# Chapter 7. Virtual Work Principles in Framework Analysis

The principle of virtual work can be conveniently used in the formulation of approximate solutions which the direct formulation can't achieve.

various applications $\left\{ \begin{array}{l} \text{tapered member} \\ \text{distributed load} \\ \text{Geometric nonlinear and elastic critical load} \end{array} \right.$

exact or approximate displacement function

$\Rightarrow$ shape function

## 7.1 Description of the displaced state of elements

### 7.1.1 Definition of the shape function Mode of Description

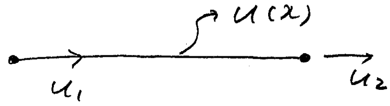
Ingredients for construction of $\underset{\sim}{k}$

1) Elastic constants $\underset{\sim}{\varepsilon} - \underset{\sim}{\sigma}$ relationship

2) real and virtual displaced states $\Rightarrow$ shape function

3) $\underset{\sim}{\varepsilon} - \underset{\sim}{u}$ relationship

## 7.1.2 Formulation of shape function




$$u = N_1 u_1 + N_2 u_2$$

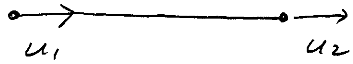
$$= [N_1 \;\; N_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad : \text{ nodal displacement}$$

$$\underset{\sim}{f} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \quad : \text{ shape function}$$

$u$ : Generic displacement

Axial force member




Equilibrium : $\frac{dF}{dx} = g_u = 0 \qquad F = EA\frac{du}{dx}$

$$\Rightarrow \frac{d^2u}{dx^2} = 0$$

$u = a_1 + a_2 x \quad \Rightarrow \quad$ conditions : $\begin{cases} 1) & \text{1st order polynomial} \\ 2) & \text{two displ. B/c} \end{cases}$

B/c $\quad x = 0 \qquad u = a_1 = u_1$

$\quad x = L \qquad u = a_1 + a_2 L = u_2$

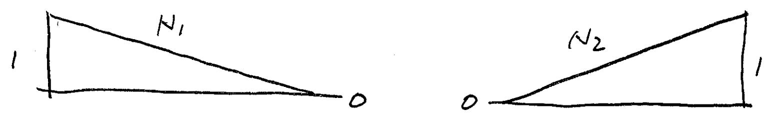
$\quad a_2 = \frac{1}{L}(u_2 - u_1)$

$$u = a_1 + a_2 x$$

$$= u_1 + \frac{1}{L}(u_2 - u_1)\, x$$

$$= \underbrace{\left(1 - \frac{x}{L}\right)}_{N_1} u_1 + \underbrace{\frac{x}{L}}_{N_2} u_2$$

$$N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$




shape function

Torsional Action

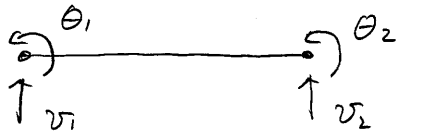

Equilibrium $\quad \frac{dM_x}{dx} = m = 0 \qquad M_x = GJ\frac{d\theta_x}{dx}$

$$\Rightarrow \frac{d^2\theta_x}{dx^2} = 0$$

$$\theta_x = a_1 + a_2 x$$

$$= \underbrace{\left(1 - \frac{x}{L}\right)}_{N_1} \theta_1 + \underbrace{\frac{x}{L}}_{N_2} \theta_2$$

flexural Action




Equilibrium $\qquad \dfrac{d^2M}{dx^2} = \gamma_v = 0 \qquad \qquad \dfrac{M}{EI} = \phi = \dfrac{d^2v}{dx^2}$

$$\Rightarrow \quad \frac{d^4v}{dx^2} = 0$$

$$v = a_1 + a_2x + a_3x^2 + a_4x^3 \Rightarrow \text{conditions} \begin{cases} 1) \ \dfrac{d^4v}{dx^4} = 0 \\ 2) \ \text{four displ. B/c} \end{cases}$$

B/c

$x = 0 \qquad v = a_1 = v_1$

$x = L \qquad v = a_1 + a_2L + a_3L^2 + a_4L^3 = v_2$

$v' = a_2 + 2a_3x^2 + 3a_4x^2$

$x = 0 \qquad v'(0) = a_2 = \theta_1$

$x = L \qquad v'(L) = a_2 + 2a_3L^2 + 3a_4L^2 = \theta_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

$a_1 = v_1$

$a_2 = \theta_1$

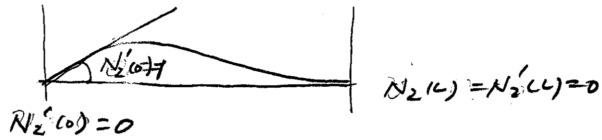
$a_3 = \dfrac{1}{L^2}(-3v_1 + 3v_2 - 2\theta_1L + \theta_2L)$

$a_4 = \dfrac{1}{L^3}(2v_1 - 2v_2 + \theta_1L + \theta_2L)$

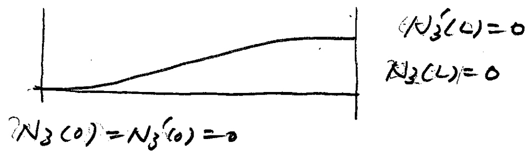
$$v = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2$$

$$N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

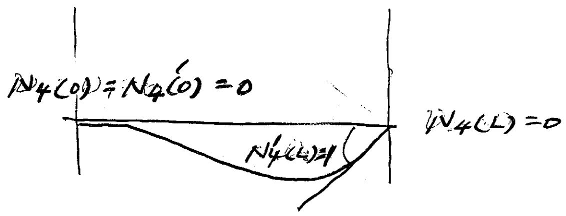

$$N_2 = x\left(1 - \frac{x}{L}\right)^2$$




$$N_3 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$




$$N_4 = x\left[\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right]$$




$x=0, \quad v = v_1 \quad \Rightarrow \quad N_1 = 1 \quad N_2 = N_3 = N_4 = 0$

$x=0, \quad v' = \theta_1 \quad \Rightarrow \quad N_2' = 1 \quad N_1' = N_3' = N_4' = 0$

7.1.3 characteristics of shape function

$$u = N_1 u_1 + N_2 u_2$$

$$\left.\begin{aligned} N_1 &= 1 - \frac{x}{L} \\ N_2 &= \frac{x}{L} \end{aligned}\right\} \text{ shape function}$$

$\Rightarrow$ weighting function

showing influence of $u_1$ and $u_2$

## 7.2 Virtual Displacements in the formulation of element stiffness equations

### 7.2.1 Construction of expressions for real and virtual Displacements

axial force member

Generic displacement - nodal displacement

$$u = N_1 u_1 + N_2 u_2$$

$$= [N_1 \quad N_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \underset{\sim}{N} \, \underset{\sim}{q} \qquad \underset{\sim}{q} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \text{nodal displacement vector}$$

Strain - Generic displacement

$$\varepsilon = \frac{du}{dx}$$

Strain - nodal displacement

$$\varepsilon = \frac{du}{dx} = \underset{\sim}{N}' \, \underset{\sim}{q}$$

$$= [N_1' \quad N_2'] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \left[-\frac{1}{L} \quad \frac{1}{L}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underset{\sim}{N}' = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$$

Stress - strain

$$\sigma = E \varepsilon$$

$$= E \, \underset{\sim}{N}' \, \underset{\sim}{q}$$

Virtual strain Energy

$$\delta U = \int \delta \underset{\sim}{\varepsilon} \cdot \underset{\sim}{\sigma} \, dV = \int \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma} \, dV$$

$$\delta \varepsilon = \underset{\sim}{N}' \delta \underset{\sim}{q}$$

$$\sigma = E \varepsilon = E \underset{\sim}{N}' \underset{\sim}{q}$$

$$\delta U = \int \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma} \, dV$$

$$= \int \delta \underset{\sim}{q}^T \underset{\sim}{N}'^T E \underset{\sim}{N}' \underset{\sim}{q} \, dV$$

$$= \delta \underset{\sim}{q}^T \left[ \int \underset{\sim}{N}'^T E \underset{\sim}{N}' \, dV \right] \underset{\sim}{q}$$

virtual external work due to nodal forces

$$\delta W = \delta \underset{\sim}{q}^T \underset{\sim}{P}$$

$$\delta U = \delta W \Rightarrow$$

$$\delta \underset{\sim}{q}^T \left[ \int \underset{\sim}{N}'^T E \underset{\sim}{N}' \, dV \right] \underset{\sim}{q} = \delta \underset{\sim}{q}^T \underset{\sim}{P}$$

$$\Rightarrow \underbrace{\left[ \int \underset{\sim}{N}'^T E \underset{\sim}{N}' \, dV \right]}_{\underset{\sim}{K}} \underset{\sim}{q} = \underset{\sim}{P}$$

$$\underset{\sim}{K} = \int \underset{\sim}{N}'^T E \underset{\sim}{N}' \, dV$$

$$= E \int \underset{\sim}{N}'^T \underset{\sim}{N}' \, dV = EA \int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx$$

$$= \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

# Torsional member

Generic displacement - nodal displacement

$$\theta = N_1 \theta_1 + N_2 \theta_2$$

$$= \underset{\sim}{N} \cdot \underset{\sim}{q} \qquad \underset{\sim}{N} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \qquad \underset{\sim}{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$N_1 = 1 - \frac{x}{L}, \qquad N_2 = \frac{x}{L}$$

Strain - Generic displacement - nodal displacement

$$\gamma = r \frac{d\theta}{dx}$$

$$= r \underset{\sim}{N}' \underset{\sim}{q}$$

Stress - Strain relationship

$$\tau = G\gamma = G r \underset{\sim}{N}' \underset{\sim}{q}$$

Virtual Strain Energy

$$\delta U = \int \delta\gamma \, \tau \, dV$$

$$= \delta \underset{\sim}{q}^T \left[ \int G \underset{\sim}{N}'^T \underset{\sim}{N}' r^2 dV \right] \underset{\sim}{q}$$

$$= \delta \underset{\sim}{q}^T \left[ \int GJ \underset{\sim}{N}'^T \underset{\sim}{N}' dx \right] \underset{\sim}{q}$$

or

$$\delta U = \int \delta\beta^T M \, dx \qquad \beta = \frac{d\theta}{dx} = \underset{\sim}{N}' \underset{\sim}{q}$$

$$M = GJ\beta = GJ \underset{\sim}{N}' \underset{\sim}{q}$$

$$= \delta \underset{\sim}{q}^T \left[ \int GJ \underset{\sim}{N}'^T \underset{\sim}{N}' dx \right] \underset{\sim}{q}$$

External virtual work

$$\delta W = \delta \underset{\sim}{\delta}^T \underset{\sim}{M}$$

$$\delta U = \delta W$$

$$\left[ \int GJ \, \underset{\sim}{N}'^T \underset{\sim}{N}' \, dx \right] \underset{\sim}{\delta} = \underset{\sim}{M}$$

$$\underset{\sim}{K} = GJ \int \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx$$

$$= \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Flexural member

Generic displacement - Nodal displacement

$$v = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2$$

$$N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \qquad N_2 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$N_3 = x\left(1 - \frac{x}{L}\right)^2 \qquad N_4 = x\left[\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right]$$

$$v = \underset{\sim}{N}^T \underset{\sim}{\delta}$$

$$\underset{\sim}{N} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \qquad \underset{\sim}{\delta} = [v_1 \; \theta_1 \; v_2 \; \theta_2]$$

strain - generic displacement - nodal displacement

$$\varepsilon = -y\varphi = -y \frac{d^2 v}{dx^2}$$

$$= -y \, \underset{\sim}{N}'' \underset{\sim}{\delta}$$

$$= -y \begin{bmatrix} N_1'' & N_2'' & N_3'' & N_4'' \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

$$N_1'' = \frac{6}{L^2}\left(1 - \frac{2x}{L}\right) \qquad N_2'' = \frac{2}{L}\left(\frac{3x}{L} - 1\right)$$

$$N_3'' = \frac{6}{L^2}\left(\frac{2x}{L} - 1\right) \qquad N_4'' = \frac{2}{L}\left(\frac{3x}{L} - 2\right)$$

stress - strain relationship

$$\sigma = E\varepsilon = -E \cdot y \underset{\sim}{N}'' \underset{\sim}{q}$$

virtual strain energy

$$\delta U = \int \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma} \, dV$$

$$= \delta \underset{\sim}{q}^T \left[ \int E y^2 \underset{\sim}{N}''^T \underset{\sim}{N}'' \, dV \right] \underset{\sim}{q}$$

$$= \delta \underset{\sim}{q}^T \left[ \int EI \underset{\sim}{N}''^T \underset{\sim}{N}'' \, dx \right] \underset{\sim}{q}$$

virtual external energy

$$\delta W = \delta \underset{\sim}{q}^T \underset{\sim}{P} \qquad \underset{\sim}{q} = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

$$\delta U = \delta W \Rightarrow$$

$$\left[ \int EI \underset{\sim}{N}''^T \underset{\sim}{N}'' \, dx \right] \underset{\sim}{q} = \underset{\sim}{P}$$

$$\underset{\sim}{K} = \int EI\, \underset{\sim}{N}''^{T} \underset{\sim}{N}''\, dx$$

$$= \frac{EI}{L} \begin{bmatrix} \frac{12}{L^2} & \frac{-6}{L} & \frac{-12}{L^2} & \frac{-6}{L} \\ & 4 & \frac{6}{L} & 2 \\ & \text{sym.} & \frac{12}{L^2} & \frac{6}{L} \\ & & & 4 \end{bmatrix}$$

Generally,

$$\delta U = \int \delta \underline{\varepsilon}^T \underline{\sigma} \, dV$$

Generic Displacement – nodal displacement

$$\underset{\sim}{u} = \underset{\sim}{f} \, \underline{q} \qquad \underset{\sim}{f} = \text{shape function}$$

$$u = [f_1 \ f_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$v = [f_1 \ f_2 \ f_3 \ f_4] \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Strain – Generic Displacement

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d} \, \underset{\sim}{u}$$

$$\varepsilon = \frac{du}{dx} \qquad \left( \underset{\sim}{d} = \frac{d}{dx} \right)$$

$$\varepsilon = -y \frac{d^2 v}{dx^2} \qquad \left( \underset{\sim}{d} = -y \frac{d^2}{dx^2} \right)$$

Strain – nodal Displacement

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d} \, \underset{\sim}{u} = \underset{\sim}{d} \, \underset{\sim}{f} \, \underline{q} = \underset{\sim}{B} \, \underline{q}$$

$$\varepsilon = \frac{df}{dx} \underline{q}_u \qquad \left( \underset{\sim}{B} = \frac{df}{dx} = [f_1' \ f_2'] \right)$$

$$\varepsilon = -y \frac{d^2 f}{dx^2} \underline{q}_v \qquad \left( \underset{\sim}{B} = -y \frac{d^2 f}{dx^2} = -y [f_1'' \ f_2'' \ f_3'' \ f_4''] \right.$$

virtual strain

$$\delta \underline{\varepsilon} = \underset{\sim}{B} \, \delta \underline{q}$$

stress

$$\underline{\sigma} = \underline{E}\,\underline{\varepsilon} = \underline{E}\,\underline{B}\,\underline{q}$$

$$\delta U = \int \delta \underline{\varepsilon}^T \underline{\sigma}\, dV$$

$$= \int \delta \underline{q}^T \underline{B}^T \underline{E}\,\underline{B}\,\underline{q}\, dV$$

$$= \delta \underline{q}^T \underbrace{\left[ \int \underline{B}^T \underline{E}\,\underline{B}\, dV \right]}_{\underline{k}} \underline{q} = \delta \underline{q}^T \underline{P} \quad (= \delta W)$$

$$\underline{k} = \int \underline{B}^T \underline{E}\,\underline{B}\, dV$$

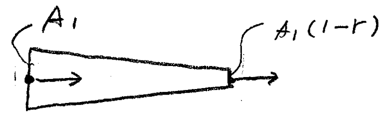
$$\underline{k} = \int \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} [f_1' f_2'] \, E\, dA\, dx$$

$$= \int EA \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} [f_1' f_2']\, dx$$

$$\underline{k} = \int \begin{bmatrix} f_1'' \\ f_2'' \\ f_3'' \\ f_4'' \end{bmatrix} [f_1'' f_2'' f_3'' f_4''] \cdot E\, y^2 dA\, dx$$

$$= \int_0^l EI \begin{bmatrix} f_1'' \\ f_2'' \\ f_3'' \\ f_4'' \end{bmatrix} [f_1'' f_2'' f_3'' f_4'']\, dx$$

## 7.3 Nonuniform Elements




$$u = a_1 + a_2 x$$

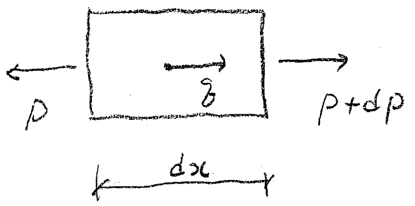
$$A = A_1\left(1 - \frac{rx}{L}\right) \qquad u = \left(1-\frac{x}{L}\right)u_1 + \frac{x}{L}u_2$$

if use $\underset{\sim}{k} = \int EA\, \underset{\sim}{N}'^T \underset{\sim}{N}'\, dx$

$$= \int \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} EA_1\left(1 - \frac{rx}{L}\right) dx$$

$$= \frac{EA_1}{L}\left(1 - \frac{r}{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A_{effective} = A_1\left(1 - \frac{r}{2}\right)$$




By equilibrium $\quad -p + p + dp + g\,dx = 0$

$$\frac{dp}{dx} = -g$$

if $g = 0$, $\quad \frac{dp}{dx} = 0$

$$\Rightarrow \frac{d}{dx}(A\sigma_x) = 0 \quad \text{or} \quad \frac{d}{dx}(EA\varepsilon_x) = 0$$

$$\Rightarrow \frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

$$EA\frac{du}{dx} = a_1 \qquad \frac{du}{dx} = \frac{1}{EA}a_1 = \frac{a_1}{EA_1\left(1 - r\frac{x}{L}\right)}$$

$$u = \frac{a_1}{EA_1} \ln\left(1 - \frac{rx}{L}\right) \times \left(-\frac{L}{r}\right) + a_2$$

$$= \frac{a_1}{EA_1}\left(-\frac{L}{r}\right) \ln\left(1 - \frac{rx}{L}\right) + a_2 \Rightarrow$$ Exact displ. function satisfying the force-equilibrium

$x = 0, \quad u = u_1 = a_2$

$x = L, \quad u = \frac{a_1}{EA_1}\left(-\frac{L}{r}\right) \ln(1-r) + u_1 = u_2$

$$a_1 = (u_2 - u_1) \frac{EA_1}{\left(-\frac{L}{r}\right) \ln(1-r)}$$

$$u = \frac{\ln\left(1 - \frac{rx}{L}\right)}{\ln(1-r)}(u_2 - u_1) + u_1$$

$$= \underbrace{\left[1 - \frac{\ln(1 - rx/L)}{\ln(1-r)}\right]}_{N_1} u_1 + \underbrace{\left[\frac{\ln(1 - rx/L)}{\ln(1-r)}\right]}_{N_2} u_2$$

$$N_1' = \frac{+\left(\frac{r}{L}\right)}{\ln(1-r)} \frac{1}{(1 - rx/L)} = -N_2'$$

$$\underset{\sim}{k} = \int EA_1 \frac{\left(1 - \frac{rx}{L}\right)}{\left(1 - \frac{rx}{L}\right)^2} \cdot \frac{\left(\frac{r}{L}\right)^2}{\ln(1-r)^2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1 \quad -1]\, dx$$

$$= \frac{EA_1}{L^2} \frac{r^2}{\ln(1-r)^2} \int \frac{1}{\left(1 - \frac{rx}{L}\right)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

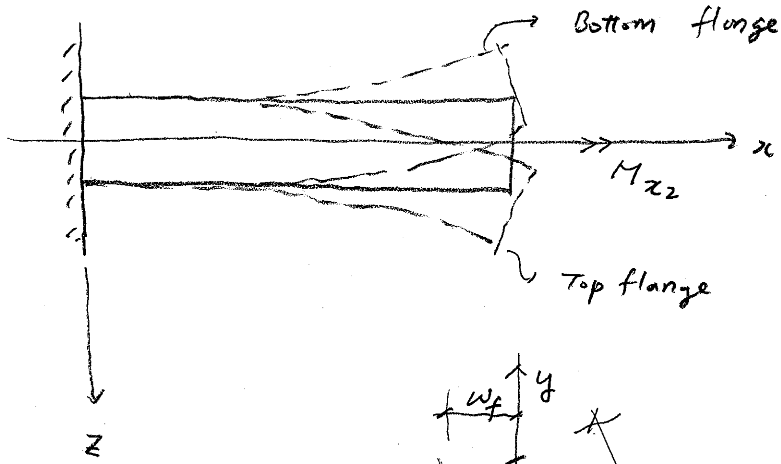
$$= \frac{EA_1}{\ln(1-r)^2}\left(\frac{r}{L}\right)^2 \left[ -\frac{1}{\left(\frac{r}{L}\right)} \ln\left(1 - \frac{rx}{L}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right]_0^L$$

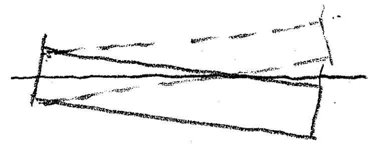
$$= -\frac{EA_1}{L} \frac{r}{\ln(1-r)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

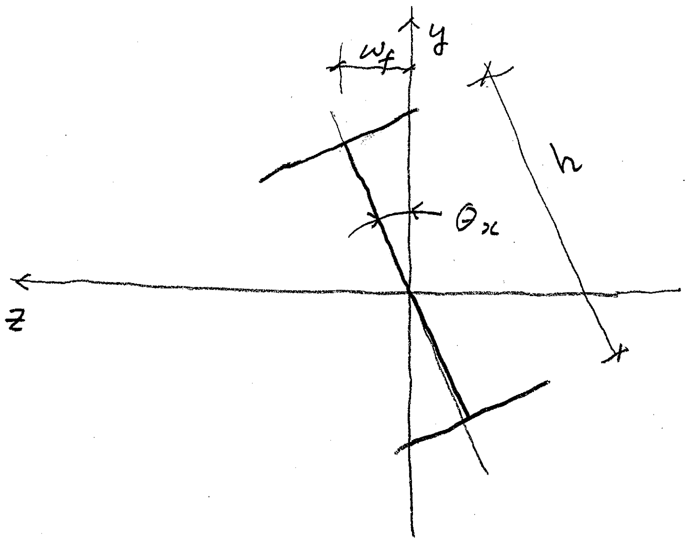
## 7.4 Nonuniform Torsion

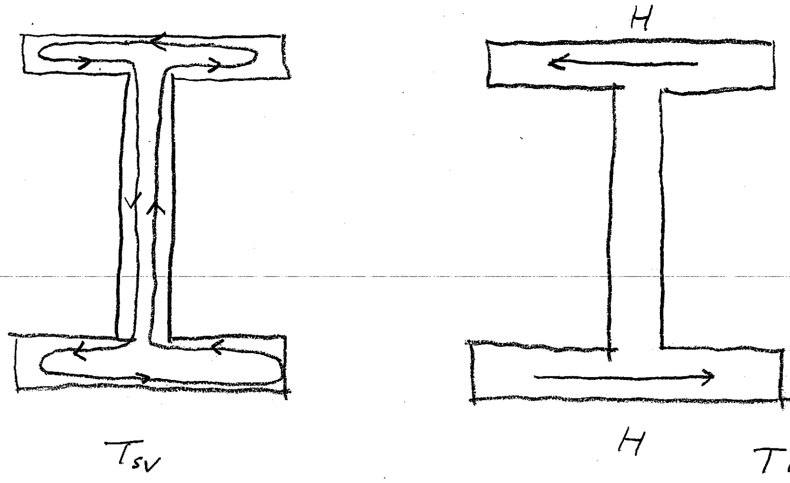
Torsional Resistance

- Pure Torsion - closed cross-section
- warping Torsion - open cross-section




$T_{sv}$ Saint-Venant Torsion

$T_{wr}$ warping Torsion

St. Venant Torsion

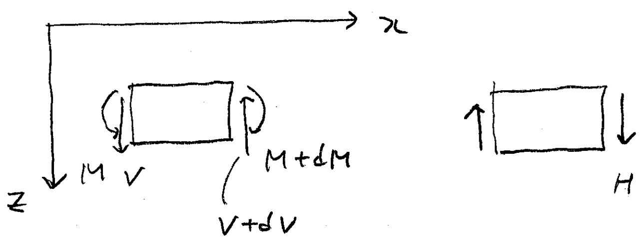
$$T_{sv} = GJ\theta_x' = GJ\frac{d\theta_x}{dx}$$

Warping Torsion

$$M_f = EI_f\frac{d^2w_f}{dx^2} = E\frac{I_y h}{4}\left(\frac{d^2\theta_x}{dx^2}\right)$$

$$w_f = \theta_x\frac{h}{2} \qquad I_f = \frac{I_y}{2}$$

$$\frac{dM_f}{dx} = V = -H$$




$$T_{wr} = H\cdot h = -\frac{dM_f}{dx}\cdot h$$

$$= -E\frac{I_y h^2}{4}\theta_x'''$$

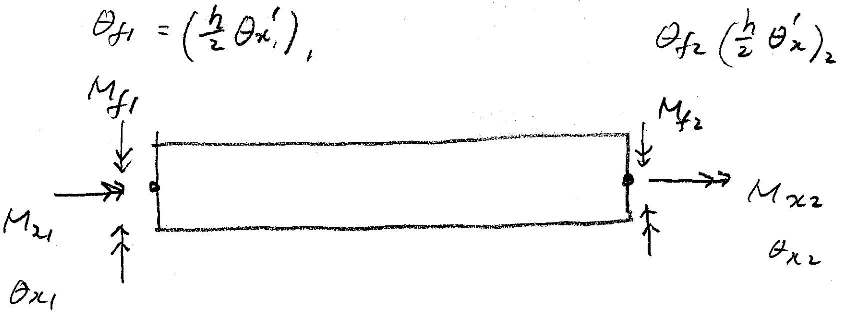
$$= -EC_w\theta_x''' \qquad C_w = \frac{I_y h^2}{4} \text{ : warping constant}$$

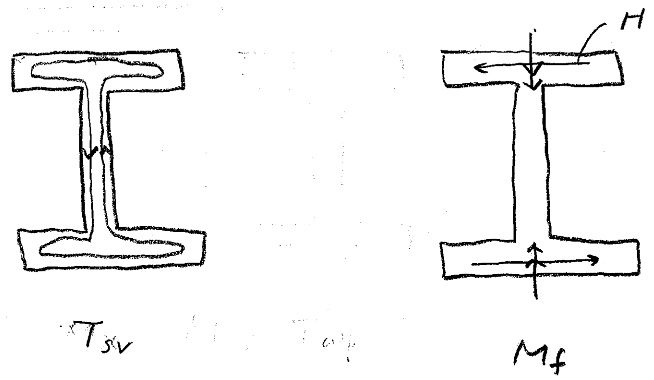
$$M_x = T_{sv} + T_{wr}$$

$$= GJ\theta_x' - EC_w\theta_x'''$$

if $m_x = 0 \Rightarrow \frac{dM_x}{dx} = 0 \Rightarrow$ Equilibrium

$$GJ\theta_x'' - EC_w\theta_x^{IV} = 0 \Rightarrow \text{Governing Equation}$$




$$\underset{\sim}{P} = \begin{bmatrix} M_{x1} \\ M_{x2} \\ M_{f1} \\ M_{f2} \end{bmatrix}$$

$M_{x1}, M_{x2}$ = Torque ($= T_{sv} + T_{wp}$)

$M_{f1}, M_{f2}$ = moment of flanges

$$\theta_f = \frac{d w_f}{dx} = \frac{h}{2}\frac{d\theta_x}{dx} = \frac{h}{2}\theta_x'$$

$$\underset{\sim}{\delta} = \begin{bmatrix} \theta_{x1} \\ \theta_{x2} \\ \left(\frac{h}{2}\theta_x'\right)_1 \\ \left(\frac{h}{2}\theta_x'\right)_2 \end{bmatrix}$$

$$\delta W_{ext} = \delta\underset{\sim}{\delta}^T \cdot \underset{\sim}{P} = \delta\theta_{x1} M_{x1} + \delta\theta_{x2} M_{x2}$$

$$\underbrace{+ \, \delta\theta_{x1}' \underbrace{hM_{f1}}_{B_1}}_{\Downarrow \; 2\delta\theta_f M_{f1}} + \delta\theta_{x2}' \underbrace{hM_{f2}}_{B_2}$$

$$\delta W_{int} = \int \delta\theta_x' \, T_{sv} \, dx + 2\int \delta\theta_f' \, M_f \, dx$$

neglecting virtual work by shearforce H

$$T_{sv} = GJ\theta_x' \qquad M_f = EC_w\left(\frac{1}{h}\right)\theta_x''$$

$$\theta_f' = \frac{h}{2}\theta_x''$$

$$\delta W_{int} = \int \delta\theta_x' \, GJ\theta_x' \, dx + \int \delta\theta_x'' \, EC_w \, \theta_x'' \, dx$$

Reset

$$\underset{\sim}{P} = \begin{bmatrix} M_{x1} \\ M_{x2} \\ B_1 \\ B_2 \end{bmatrix} \qquad \underset{\sim}{q} = \begin{bmatrix} \theta_{x1} \\ \theta_{x2} \\ \theta_{x1}' \\ \theta_{x2}' \end{bmatrix}$$

$$\left( 2\,\delta\underset{\sim}{\theta_f}\,\underset{\sim}{M_f} = \delta\underset{\sim}{\theta_x'}\cdot\underset{\sim}{B} \qquad B = hM_f \qquad \theta_f = \frac{h}{2}\theta_x' \right)$$

$$\theta_x = f_1\theta_{x1} + f_2\theta_{x2} + f_3\theta_{x1}' + f_4\theta_{x2}'$$

$$= \underset{\sim}{f}\,\underset{\sim}{q}$$

$$\theta_x = a_1 + a_2x + a_3x^2 + a_4x^3$$

$\Rightarrow$ <This function does not satisfy equilibrium for pure torsional action.>

$$\frac{dT_{sv}}{dx} = m_x = 0$$

$$\frac{d}{dx}\left(GJ\frac{d\theta_x}{dx}\right) = m_x$$

and

$$GJ\theta_x'' - EC_w\theta_x^{IV} = 0$$

when $m_x = 0$

$$\begin{cases} x = 0. & \theta_x = \theta_{x1}, \quad \theta_x' = \theta_{x1}' \\ x = L & \theta_x = \theta_{x2}, \quad \theta_x' = \theta_{x2}' \end{cases}$$

$$\Rightarrow \quad f_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

$$f_2 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$f_3 = x\left(1 - \frac{x}{L}\right)^2$$

$$f_4 = x\left[\left(\frac{x}{L}\right)^2 - \frac{x}{L}\right]$$

$$\delta W_{ext} = \delta W_{int}$$

$$\int \delta\theta_x' \, GJ \, \theta_x' \, dx + \int \delta\theta_x'' \, EC_w \, \theta_x'' \, dx = \delta \underset{\sim}{q}^T \underset{\sim}{P}$$

$$\delta \underset{\sim}{q}^T \left[ \int \underset{\sim}{f}'^T GJ \underset{\sim}{f}' dx \right] \underset{\sim}{q} + \delta \underset{\sim}{q}^T \left[ \int \underset{\sim}{f}''^T EC_w \underset{\sim}{f}'' dx \right] \underset{\sim}{q} = \delta \underset{\sim}{q}^T \underset{\sim}{P}$$

$$\therefore \underset{\sim}{k} = \underbrace{\int \underset{\sim}{f}'^T GJ \underset{\sim}{f}' dx}_{\underset{\sim}{k}_{sv}} + \underbrace{\int \underset{\sim}{f}''^T EC_w \underset{\sim}{f}'' dx}_{\underset{\sim}{k}_{wp}}$$

See eq. 7.28, 29, 30

$$\underset{\sim}{k}_{sv} = GJ \int_0^l \begin{bmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{bmatrix} [f_1' \ f_2' \ f_3' \ f_4'] \, dx$$

$$\underset{\sim}{k}_{wp} = EC_w \int_0^l \begin{bmatrix} f_1'' \\ f_2'' \\ f_3'' \\ f_4'' \end{bmatrix} [f_1'' \ f_2'' \ f_3'' \ f_4''] \, dx$$

$$T_{sv} = GJ\theta_x'$$

$T_{wr} = -EC_w \theta_x''' \Rightarrow$ 부정확 $\theta_x$ = 3차식

$\Rightarrow$ Approximation.

$T_{sv}$ 구하고 $T - T_{sv} = T_{wr}$ 이 보다 정확함

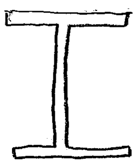
## 7.4.2 Applications and Examples

1. The relative magnitude of the warping restraint and St. Venant effects

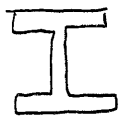
1) $\dfrac{C_w}{J l^2}$ $\Rightarrow$ warping torsion decreases with increase in beam length

$$k_{wp} \propto \frac{C_w}{l^3} \quad \text{vs} \quad k_{sv} \propto \frac{J}{l}$$

2)

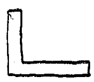
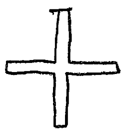


warping torsional rigidity is prevalent



Saint Venant Torsional rigidity is prevalent.

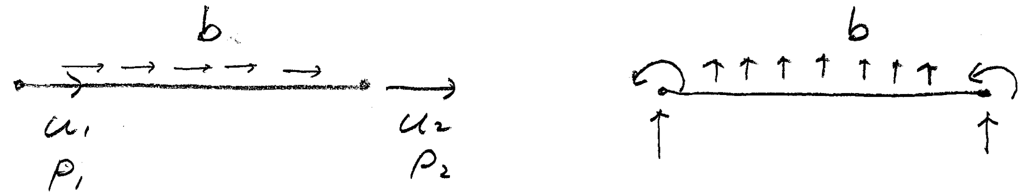
3) For warping torsion, boundary conditions for flanges & length play major role

4)  and  $\Rightarrow$ warping resistance is negligible.

## 7.5 Loads between nodal points and initial strain effects

### Body forces

Axial force member          flexural member




$\delta W_b$ = virtual work due to virtual displacement on body forces

$$\delta W_b = \int_0^l \delta u^T b \, dx \quad \left( = \int \delta \underline{u}^T \underline{b} \, dv \text{ in case of body force per unit volume} \right)$$

$$= \delta \underline{q}^T \int \underline{f}^T b \, dx$$

$$\delta U = \delta W$$

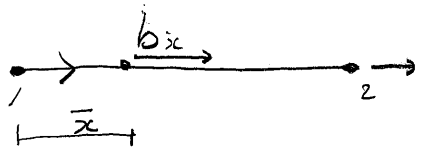
$$\delta \underline{q}^T \left[ \int \underline{B}^T \underline{E} \, \underline{B} \, dv \right] \underline{q} = \delta \underline{q}^T \underline{P} + \delta \underline{q}^T \int \underline{f}^T b \, dx$$

$$\left[ \int \underline{B}^T \underline{E} \, \underline{B} \, dv \right] \underline{q} = \underline{P} + \int \underline{f}^T b \, dx$$

$$= \underline{P} + \underline{P}_b$$

$\underline{P}_b$ = equivalent nodal force due to body forces

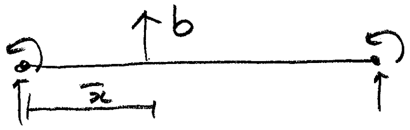
$= - \underline{FEF}$




$b_x$ = concentrated load at $x = \bar{x}$

$$\delta W_b = \delta u^T_{x=\bar{x}} \cdot b = \delta \underset{\sim}{q}^T \underset{\sim}{f}^T_{x=\bar{x}} \cdot b$$

$$\underset{\sim}{P}_b = \underset{\sim}{f}^T_{x=\bar{x}} \cdot b = -\underset{\sim}{FEF}$$



## Initial Strain

$$\sigma = E(\varepsilon - \varepsilon_0)$$

initial imperfection $\quad \varepsilon_0 = \frac{\Delta_0}{L}$ $\quad$ for ~~truss~~ ~~truss~~ element

temperature change $\quad \varepsilon_0 = \alpha \Delta T$

$$\delta U = \int \delta \underline{\varepsilon}^T \underline{\sigma} \, dV$$

$$= \int \delta \underline{\varepsilon}^T \underline{E} (\underline{\varepsilon} - \underline{\varepsilon}_0) \, dV$$

$$= \int \delta \underline{\varepsilon}^T \underline{E} \, \underline{\varepsilon} \, dV - \int \delta \underline{\varepsilon}^T \underline{E} \, \underline{\varepsilon}_0 \, dV$$

$$(\underline{\varepsilon} = \underline{B} \, \underline{\delta} \, , \quad \delta \underline{\varepsilon} = \underline{B} \, \delta \underline{\delta})$$

$$= \delta \underline{\delta}^T \left[ \int \underline{B}^T \underline{E} \, \underline{B} \, dV \right] \underline{\delta} - \delta \underline{\delta}^T \left[ \int \underline{B}^T \underline{E} \, \underline{\varepsilon}_0 \, dV \right]$$

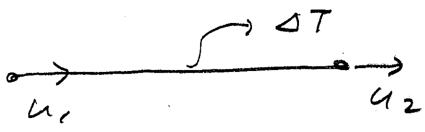
$$\delta U = \delta W$$

$$\left[ \int \underline{B}^T \underline{E} \, \underline{B} \, dV \right] \underline{\delta} = \underline{P} + \underline{P}_b + \underline{P}_0$$

$$\underline{P}_0 = \int \underline{B}^T \underline{E} \, \varepsilon_0 \, dV = -FEF$$

$=$ equivalent nodal load due to initial strain

Truss element

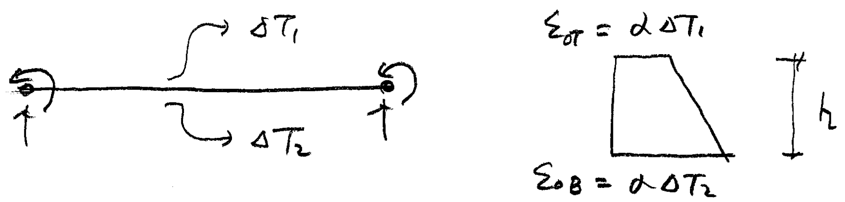



$$\varepsilon_o = \alpha \Delta T$$

$$\underset{\sim}{P_o} = \int \underline{B}^T \underline{E}\, \underline{\varepsilon_o}\, dV = \int \underline{B}^T E\, \varepsilon_o A\, dx$$

$$= \underline{B}^T E A \alpha \Delta T L \qquad \underline{B} = \text{constant}$$

flexural element




$$\varphi_T = \frac{\alpha}{h}(\Delta T_2 - \Delta T_1)$$

$$\varepsilon_T = \frac{\alpha}{h}(\Delta T_2 - \Delta T_1)(-y) = \varepsilon_o$$

$$\underset{\sim}{P_o} = \int \underline{B}^T \underline{E}\, \underset{\sim}{\varepsilon_o}\, dV = \iint \underline{B}^T E\, \varepsilon_o\, dA\, dx$$

$$= \int (-y)\, \underset{\sim}{f}''^T E\, \frac{\alpha}{h}(\Delta T_2 - \Delta T_1)(-y)\, dA\, dx$$

$$= EI \int \underset{\sim}{f}''^T \varphi_T\, dx$$

## 7.6 Virtual Forces in the formulation of Element Force - Displacement Equations

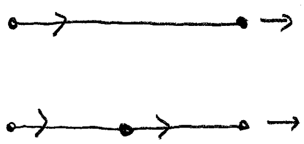
### 7.6.1 Construction of Element Equations by the Principle of Virtual Forces

principle of virtual force is a basis of the direct formulation of element flexibility equations

The principle can be used to drive the element stiffness matrix.

Since, the flexibility formulation is more straightforward, it is useful to drive the element stiffness particularly for complex conditions

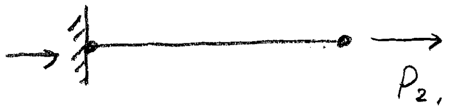
- combined shear and bending deformation
- tapered element
- elements with curved or irregular axes



( o )

( x ) inapplicable

Truss element




$$\delta U^* = \int \delta \underset{\sim}{\sigma}^T \underset{\sim}{\varepsilon}\, dV$$

$$= \int \delta \underline{\sigma}^T \underline{E}^{-1} \underline{\sigma}\, dV$$

$$= \int \delta \sigma_x^T E^{-1} \sigma_x\, dV$$

$$= \int \delta F_x^T \cdot \frac{F_x}{EA}\, dx$$

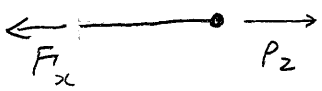
$(F_x = Q \underset{\sim}{P_f})$ $\quad Q$ = relationship between internal forces and nodal forces

$$= \delta \underline{P}_f^T \left[ \int \underset{\sim}{Q}^T \frac{1}{EA} \underset{\sim}{Q}\, dx \right] \underline{P}_f$$

$$\delta W^* = \delta \underset{\sim}{P_f}^T \cdot \underset{\sim}{\delta}$$

$$\delta U^* = \delta W^*$$

$$\underbrace{\left[ \int \underset{\sim}{Q}^T \frac{1}{EA} \underset{\sim}{Q}\, dx \right]}_{\underline{d}} \underline{P}_f = \underset{\sim}{\delta}$$

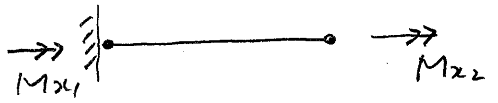



$\underset{\sim}{P_f} = P_2 = F_x \Rightarrow Q = 1$

$$\underset{\sim}{k} = \left[ \begin{array}{c|c} \underline{d}^{-1} & \underline{d}^{-1}\underline{\Phi}^T \\ \hline \underline{\Phi}\,\underline{d}^{-1} & \underline{\Phi}\,\underline{d}\,\underline{\Phi}^T \end{array} \right] \qquad \underset{\sim}{P_s} = \underline{\Phi}\, \underline{P}_f$$

$\Phi$ = relationship between nodal forces

Torsional Element




$P_f = M_{x_2}$ . $P_s = M_{x_1}$

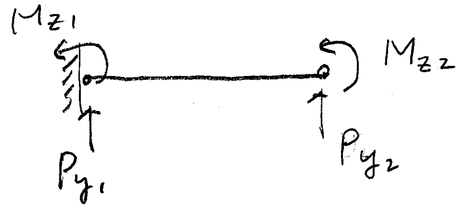
$$\delta U^* = \int \delta \tau^T \gamma \, dV$$

$$= \int \delta M_x \frac{M_x}{GJ} dx \qquad M_x = \underset{\sim}{Q} \, P_f = \underset{\sim}{Q} \, M_{x_2}$$

$$Q = 1$$

$$= \delta \underset{\sim}{P_f}^T \Big[ \underbrace{\int \underset{\sim}{Q}^T \frac{1}{GJ} \underset{\sim}{Q} \, dx}_{\underset{\sim}{d}} \Big] \underset{\sim}{P_f}$$

flexural element




$$\underset{\sim}{P_f} = \begin{bmatrix} P_{y_2} \\ M_{z_2} \end{bmatrix} \qquad \underset{\sim}{P_s} = \begin{bmatrix} P_{y_1} \\ M_{z_1} \end{bmatrix}$$

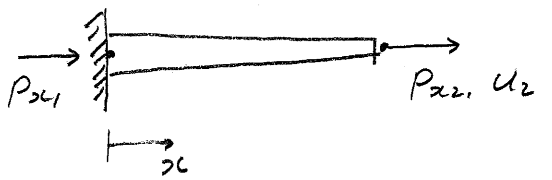
$$\delta U^* = \int \delta \underset{\sim}{\sigma}^T \underset{\sim}{\varepsilon} \, dV$$

$$= \int \delta \sigma \cdot \frac{1}{E} \sigma \, dV$$

$$= \int \delta M_z \frac{M_z}{EI} dx \qquad M_z = \underset{\sim}{Q} \, \underset{\sim}{P_f}$$

$$= \delta \underset{\sim}{P_f}^T \Big[ \underbrace{\int \underset{\sim}{Q}^T \frac{1}{EI} Q \, dx}_{\underset{\sim}{d}} \Big] \underset{\sim}{P_f}$$

Tapered axial member




$$A = A_1\left(1 - \frac{rx}{L}\right)$$

$$P_f = P_{x2}, \quad P_s = P_{x1} \qquad P_{x1} = -P_{x2} \Rightarrow \underset{\sim}{\Phi} = -1$$

$$F_x = P_{x2} \Rightarrow \underset{\sim}{Q} = 1$$

$$\underset{\sim}{d} = \int \underset{\sim}{Q}^T \frac{1}{EA} \underset{\sim}{Q}\, dx$$

$$= \int \frac{1}{EA} dx$$

$$= \frac{1}{EA_1} \int \frac{1}{\left(1 - \frac{rx}{L}\right)} dx$$

$$= -\frac{L}{rEA_1} \left[ \ln\left(1 - \frac{rx}{L}\right) \right]_0^l$$

$$= -\frac{L}{rEA_1} \ln(1 - r)$$

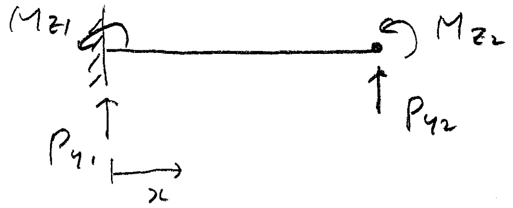
$$\underset{\sim}{k} = \left[ \begin{array}{c|c} \underset{\sim}{d}^{-1} & \underset{\sim}{d}^{-1}\underset{\sim}{\Phi}^T \\ \hline \underset{\sim}{\Phi}\,\underset{\sim}{d}^{-1} & \underset{\sim}{\Phi}\,\underset{\sim}{d}^{-1}\underset{\sim}{\Phi}^T \end{array} \right]$$

$$= -\frac{rEA_1}{L \ln(1-r)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

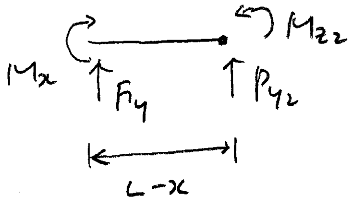
Why this exact solution can be obtained with ease.

- The virtual and real forces are given by the equation of statics. The variations of A, I, E do not affect these equations. in case of statically determinate structure
- whereas exact displacement functions can only be obtained through solution of differential equations with varying coefficients.

flexural element




$$\underset{\sim}{P_f} = \begin{bmatrix} P_{y2} \\ M_{z2} \end{bmatrix}$$




$$M_x = M_{z2} + (L-x) P_{y2}$$

$$= \underbrace{\begin{bmatrix} (L-x) & 1 \end{bmatrix}}_{\underset{\sim}{Q}} \begin{bmatrix} P_{y2} \\ M_{z2} \end{bmatrix}$$

$$\underset{\sim}{d} = \int \underset{\sim}{Q}^T \frac{1}{EI} \underset{\sim}{Q} \, dx$$

$$= \int \begin{bmatrix} (L-x) \\ 1 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} (L-x) & 1 \end{bmatrix} dx$$

$$= \frac{L}{EI} \begin{bmatrix} \frac{L^2}{3} & \frac{L}{2} \\ \frac{L}{2} & 1 \end{bmatrix}$$

$$\underset{\sim}{P_s} = \begin{bmatrix} P_{y1} \\ M_{z1} \end{bmatrix}$$

$$P_{y1} = -P_{y2} \qquad M_{z1} = -M_{z2} - L P_2$$

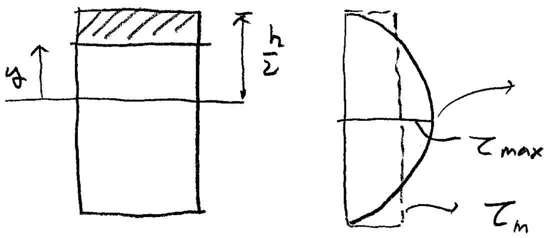
$$\begin{bmatrix} P_{y1} \\ M_{z1} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ -L & -1 \end{bmatrix}}_{\Phi} \begin{bmatrix} P_{y2} \\ M_{z2} \end{bmatrix}$$

$$\underset{\sim}{k} = \left[ \begin{array}{c|c} \underset{\sim}{d}^{-1} & \underset{\sim}{d}^{-1} \Phi^T \\ \hline \Phi \underset{\sim}{d}^{-1} & \Phi \underset{\sim}{d}^{-1} \Phi^T \end{array} \right]$$

Element with Combined Mode

$$\delta U^* = \int \delta F_x \frac{F_x}{EA}\,dx + \int \delta M_z \frac{M_z}{EI}\,dx + \int \delta M_x \frac{M_x}{GJ}\,dx + \cdots$$

Shear Area




$$\tau = \frac{VQ}{Ib}$$

$$\tau_m = \frac{V}{bh}$$

$$\tau_{max} = 1.5\,\tau_m$$

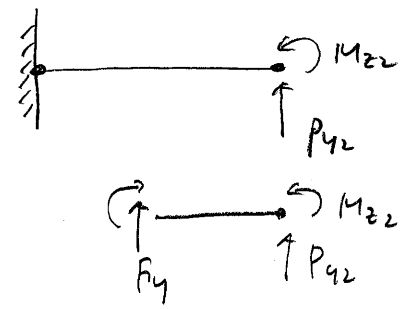
$$\int \delta\tau\,\gamma\,dA = \alpha A\,\delta\tau_m \cdot \gamma_m \qquad \gamma_m = \frac{\tau_m}{G}$$

$$Q = b\cdot\left(\frac{h}{2} - y\right) \times \frac{1}{2}\left(\frac{h}{2} + y\right) = b.\left(\frac{h^2}{4} - y^2\right)$$

$$2\int_0^{\frac{h}{2}} \frac{\delta V\,Q^2}{GI^2b^2} \cdot V\,b\,dy = \alpha\,(bh)\,\frac{\delta V}{\alpha bh} \cdot \frac{V}{Gbh\alpha}$$

$$2\,\frac{h}{I^2}\int Q^2 \cdot dy = \frac{1}{\alpha}$$

# Shearing Deformation of a Beam




$$F_y = -P_{y2} \qquad \underset{\sim}{Q} = -1$$

$$\tau_{av} = \frac{F_y}{A_s} \qquad \gamma = \frac{\tau_{av}}{G} = \frac{F_y}{GA_s}$$

$$\begin{aligned}
\delta U^* &= \int \delta \underset{\sim}{\sigma}^T \underset{\sim}{\varepsilon}\, dV \\
&= \int \delta\tau \cdot \gamma \, dV \\
&= \delta \underset{\sim}{P}_f \int \underset{\sim}{Q}^T \frac{1}{GA_s} \underset{\sim}{Q}\, dx \; \underset{\sim}{P}_f \\
&= \delta \underset{\sim}{P}_{y2} \frac{L}{GA_s} P_{y2}
\end{aligned}$$

$$\begin{aligned}
\delta U^* &= \delta U^*_{bending} + \delta U^*_{shear} \\
&= [\delta P_{y2} \;\; \delta M_{z2}] \underbrace{\begin{bmatrix} \frac{L^3}{3EI} + \frac{L}{A_s G} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}}_{\underset{\sim}{d}} \begin{bmatrix} P_{y2} \\ M_{z2} \end{bmatrix}
\end{aligned}$$

$$\underset{\sim}{k} = \left[\begin{array}{c|c} \underset{\sim}{d}^{-1} & \underset{\sim}{d}^{-1}\underset{\sim}{\Phi}^{T} \\ \hline \underset{\sim}{\Phi}\,\underset{\sim}{d}^{-1} & \underset{\sim}{\Phi}\,\underset{\sim}{d}^{-1}\underset{\sim}{\Phi}^{T} \end{array}\right]$$

$$\begin{bmatrix} P_{y1} \\ M_{z1} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ -L & -1 \end{bmatrix}}_{\underset{\sim}{\Phi}} \begin{bmatrix} P_{y2} \\ M_{z2} \end{bmatrix}$$

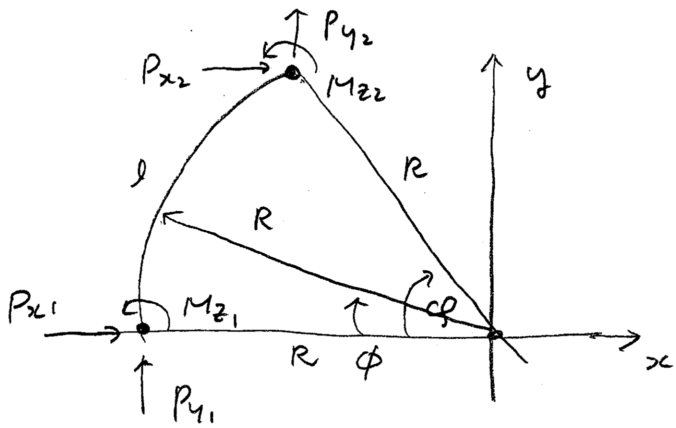
$$\begin{bmatrix} P_{y2} \\ M_{z2} \\ P_{y1} \\ M_{z1} \end{bmatrix} = \underbrace{\frac{EI}{L\left(\frac{L^2}{12}+\eta\right)} \begin{bmatrix} 1 & -\frac{L}{2} & -1 & -\frac{L}{2} \\ & \left(\frac{L^2}{3}+\eta\right) & \frac{L}{2} & \left(\frac{L^2}{6}-\eta\right) \\ & & 1 & \frac{L}{2} \\ & \text{Sym.} & & \left(\frac{L^2}{3}+\eta\right) \end{bmatrix}}_{\underset{\sim}{k}} \begin{bmatrix} v_2 \\ \theta_{z2} \\ v_1 \\ \theta_{z1} \end{bmatrix}$$

$$\eta = \frac{EI}{A_s G}$$

# Circular Ring Beam

Elementary flexure theory is used.

axial behavior, transverse shear deformation, and curved beam theory is disregarded.




$$\underset{\sim}{P_f} = \begin{bmatrix} P_{x1} \\ P_{y1} \\ M_{z1} \end{bmatrix}$$

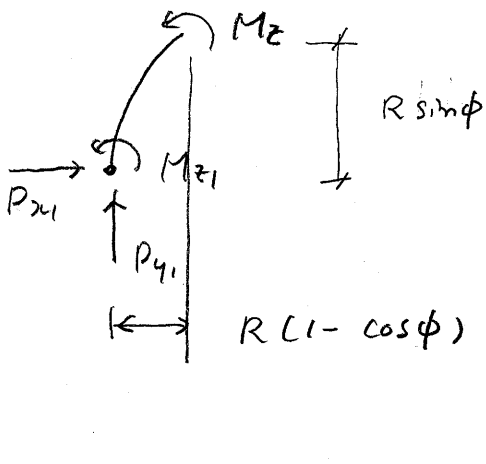
$$\underset{\sim}{P_s} = \begin{bmatrix} P_{x2} \\ P_{y2} \\ M_{z2} \end{bmatrix}$$

$$\delta U^* = \int \delta \underset{\sim}{\sigma}^T \underset{\sim}{\varepsilon}\, dV$$

$$= \int_0^l \delta M_z \frac{M_z}{EI} ds$$

$$(ds = R\, d\phi)$$

$$= \int_0^{\varphi} dM_z \frac{M_z}{EI} R\, d\phi$$



$$M_z = -P_{x1} R \sin\phi + P_{y1} R(1-\cos\phi) - M_{z1}$$

$$= \underbrace{\begin{bmatrix} -\sin\phi & (1-\cos\phi) & -1 \end{bmatrix}}_{\underset{\sim}{Q}} \begin{bmatrix} P_{x1}R \\ P_{y1}R \\ M_{z1} \end{bmatrix}$$

$$\delta U^* = \delta \underset{\sim}{P_f} \underbrace{\left[ \int_0^{\varphi} \underset{\sim}{Q}^T \frac{1}{EI} \underset{\sim}{Q} \; R\, d\phi \right]}_{\underset{\sim}{d}} \underset{\sim}{P_f}$$

$$\underset{\sim}{d} = \int_0^{\varphi} \frac{R}{EI} \begin{bmatrix} -\sin\phi \\ (1-\cos\phi) \\ -1 \end{bmatrix} \begin{bmatrix} -\sin\phi & (1-\cos\phi) & -1 \end{bmatrix} d\phi$$

$$= \text{Eq. } 7.50$$

$$\underset{\sim}{q_f} = \begin{bmatrix} u_1/R \\ v_1/R \\ \theta_{z1} \end{bmatrix}$$

$$\underset{\sim}{P_s} = \underset{\sim}{\Phi} \; \underset{\sim}{P_f}$$

$$\begin{bmatrix} P_{x2}R \\ P_{y2}R \\ M_{z2} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -\sin\varphi & (1-\cos\varphi) & -1 \end{bmatrix}}_{\underset{\sim}{\Phi}} \begin{bmatrix} P_{x1}R \\ P_{y1}R \\ M_{z1} \end{bmatrix}$$

$\Rightarrow$ $\underset{\sim}{k}$ can be calculated with $\underset{\sim}{d}$ and $\underset{\sim}{\Phi}$