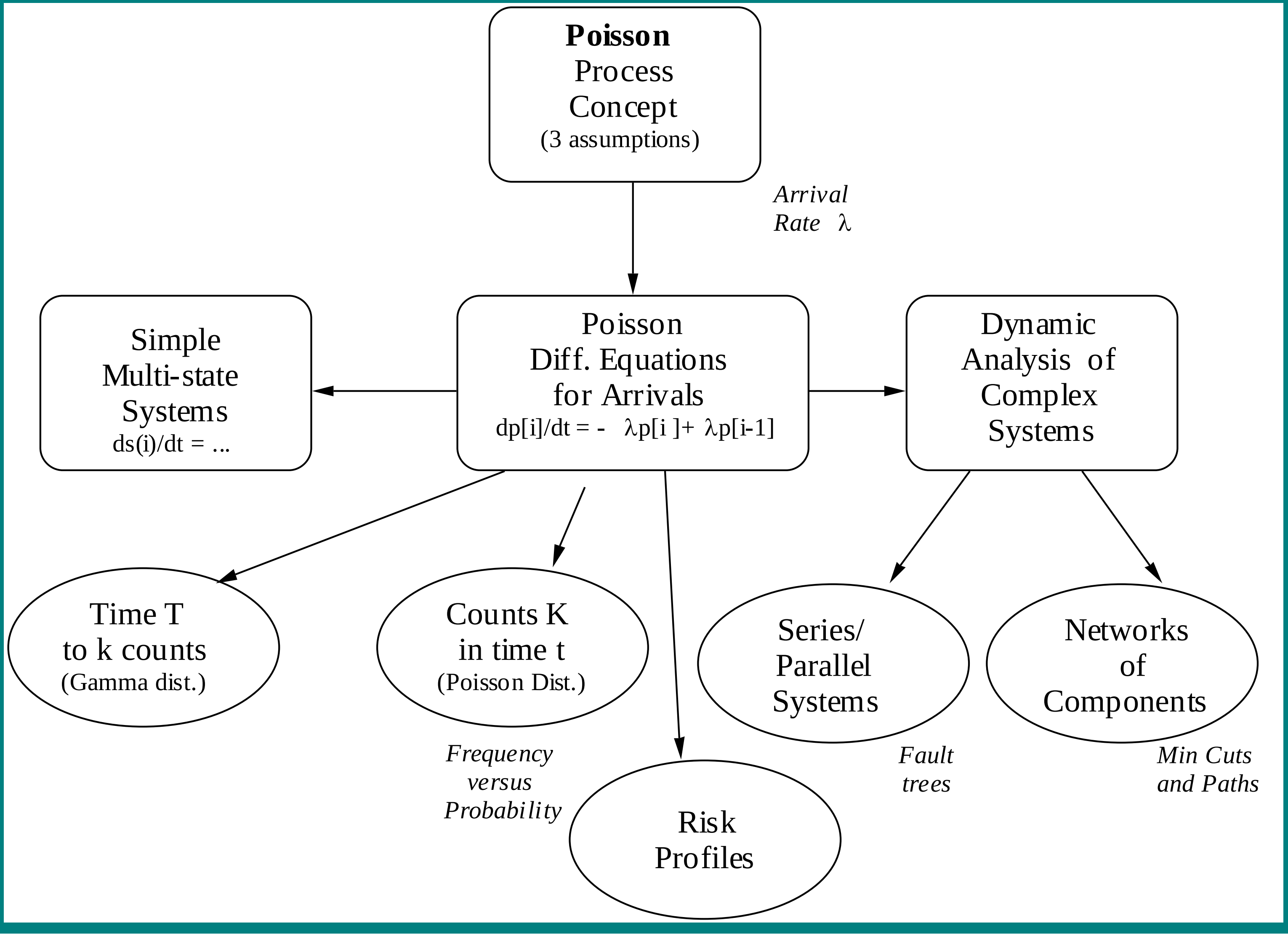
$\int e^{-\lambda x}\, dx = -\exp(-\lambda x)/\lambda + c$

where $\exp(x) = e^x$ and $e = 2.718282\ldots$

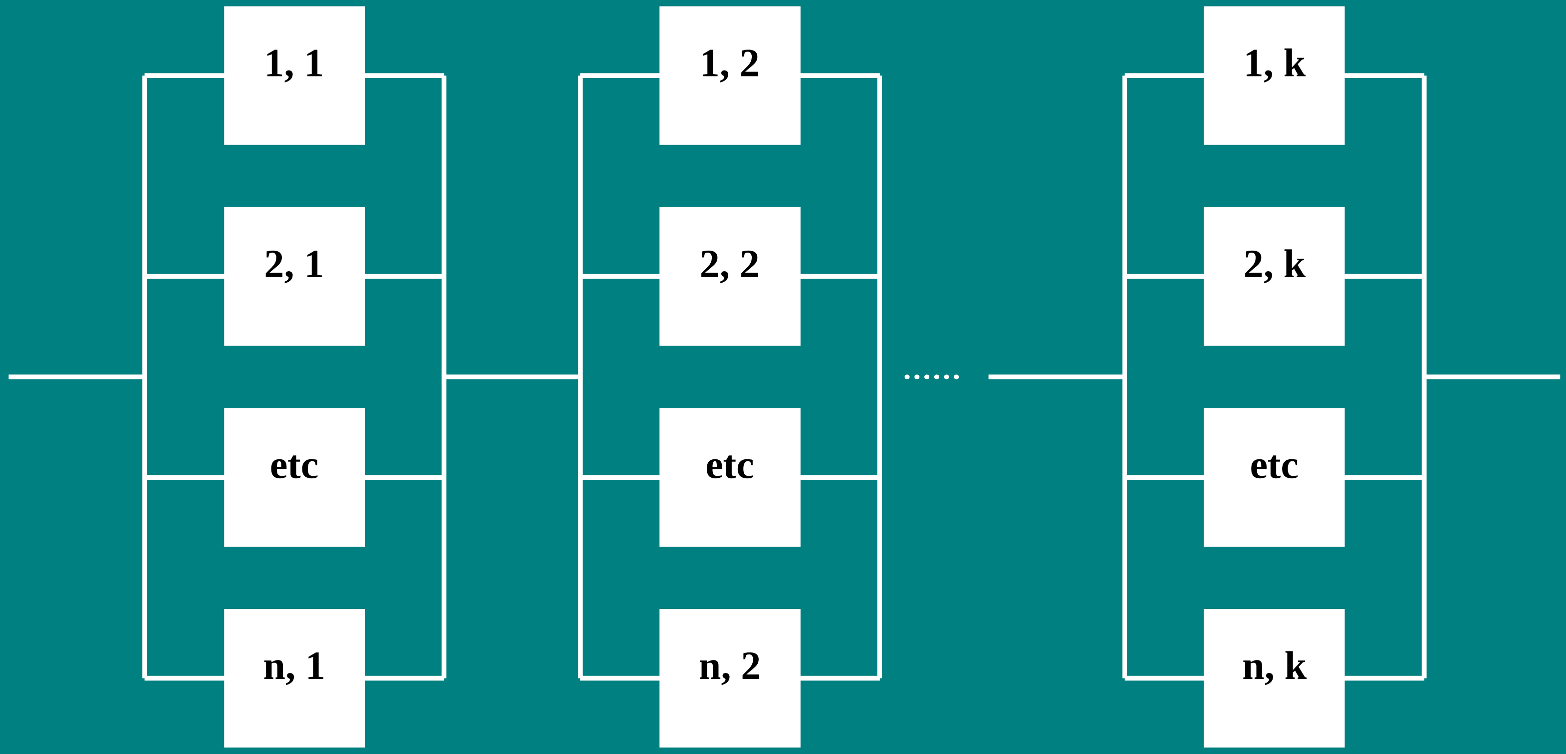
$\int a \exp(bx)\, dx = (a/b) \exp(bx) + c$

$\int x \exp(bx)\, dx = x \exp(bx)/b - \exp(bx)/b^2 + c$

Poisson Process Outline



Dynamic Reliability: Series & Parallel Systems



Moral of the Story

Avoid series systems

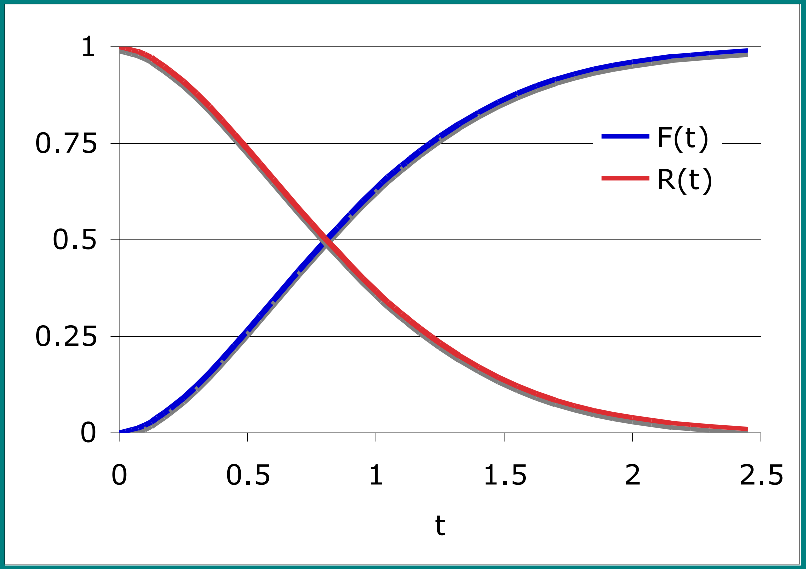
Build systems with parallel redundancy

Example: Meeting for Lunch

a) How about next Thursday I meet you in front of the Laibrary at 12:25 and we can have lunch?

b) How about next Thursday we have lunch a little past noon? I will call you at your office, and after we talk we can both walk over and meet at the Laibrary.

Definitions



t = time

$F_i(t)$ = probability component/arc i has failed before time t;
= CDF or probability that time of failure $T \leq t$ where T is "the time to failure" random variable

$R_i(t)$ = reliability of component/arc i at time t =1 - $F_i(t)$

λ_i = Poisson arrival rate for failures of component/arc i if components have exponential-time-to failure corresponding to Poisson-failure-rate model

MTBF and Example

MTBF = mean time-before failure = $\int_0^\infty t\,dF(t) = \int_0^\infty R(t)\,dt$

why?

(Integrate by parts to obtain result: $\int u\,dv = uv - \int v\,du$)

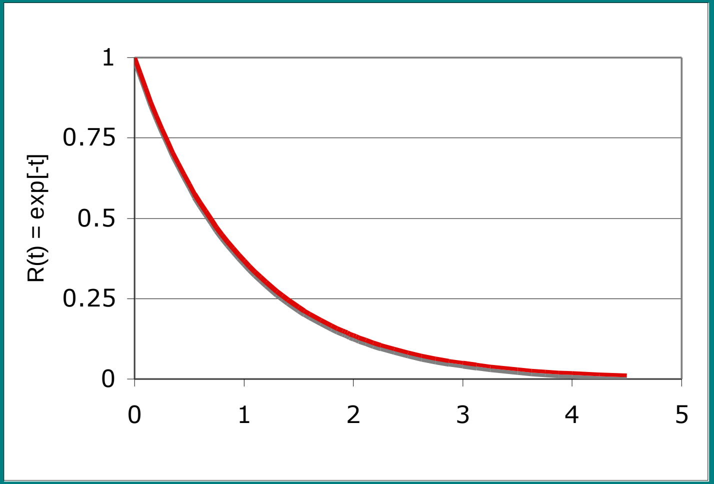
Example: exponential time-to-failure distribution

F(t) = 1 – exp[-λt]

R(t) = exp[-λt]

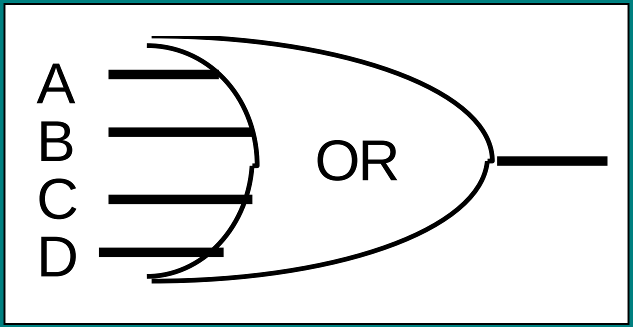
MTBF = 1/ λ

why?



Systems with n components in Series

One fails→ total system fails



$$R_T(t) = \prod_{i=1}^{n} R_i(t); F_T(t) = 1 - \prod_{i=1}^{n} R_i(t)$$

[Π means product of the terms.]

Series System Examples

Example 1; n = 2

$F_i(t) = 0.01$ for each component:

$R_T(t) = (0.99)^2 = 0.98$; $F_T(t)=0.02$

Example 2: For exponential time-to-failure distributions:

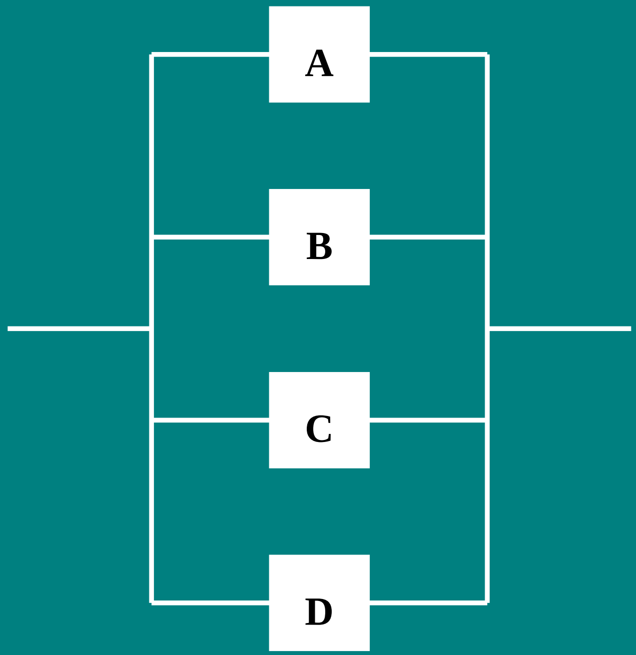
$R_i(t) = \exp[-\lambda_\iota t]$; then for each component $MTBF = 1/ \lambda_\iota$

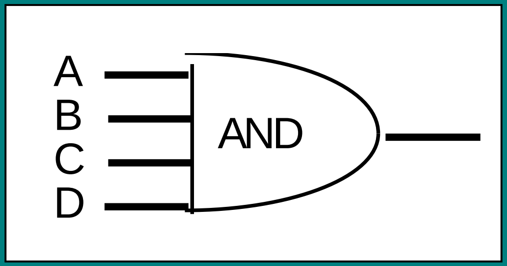
$R_T(t) = \Pi \exp[-\lambda_\iota t] = \exp[-\lambda_T t]$ where $\lambda_T = \Sigma \lambda_\iota$

$MTBF = 1 / \lambda_T$

What happened? n exponential time-to-failure components in series is equivalent to one component with $\lambda_T = \Sigma_\iota \lambda_\iota$

Systems with n components in Parallel





All must fail for total system to fail

$$F_T(t) = \prod_{i=1}^{n} F_i(t); R_T(t) = 1 - \prod_{i=1}^{n} F_i(t)$$

Parallel System Examples 1 & 2

Example 1; n = 2

$F_i(t) = 0.01$ for each component:

$F_T(t) = (0.01)^2 = 10^{-4}$

Example 2: For exponential time-to-failure distributions:

$F_i(t) = 1 - \exp[-\lambda t]$;

then

$$F_T(t) = \prod_{i=1}^{n} \{1 - \exp[-\lambda_i t]\}$$

Parallel System Example 3

Example 3: For n = 2 exponential time-to-failure distributions:

$$F_T(t) = \{1 - \exp[-\lambda_1 t]\}\{1 - \exp[-\lambda_2 t]\} = 1 - e^{-\lambda_1 t} - e^{-\lambda_2 t} + e^{-(\lambda_1+\lambda_2)t}$$

$$R_T(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

$$MTBF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{(\lambda_1+\lambda_2)}$$

$$\text{For } \lambda_1 = \lambda_2;\ MTBF = \frac{1}{\lambda} + \frac{1}{\lambda} - \frac{1}{(2\lambda)} = \frac{1.5}{\lambda}$$

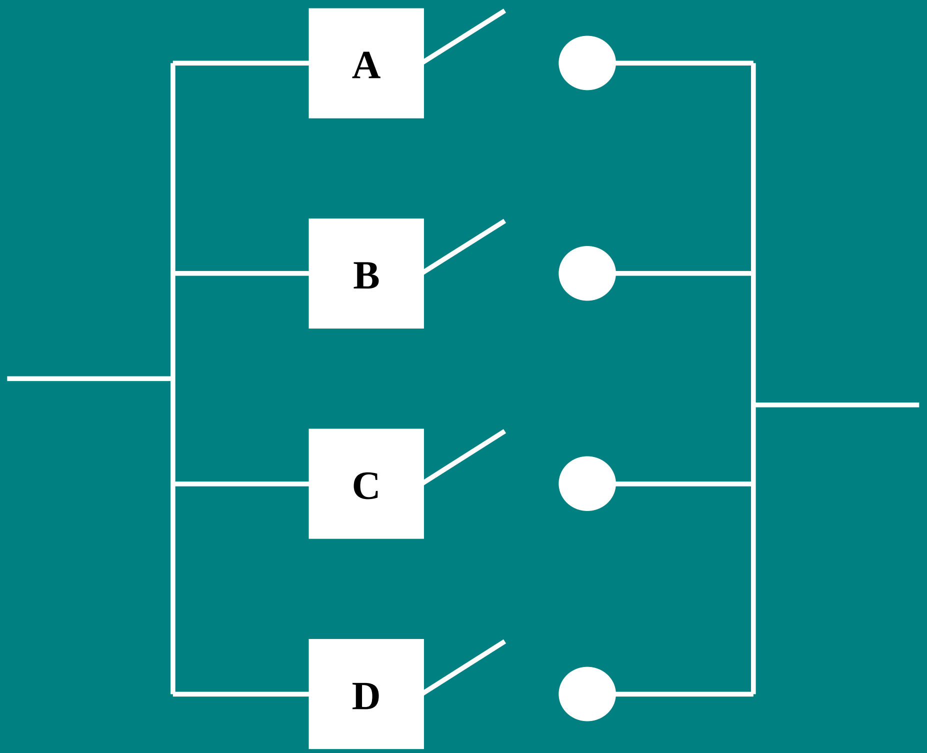
$$\text{In general, } MTBF = \frac{1}{[n\lambda]} + \frac{1}{[(n-1)\lambda]} - \frac{1}{[(n-2)\lambda]} + \ldots \frac{1}{\lambda}$$

Correct?

Off-line Redundant System

Extra components/arcs introduced when others fail.

Failure probability functions can be complex for such systems.



Off-line Redundant System Examples

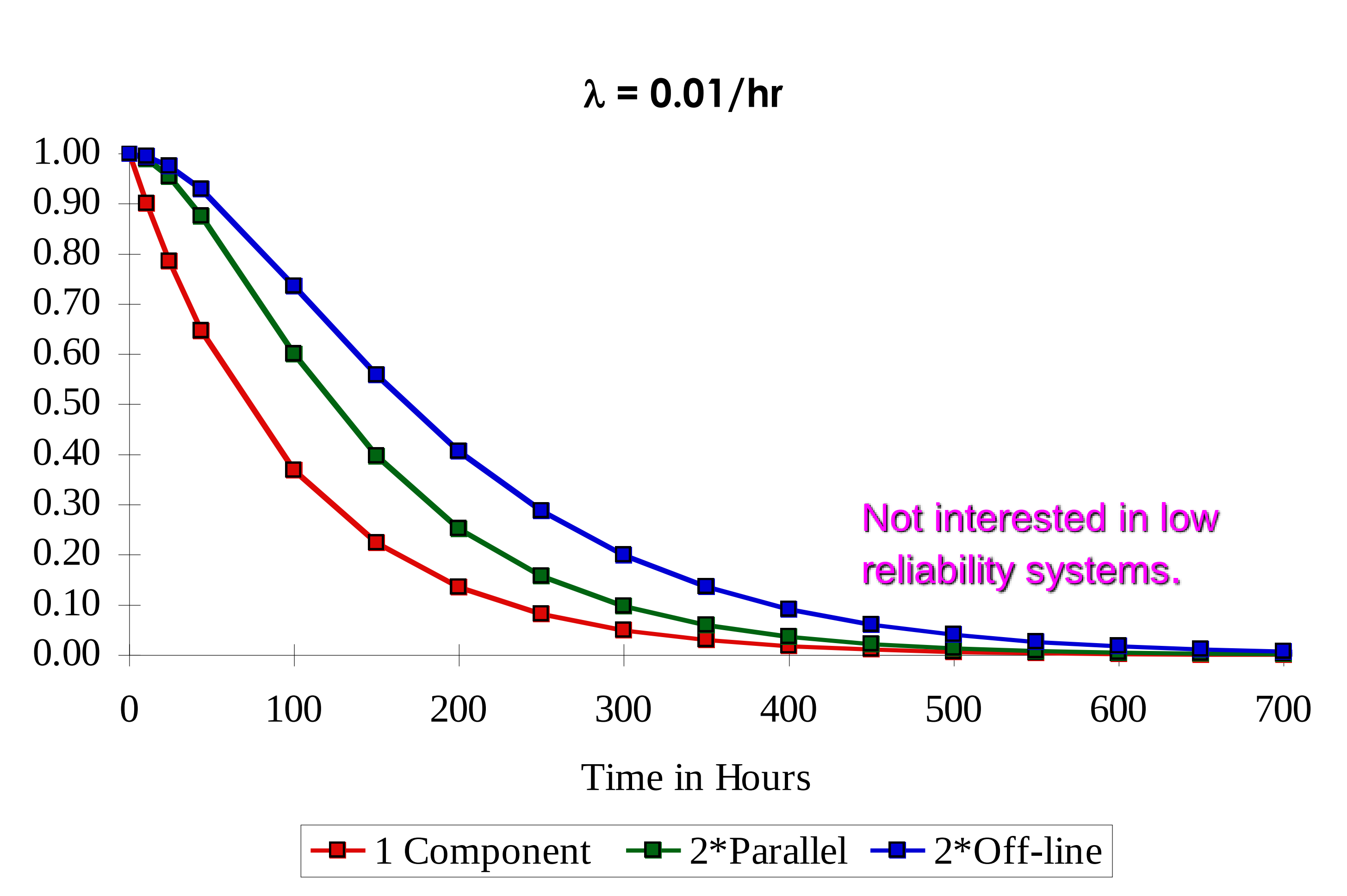
For two component system: $$f_T(t) = \int_0^t f_1(s) f_2(t-s) ds \text{ where } f_i(t) = \frac{dF_i(t)}{dt}$$

For single-component system with n components available, each with common failure rate λ, total system fails only when all n components have failed. Arrival of failures is a homogeneous Poisson process with arrival rate λ, as long as at least one component is working. Thus

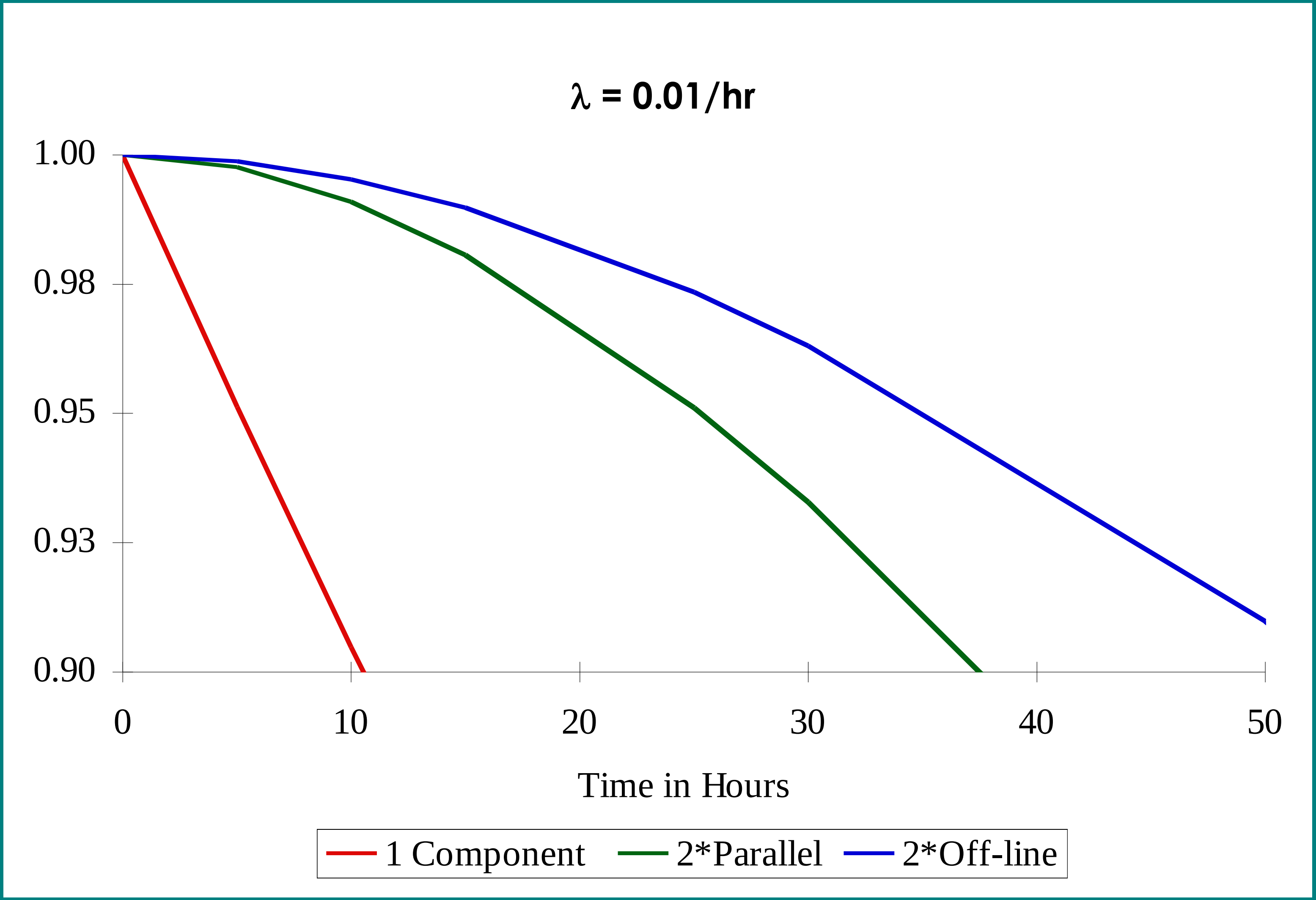
$$R_T(t) = \Pr\{\leq \text{n - 1 failures in } (0, \text{t})\} = \sum_{k=0}^{n-1} \frac{(\lambda t)^k \exp[-\lambda t]}{k!}$$

$$MTBF = \frac{1}{\lambda} + \frac{1}{\lambda} + \frac{1}{\lambda} + \ldots = \frac{n}{\lambda}$$

Reliability of Different Systems



Reliability of Different Systems - 2



Reliability of Different Systems - 3

Reliability of 1 component:

$$R_1(t) = \exp[-\lambda t]$$

$$= 1 - \lambda t + \frac{(\lambda t)^2}{2} - \frac{(\lambda t)^3}{6} + \frac{(\lambda t)^4}{24} \ldots$$

which for small λt can be written

$$\approx 1 - \lambda t$$

Reliability of 2 components in parallel:

$$R_2(t) = 1 - [F_i(t)]^2$$

$$= 1 - (1 - \exp[-\lambda t])^2 = 2\exp[-\lambda t] - \exp[-2\lambda t]$$

Using Taylor series yields

$$= 2\{1 - \lambda t + \frac{(\lambda t)^2}{2} - \frac{(\lambda t)^3}{6} \ldots\} - \{1 - 2\lambda t + \frac{(2\lambda t)^2}{2} - \frac{(2\lambda t)^3}{6} \ldots\}$$

which for small λt can be written

$$= 1 - (\lambda t)^2$$

Reliability of Different Systems - 4

Reliability Off-Line Redundant System, 2 components

$$R_{off-line}(t) = \Pr\{0 \text{ or } 1 \text{ failures in time } (0, t)\}$$

$$= e^{-\lambda t} + (\lambda t)e^{-\lambda t} = (1+\lambda t)e^{-\lambda t}$$

Using the Taylor series yields

$$= (1+\lambda t)(1-\lambda t + \frac{(\lambda t)^2}{2} - \frac{(\lambda t)^3}{6} \ldots)$$

which for small λt can be written

$$= 1 - ?$$

Hazard Function

Hazard Function h(t)

New idea. See Hillier & Lieberman (Section 14.6) or

http://en.wikipedia.org/wiki/Bathtub_curve

How to describe hazard function of systems that are still operating?

dF(t)/dt describes rate of occurrence of failures at time t,

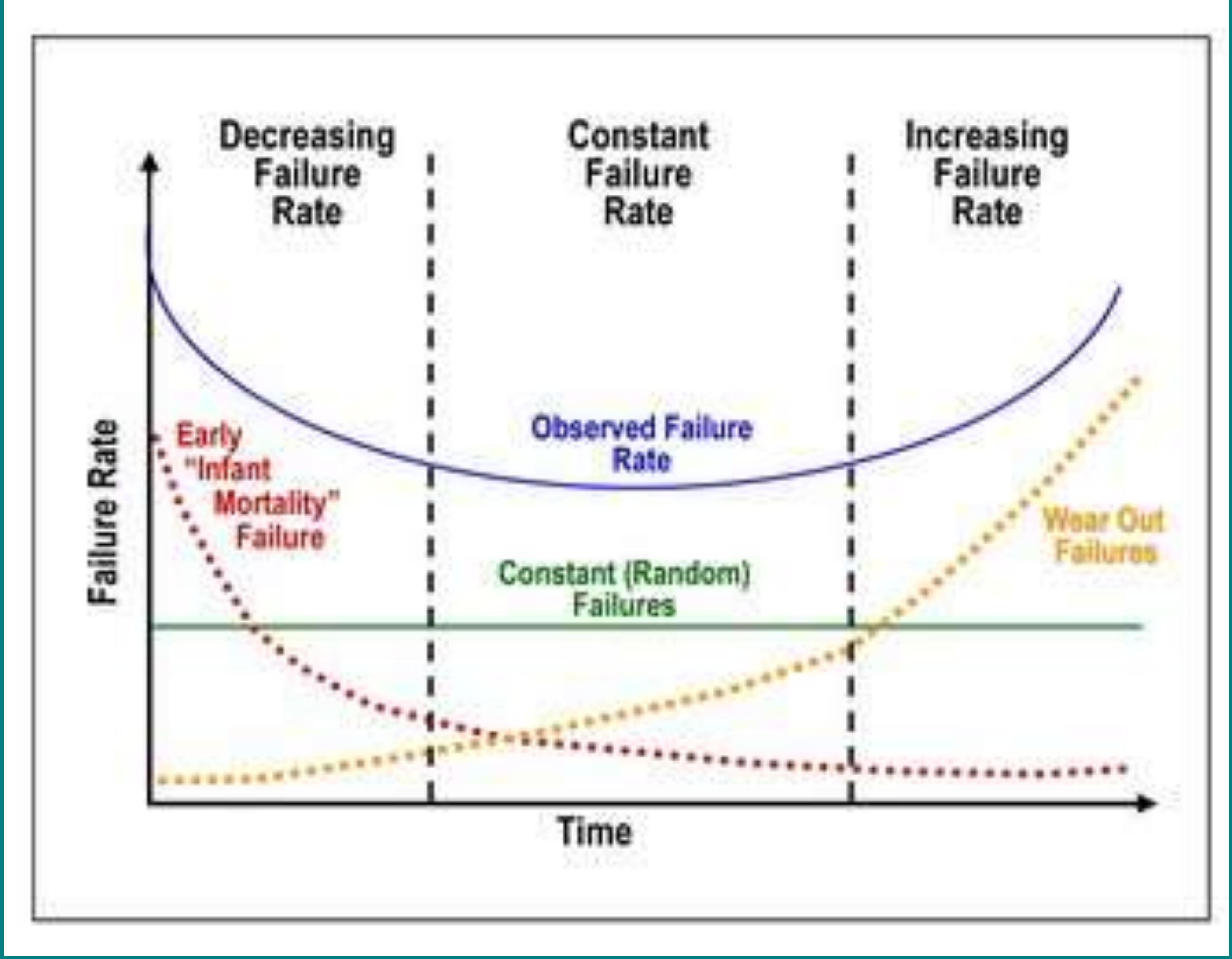
when only a fraction R(t) of units are still operating.

So hazard function of units *still operating* is:

$$h(t) = \frac{\text{rate at which system fails}}{\text{probability system still operating}} = \frac{dF(t)/dt}{R(t)} = \frac{d\{-\ln[R(t)]\}}{dt}$$

↑ *why?*

Classic Bathtub Failure Function



http://upload.wikimedia.org/wikipedia/en/thumb/6/6e/Bathtub_curve.jpg/350px-Bathtub_curve.jpg

Hazard Function Examples

Example 1:

For one component with exponential time-to-failure distribution:

$$h(t) = d\{ -\ln[\exp(-\lambda t)] \}/dt = d\{\lambda t\}/dt = \lambda$$ *CONSTANT!*

Example 2:

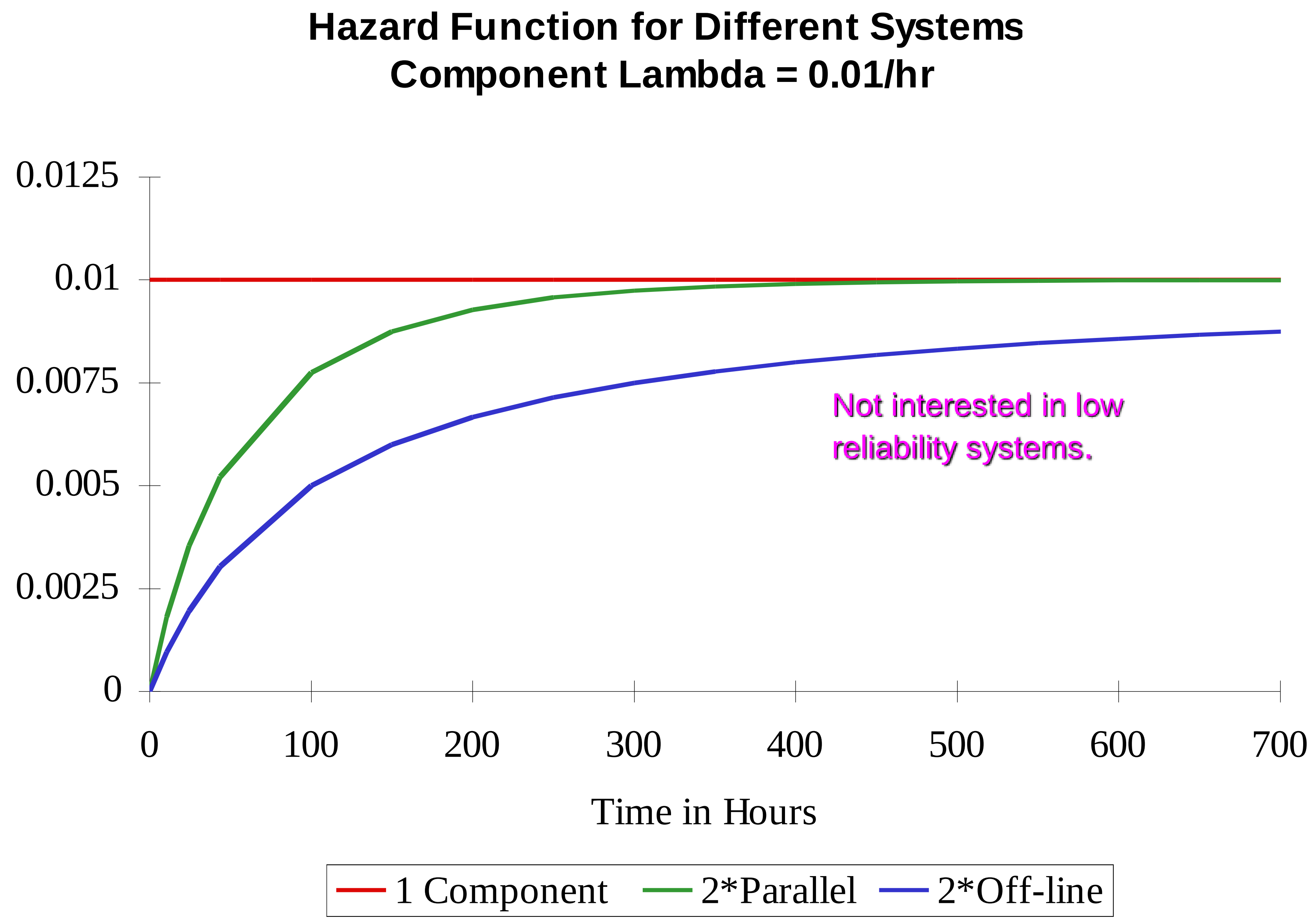
For 2-components with off-line redundancy

$$h(t) = d\{-\ln[(1 + \lambda t) \exp(-\lambda t)] \}/dt = \lambda^2 t/(1 + \lambda t)$$

Example 3:

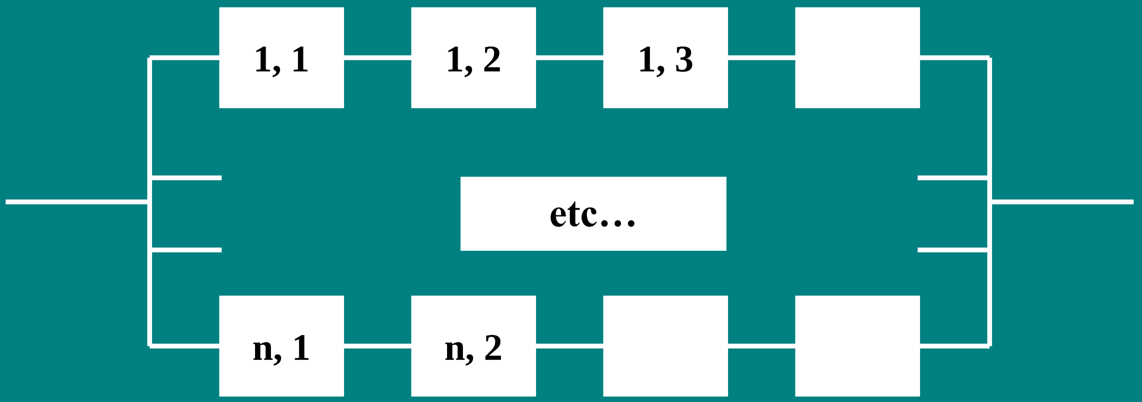
For 2-components in parallel

$$h(t) = d\{-\ln[1-(1-\exp(-\lambda t))^2] \}/dt = 2\lambda(1-\exp(-\lambda t))\exp(-\lambda t)/[1-(1-\exp(-\lambda t))^2]$$


Hazard Function for Different Systems
Component Lambda = 0.01/hr
0.0125
0.01
0.0075
0.005
0.0025
0
0
100
200
300
400
500
600
700
Time in Hours
Not interested in low reliability systems.
1 Component
2*Parallel
2*Off-line

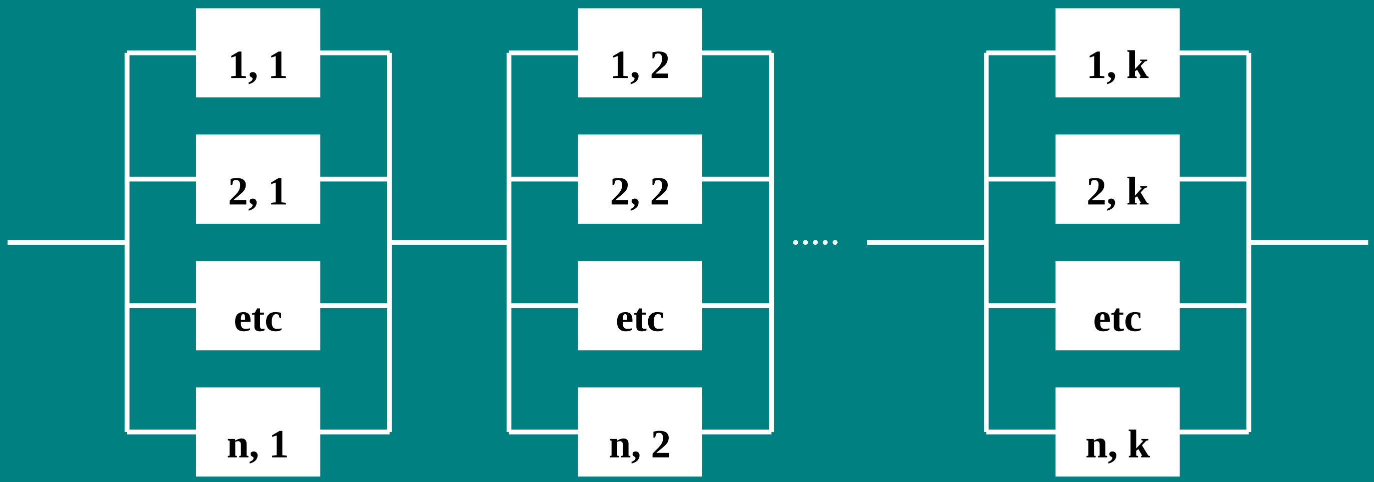
Series/Parallel Systems

(1) Parallel System of Series Components



$$F_T(t) = \prod_{i=1}^{k} \{1 - \prod_{j=1}^{k} R_{ij}(t)\}$$

(2) Series System of Parallel Components



$$R_T(t) = \prod_{j=1}^{k} \{1 - \prod_{i=1}^{k} F_{ij}(t)\}$$

Methods for Evaluation of Complex Systems

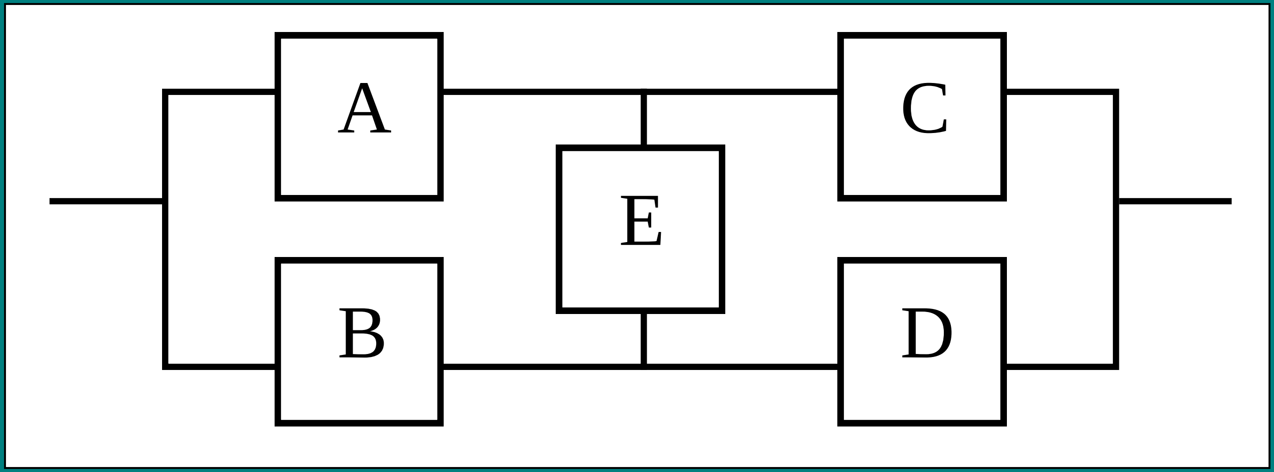
Series-Parallel &Parallel-Series networks

are equivalent to fault trees.

Analysis in previous slides are for such series & parallel systems,

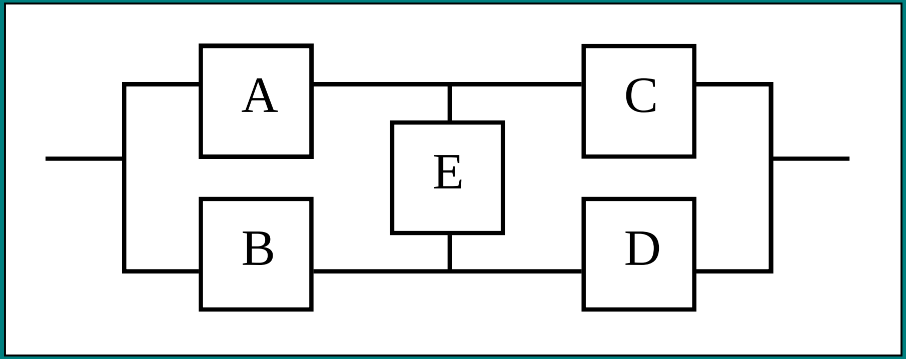
but not more complex systems.

Consider:



Conditional Probability for Complex Networks

Satisfactory if only a few critical components destroy simple structure. Here the arc E make system complex because signals/message can go either way, as can a signal/power on a wire, a car on a road, or water in pipe.



F* = Pr[Failure if E good] Pr{E good}

+ Pr[Failure if E bad] Pr{E bad}

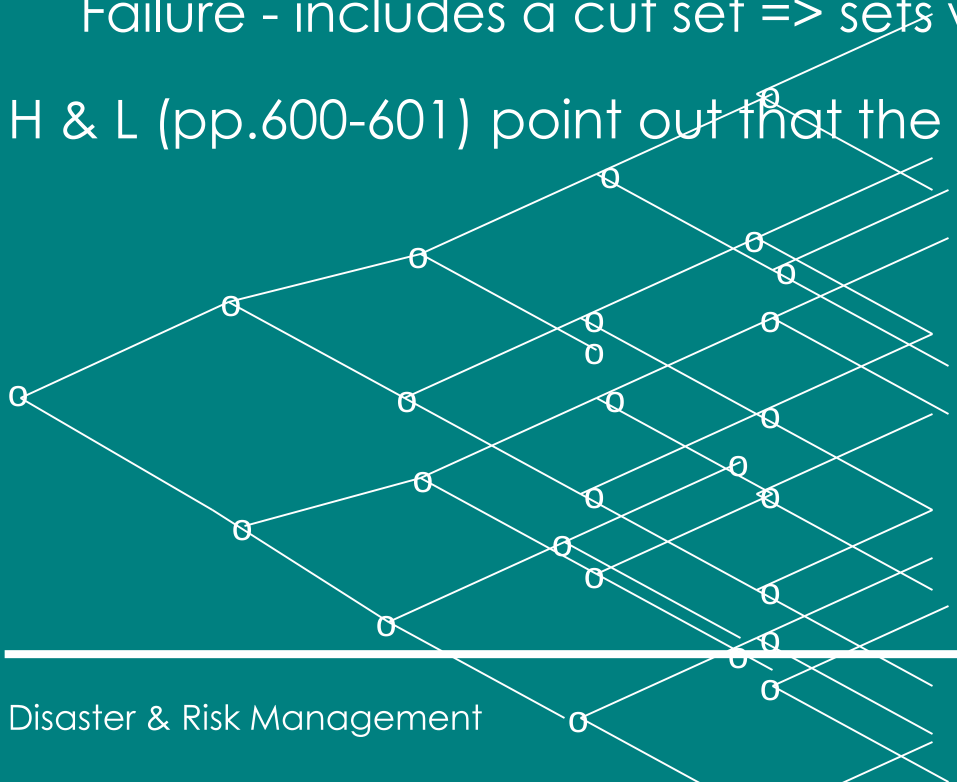
Evaluation of Complex Systems - Event Trees

Branches represent enumeration of all possible combinations of components status (operating and failure)

Success - includes a tie set => sets that ensure success;

Failure - includes a cut set => sets whose failure causes failure

H & L (pp.600-601) point out that the number of combinations is

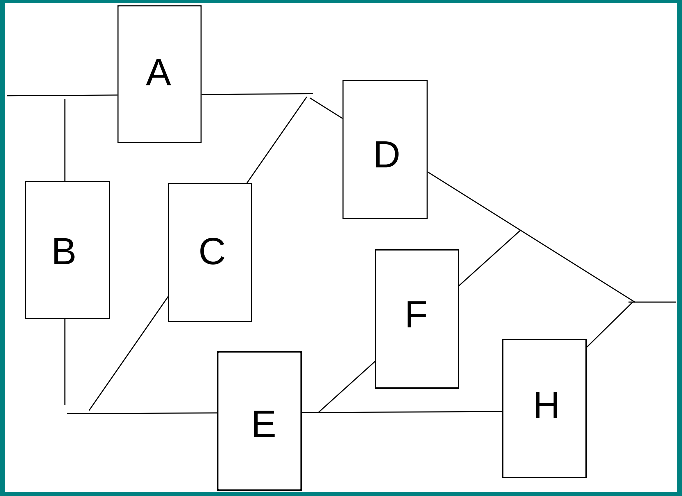


LARGE

number

of branches!

“Network” methods useful for complex systems



Network methods – Definitions

Paths: set of arcs that guarantee success.

Emphasizes operating modes.

Minimal paths: *every* component must work to assure success

(in example: AC, BD, BEC, AED)

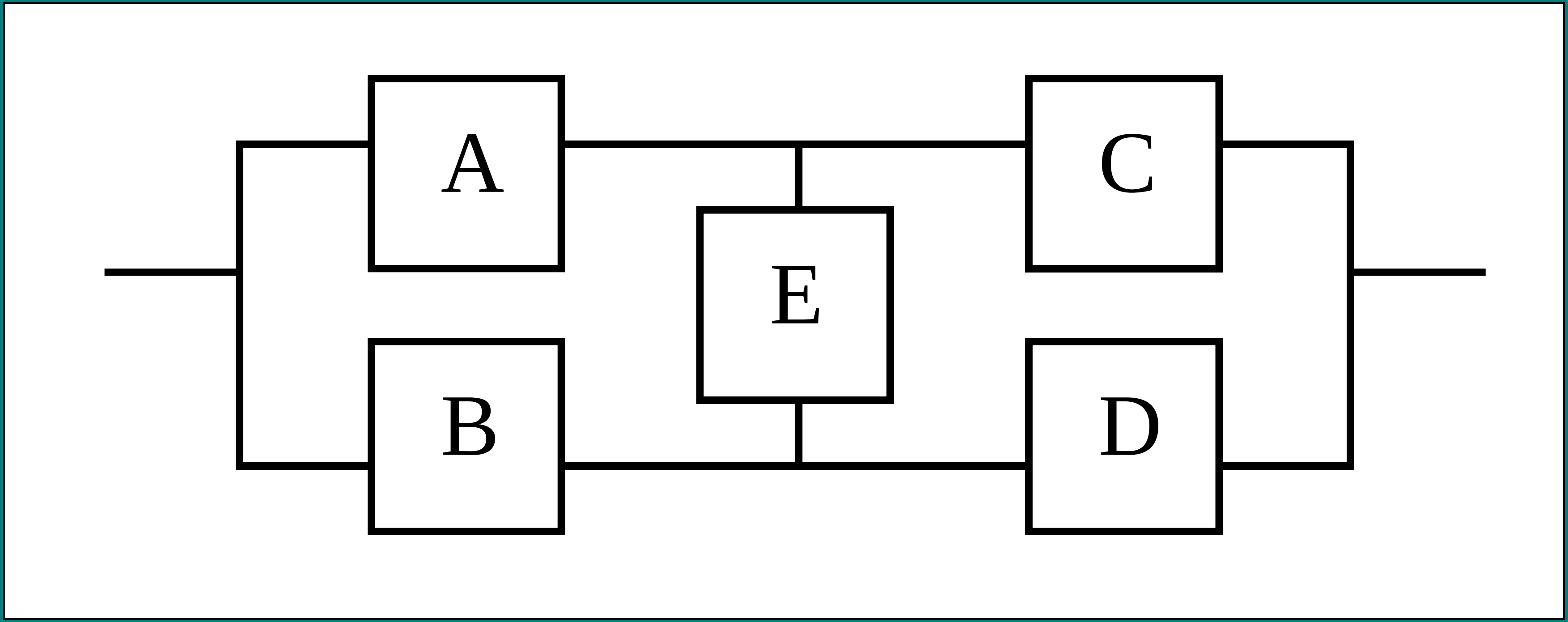
Cut sets: components whose failure ensures system failure.

Emphasizes failures.

Minimal cut set: *every member must* fail

to ensure system fails (in example: AB, DC, AEC, BED)

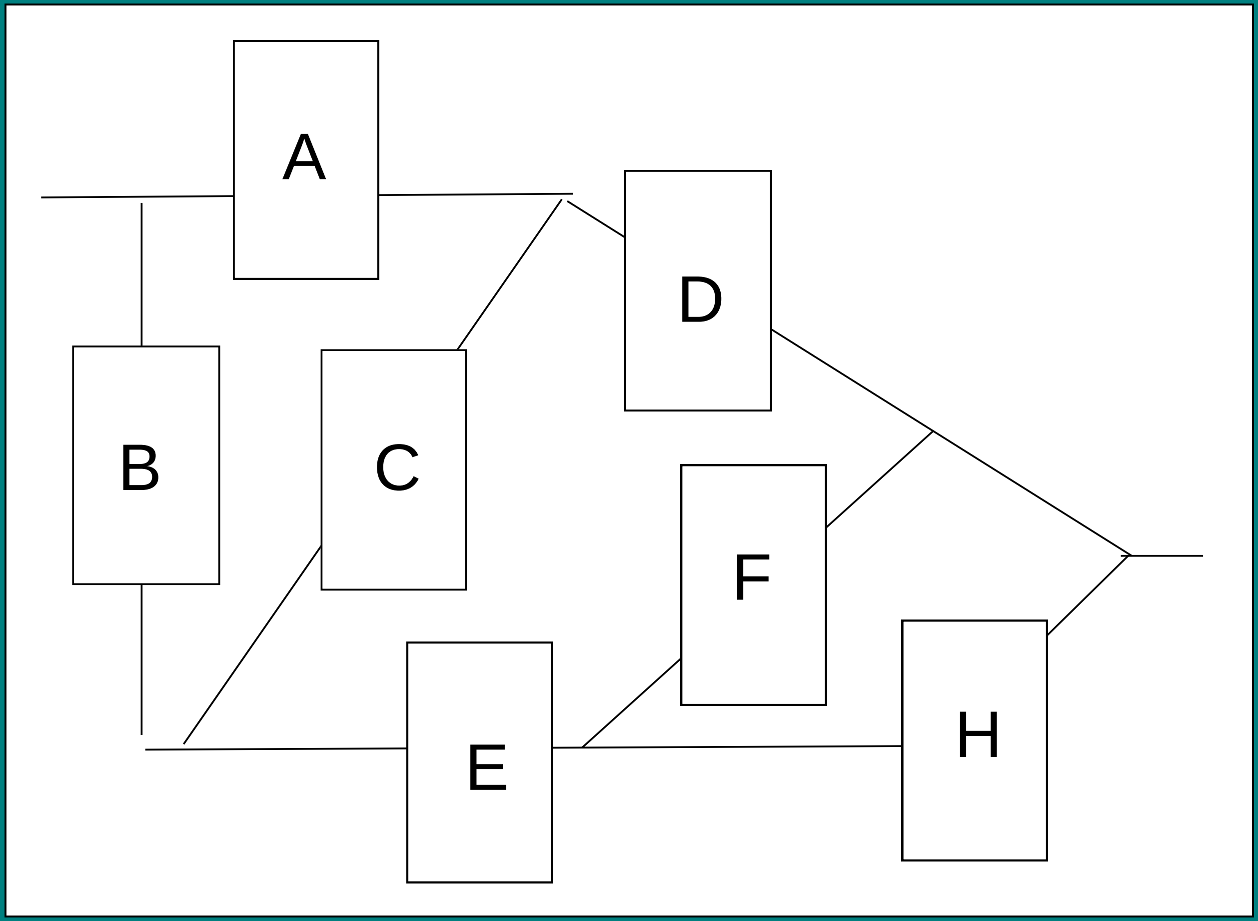
Cut Set Example



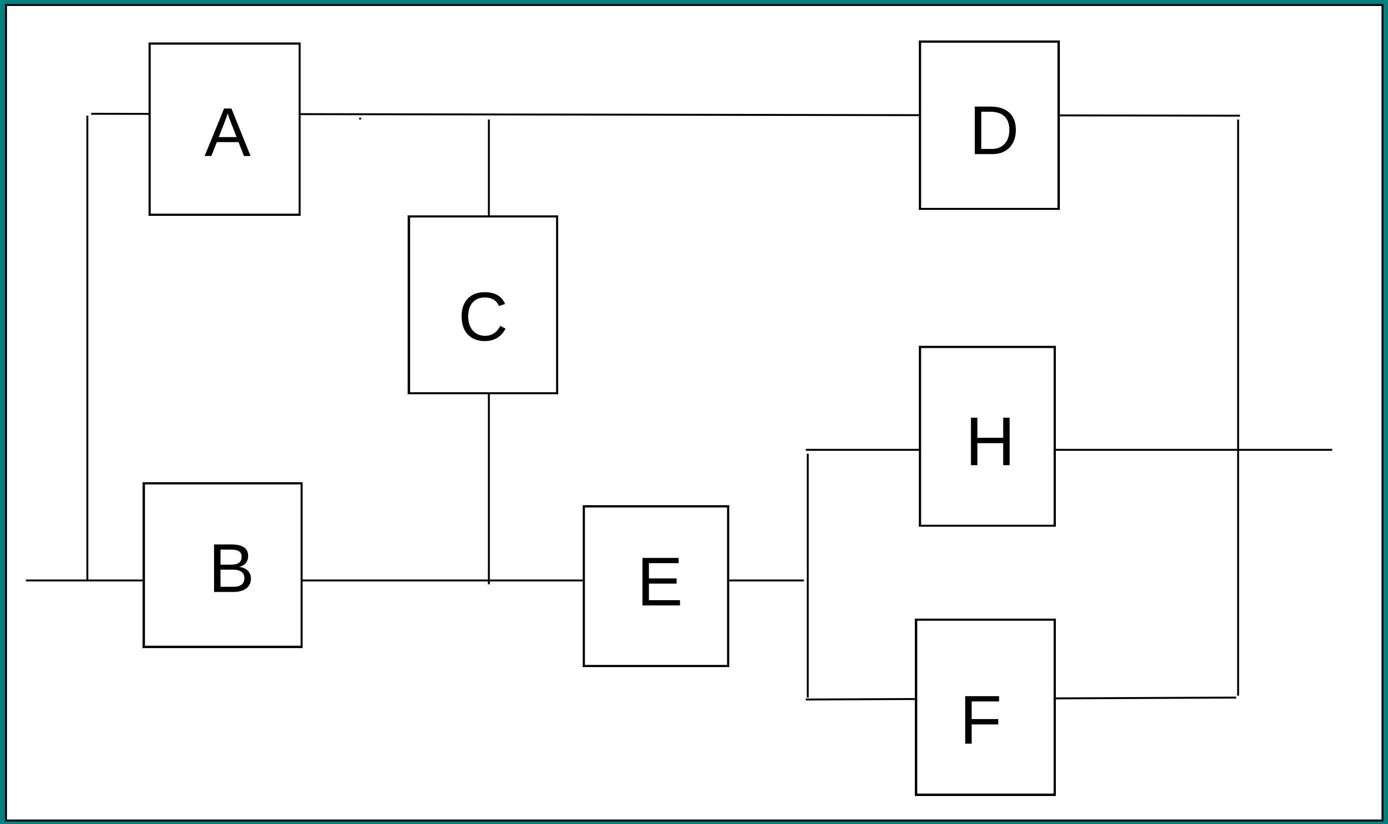
Cut Sets: AB CD AED BED

Sample Complex Network

Find the minimal cut sets of the network:



Sample Complex Network - *Redrawn*



Solution for Minimal Cut sets?

"Network" methods and Cut Sets

We will use the following approximation:

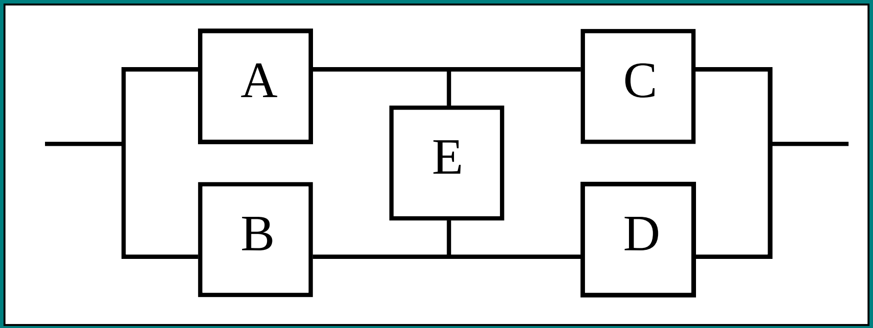
Failure Probability = P{union of minimal cut sets CS_i}

$$\leq \Sigma_{\text{all } CS_i} \Pr\{CS_i\}$$

[ignores intersections]

Can be good if P{Arc_j} < 0.01 for all j and component failures are independent so intersections with more components have smaller probabilities.

Example Why Analysis of Cut Sets Works



Example:

Cuts are C_i = AB, CD, EBC, EAD

with probabilities 10^{-4}; 10^{-4}; 10^{-6}; 10^{-6}

The intersections are ABCD, ABCE, ABDE, BCDE, ACDE, ABCDE,

with probabilities 10^{-8}; 10^{-8}; 10^{-8}; 10^{-8}; 10^{-8}; 10^{-10}

So:

$$2 \times 10^{-4} + 2 \times 10^{-6} - (5 \times 10^{-8} + 10^{-10}) \leq \Pr[\text{Fail}] \leq 2 \times 10^{-4} + 2 \times 10^{-6} = 2.02 \times 10^{-4}$$

Indeed $\Pr[\text{Fail}] \approx 2 \times 10^{-4}$

Why Analysis of Cut Sets Works

Cut Sets:

Failure Probability = P[union of minimal cut sets CS_i]

Let $f = \Sigma_{\text{all } CS_i} \Pr[Cs_i]$

then

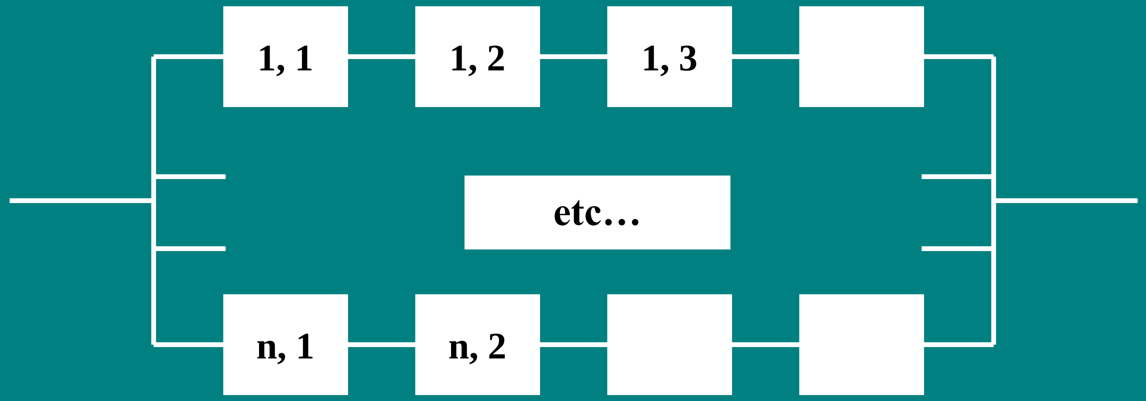
$$f - \Sigma_{\text{all } i} \Sigma_{\text{all } j<i} \Pr[CS_i \text{ and } CS_j] \leq \Pr[\text{Fail}] \leq f$$

Because the probability of intersections are small:

$$\Pr[\text{Fail}] \approx f$$

Why Paths Fail and Cuts Work

Here each series of components is a path, and all paths must fail for system to fail. But paths have common components so their failures are highly dependent: multiplying probabilities give wrong answer.

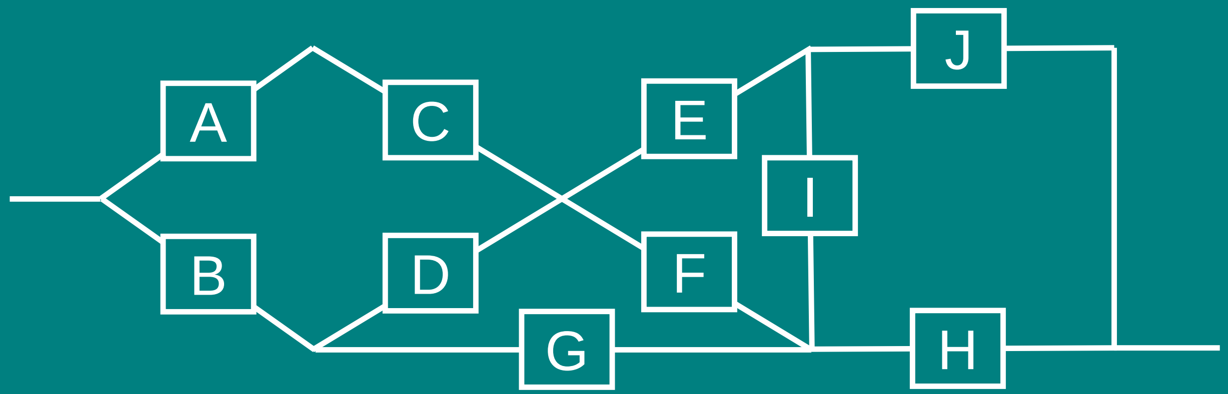


However, using cut sets one sees that the system fails if any minimal cut sets fails. So one adds their probabilities, and then subtracts the intersections. BUT the intersections have very small probabilities, so this correction can be neglected.

Homework

Due April 30

Consider the system described by the network blow:



(a) What are ALL the MINIMAL cut sets with 4 or fewer components?

(b) What are ALL the MINIMAL paths sets with 4 or fewer components?

(c) If all elements are 99% reliable, what ***approximately*** is the reliability of the system?

(d) If you are out of money and you want to use some 80% reliable components, suggest 3 components that could be replaced with the cheaper units without really compromising overall system reliability?

Assignment (cont'd)

(e) What one component is most critical to the reliability of the system?

(f) Reorganization has resulted in the removal of arcs or components G and I – those paths are no longer available. Please draw a fault tree to describe the reliability of the reduced network.

(g) For the fault tree below, if the reliability of every component is 98%, what is the reliability of the whole system?