Chapter 8. Nonlinear Analysis of Frames - An Introduction

Sources of nonlinearity

- Geometric nonlinearity $\underset{\sim}{K}(\underset{\sim}{\Delta}) \cdot \underset{\sim}{\Delta} = \underset{\sim}{P}$
- material nonlinearity $\underset{\sim}{E} \Rightarrow$ inelastic

Levels of analysis

1) first-order elastic analysis

2) analysis of critical load

   - eigenvalue problem
   - linearized stability analysis

   bifurcation
   $\Rightarrow$ deflection mode 가 바뀌는 점

3) second-order elastic analysis

   - Geometric nonlinearity
   - no material nonlinearity

4) first-order inelastic analysis

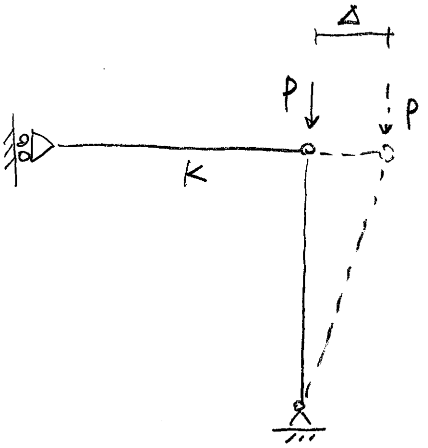
   - material nonlinearity
   - no Geometric nonlinearity

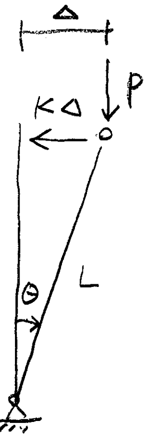
5) Second-order inelastic analysis

   - material and Geometric nonlinearity

See Fig. 8-1

Example 8.1 truss 구조의 Geometric Nonlinear Analysis



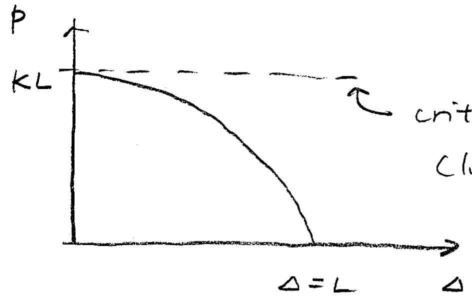


Δ가 포함된 deflected shape 을 가정하여 force-equilibrium을 취한다.

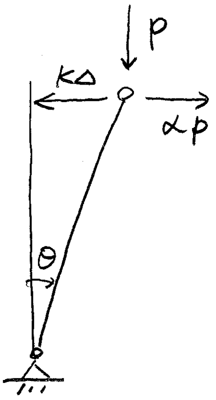
$$K\Delta L\cos\theta = P\Delta$$

$$\Delta(P - KL\cos\theta) = 0 \Rightarrow P = KL\cos\theta$$

$$= KL\frac{\sqrt{L^2-\Delta^2}}{L}$$



critical load $P = KL$ (assuming $\Delta \ll L$)
(linearized stability analysis)



$$P\cdot\Delta + \alpha P L\cos\theta - K\Delta L\cos\theta = 0$$

$$\Delta = L\sin\theta$$

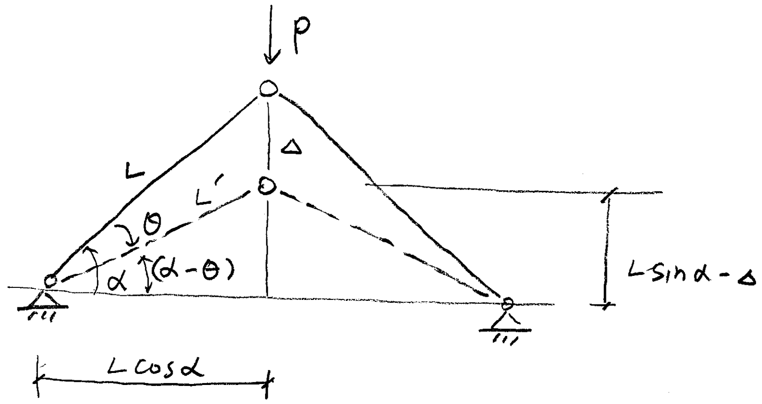
$$P = \frac{KL\cos\theta}{1+\alpha\cot\theta}$$

$$\cos\theta = \frac{\sqrt{L^2-\Delta^2}}{L} \qquad \sin\theta = \frac{\Delta}{L}$$

$$P = KL\frac{\sqrt{L^2-\Delta^2}}{L} \times \frac{1}{1+\alpha\frac{\sqrt{L^2+\Delta^2}}{\Delta}}$$

see Figure in the text.

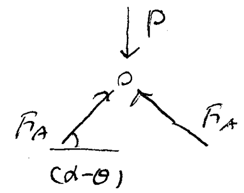
Example 8.2 truss 구조의 Geometric Nonlinear Analysis



$$L' = \frac{L\cos\alpha}{\cos(\alpha-\theta)}$$

$$F_A = \frac{(L-L')}{L} EA$$

$$= \left(1 - \frac{\cos\alpha}{\cos(\alpha-\theta)}\right) EA$$



$$P = 2 F_A \sin(\alpha-\theta)$$

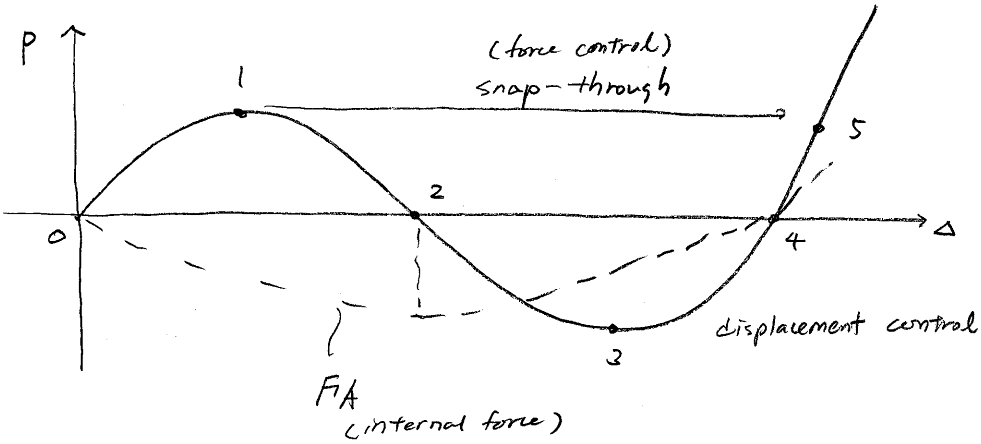
$$= 2EA \sin(\alpha-\theta)\left[1 - \frac{\cos\alpha}{\cos(\alpha-\theta)}\right]$$

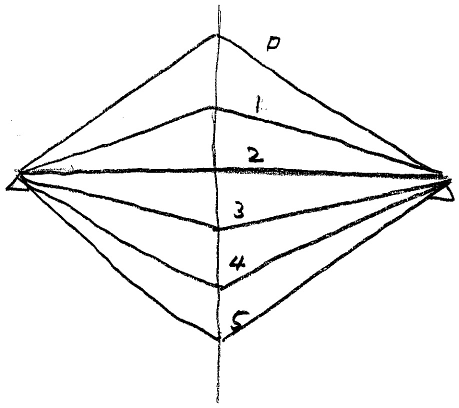
$$= 2EA\left[\sin(\alpha-\theta) - \cos\alpha \tan(\alpha-\theta)\right]$$

$$\sin(\alpha-\theta) = \frac{1}{L'}(L\sin\alpha - \Delta) = \frac{(L\sin\alpha - \Delta)}{\sqrt{L\cos^2\alpha + (L\sin\alpha - \Delta)^2}}$$

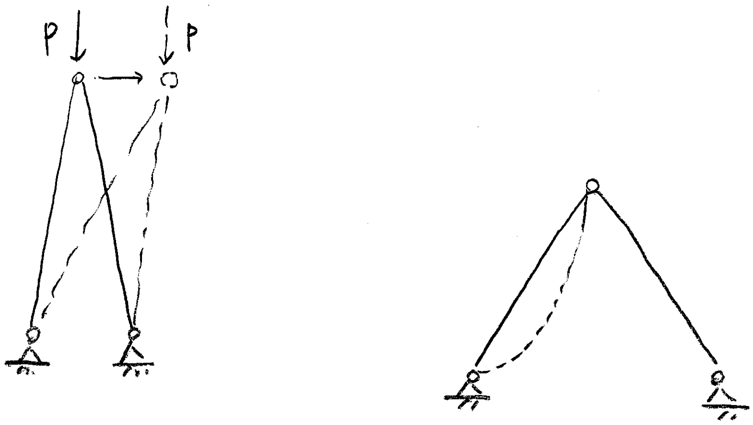
$$\tan(\alpha-\theta) = \frac{(L\sin\alpha - \Delta)}{L\cos\alpha}$$

$$P = 2EA\left[\frac{(L\sin\alpha - \Delta)}{\sqrt{L\cos^2\alpha + (L\sin\alpha-\Delta)^2}} - \frac{\cos\alpha(L\sin\alpha - \Delta)}{L\cos\alpha}\right]$$





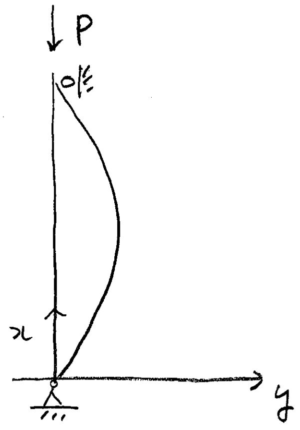
buckling modes that should be considered additionally

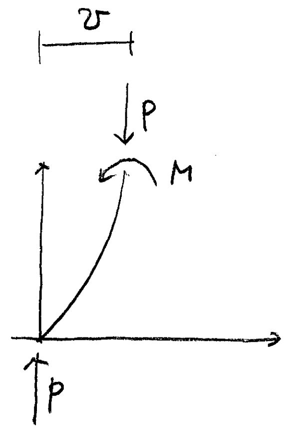


Lateral buckling

buckling of element
flexural buckling

Example 8.3 Geometric Nonlinear behavior by flexural action





$M - Pv = 0 \qquad \frac{M}{EI} = -v''$

$v'' + \frac{Pv}{EI} = 0 \qquad \text{set } k = \sqrt{\frac{P}{EI}}$

$v'' + k^2 v = 0$

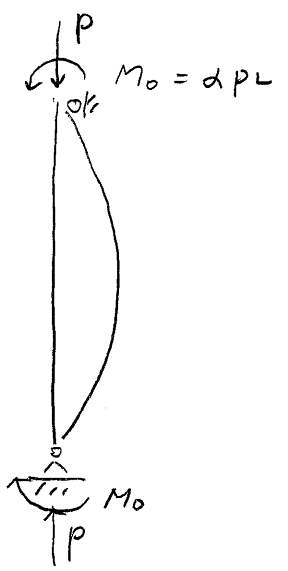
$v = c_1 \sin kx + c_2 \cos kx$

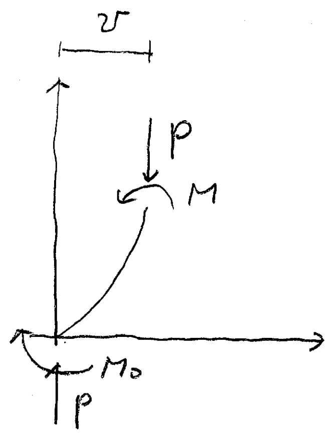
$x = 0,\ v = 0 \Rightarrow c_2 = 0$

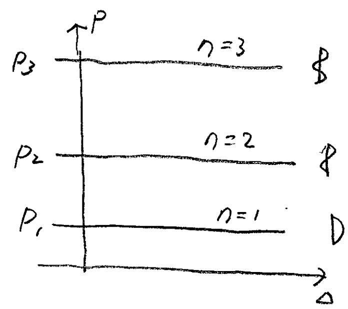
$x = L,\ v = 0 \Rightarrow c_1 \sin kL = 0 \qquad kL = n\pi$

$k^2 = \frac{P}{EI} = \frac{n^2\pi^2}{L^2}$

$P_{cr} = \frac{n^2\pi^2 EI}{L^2} \qquad P_E = P_{cr,\,n=1} = \frac{\pi^2 EI}{L^2}$







$$M - Pv - M_o = 0 \qquad \frac{M}{EI} = -v''$$

$$-EIv'' - Pv - M_o = 0$$

$$v'' + \frac{P}{EI}v = -\frac{\alpha PL}{EI}$$

$$v = v_h + v_p$$

$$v = C_1 \sin kx + C_2 \cos kx - \alpha L$$

$$x=0,\ v=0 \Rightarrow C_2 - \alpha L = 0 \qquad C_2 = \alpha L$$

$$x=L,\ v=0 \Rightarrow C_1 \sin kL + \alpha L(\cos kL - 1) = 0$$

$$C_1 = \frac{\alpha L(1-\cos kL)}{\sin kL}$$

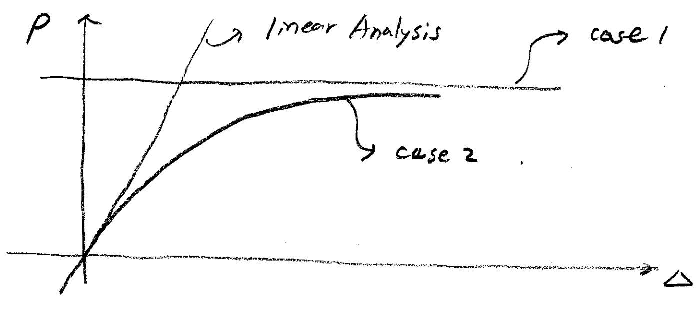
$$v = \left[\frac{\alpha L(1-\cos kL)}{\sin kL}\right]\sin kx + \alpha L \cos kx - \alpha L$$

$$v\left(x=\frac{L}{2}\right) = \delta =$$

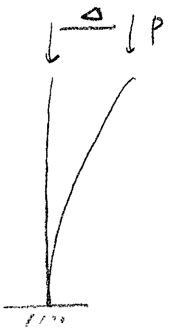
$$\frac{\alpha L}{\sin kL}\left[\sin k\frac{L}{2} - \cos kL \sin k\frac{L}{2} + \cos k\frac{L}{2}\sin kL\right] - \alpha L$$

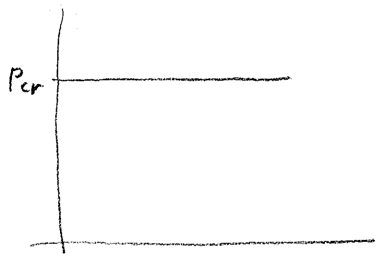
$$= \frac{\alpha L}{2\sin\frac{kL}{2}\cos\frac{kL}{2}}\left[\sin k\frac{L}{2}\, 2\sin^2\frac{kL}{2} + 2\cos k\frac{L}{2}\sin\frac{kL}{2}\cos\frac{kL}{2}\right] - \alpha L$$

$$= \alpha L\left[\sec\frac{kL}{2} - 1\right] \qquad \frac{P}{EI} \to \frac{\pi^2}{L^2} \text{ or } k \to \frac{\pi}{L} \Rightarrow v \to \infty$$



Example 8.4



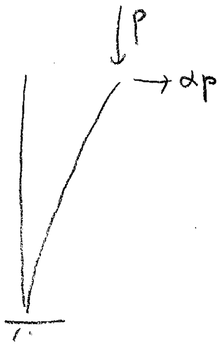


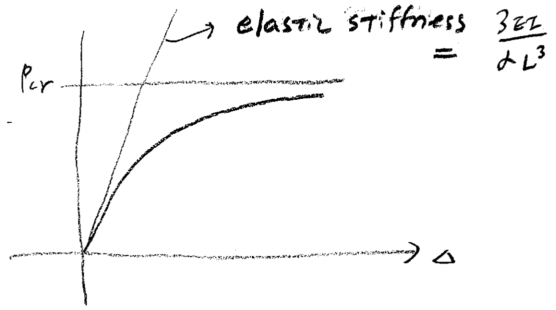
$$P_{cr} = \frac{\pi^2 EI}{4L^2} \qquad v = C_1 \sin\frac{\pi x}{2L}$$

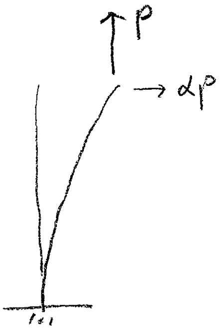
in general

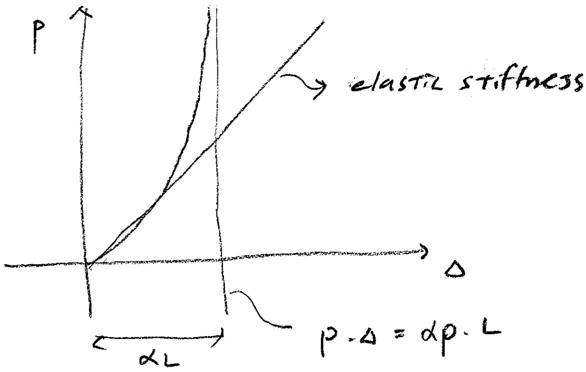
$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

K = effective length factor









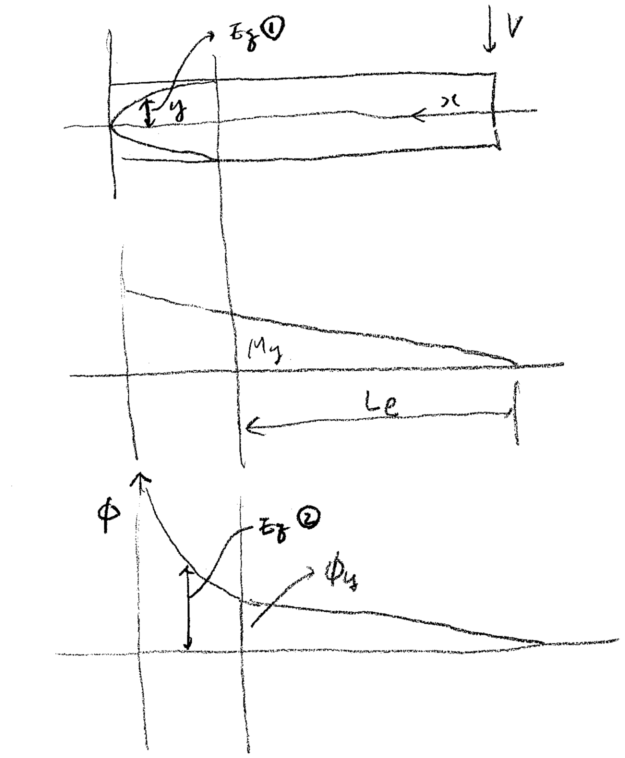
$P \cdot \Delta = \alpha P \cdot L$

$\Delta = \alpha L$

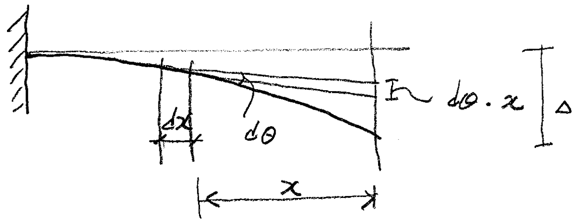
further

(Deflection from force-equilibrium)

Example 8.5 material nonlinearity – determinate structure

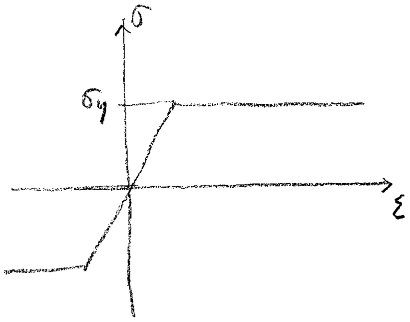


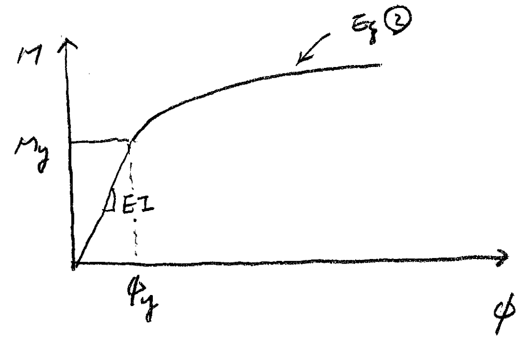
Distributed plastic hinge model

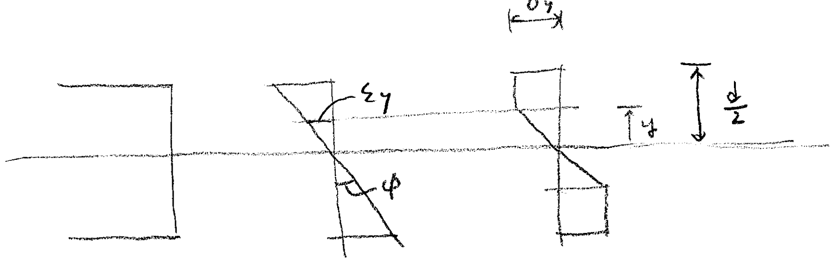


$$\Delta = \int_0^l x\,d\theta = \int_0^l x\,\phi\,dx$$

$$\phi = f(M(x))$$







$\phi = \dfrac{\varepsilon_y}{y} = \dfrac{1}{y}\dfrac{\sigma_y}{E}$ after yielding Before yielding $\phi = \dfrac{M}{EI}$

$$M = b\sigma_y\left(\frac{d}{2}-y\right)\frac{\left(\frac{d}{2}+y\right)}{2}\,2 + b\sigma_y\, y\,\frac{1}{2}\cdot 2y\,\frac{2}{3}$$

$$= b\sigma_y\left(\frac{d^2}{4}-y^2\right) + b\sigma_y\,\frac{2}{3}\,y^2$$

$$= b\sigma_y\left(\frac{d^2}{4}-\frac{y^2}{3}\right)$$

$$\frac{y^2}{3} = \frac{d^2}{4} - \frac{M}{b\sigma_y} \qquad y = \sqrt{3}\left(\frac{d^2}{4} - \frac{M}{b\sigma_y}\right)^{1/2} \quad \text{— ①}$$

$$\phi = \frac{\sigma_y}{E}\,\frac{1}{\sqrt{3}\left(\frac{d^2}{4} - \frac{M}{b\sigma_y}\right)^{1/2}} \quad \text{— ②}$$

$$M = V \cdot x = \frac{x}{L_e} M_y \qquad M_y = \sigma_y \frac{bd^2}{6}$$

$$y = \sqrt{3}\left(\frac{d^2}{4} - \frac{1}{6\sigma_y}\frac{x}{L_e}\sigma_y\frac{bd^2}{6}\right)^{1/2}$$

$$= \frac{\sqrt{3}}{2} d \left[1 - \frac{2}{3}\frac{x}{L_e}\right]^{1/2}$$

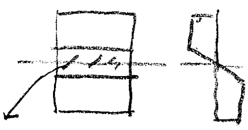
$$\phi = \frac{\sigma_y}{E}\frac{2}{\sqrt{3}d\left[1 - \frac{2}{3}\frac{x}{L_e}\right]^{1/2}}$$

$$\Delta = \Delta_{ac} + \Delta_{cb}$$

$$= \int_{L_e}^{L} \phi\, x\, dx + \frac{VL_e^3}{3EI}$$

in Text

$$= \int_{L_e}^{L} \frac{Mx}{EI_{eff}}\, dx + \frac{VL_e^3}{3EI} \qquad I_{eff} = \frac{2by^3}{3}$$

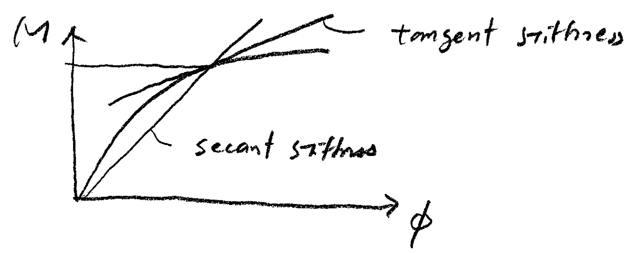


I_{eff}

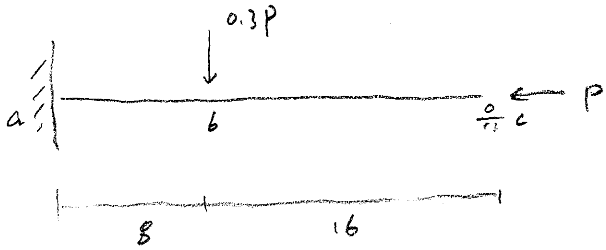
$\Rightarrow$ wrong

because $\quad I_{eff} = \dfrac{2by^3}{3} \Rightarrow$ tangent stiffness $(= dM/d\phi)$

not secant stiffness $(= M/\phi)$



Example 8.6 material nonlinearity – indeterminate structure



W12×65 $A = 19.1 \text{ in}^2$

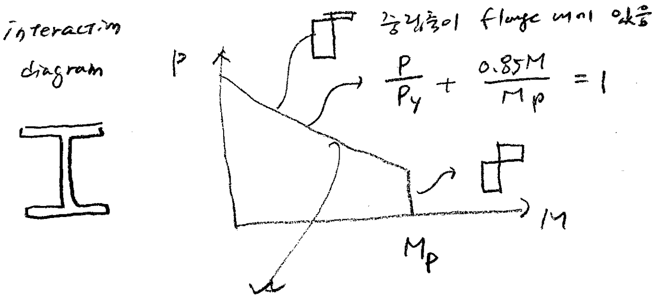
$I = 533 \text{ in}^4$

$Z = 96.8 \text{ in}^3$

$E = 29{,}000$ ksi,

$\sigma_y = 50$ ksi

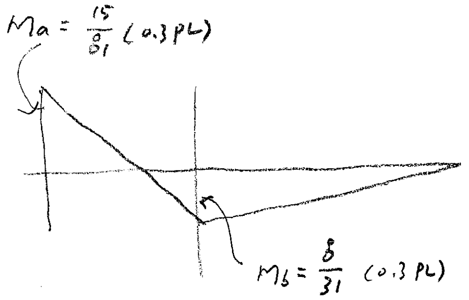
interaction diagram



$P_y = 19.1 \times 50 = 955$ kips

$M_p = 96.8 \times 50 = 4840$ kips

$$\frac{P}{955} + \frac{0.85M}{4840} = 1$$

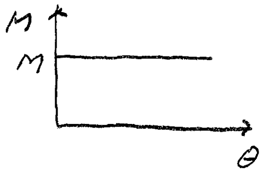


Elastic M diagram

step by step analysis

1st hinge forms at a.

Assumption
- concentrated plastic hinge model
- elasto perfectly plastic model



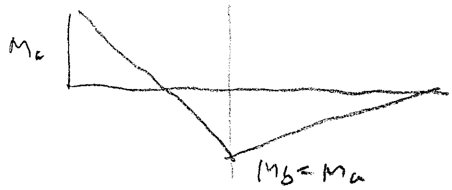
$$\frac{P}{955} + \frac{0.85\left(\frac{15}{81}(0.3PL)\right)}{4840} = 1 \Rightarrow P = 259.3 \text{ kips}$$

$M_a = 4148$

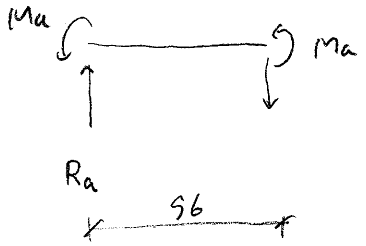
$M_b = 2212$

2nd hinge forms at b

At that time $|M_a| = |M_b|$ because the same axial load P is applied

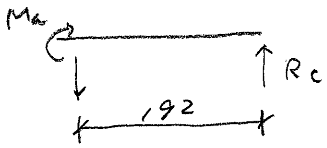


$$\Sigma M_{at\,b} = 2M_a - R_a 96 = 0$$



$$R_a = \frac{2}{96} M_a$$

$$\Sigma M_{at\,b} = -M_a + R_c(192) = 0$$



$$R_c = \frac{M_a}{192}$$

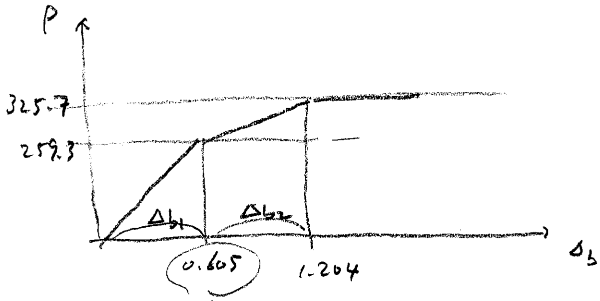
$$R_a + R_c = 0.3P \qquad \frac{2}{96} M_a + \frac{1}{192} M_a = 0.3P$$

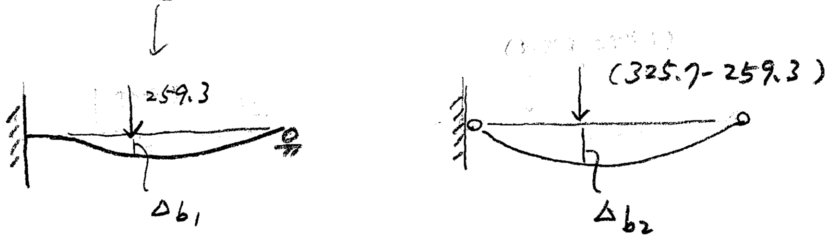
$$M_a = 11.52P = M_b$$

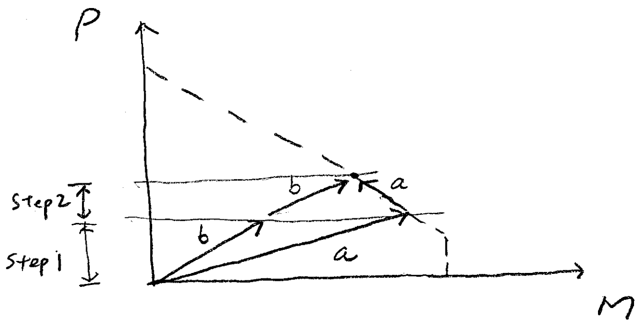
$$\frac{P}{955} + \frac{0.85(11.52P)}{4840} = 1$$

$P = 325.7$

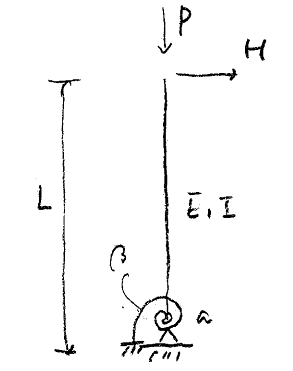
$M_a = M_b = 3752$

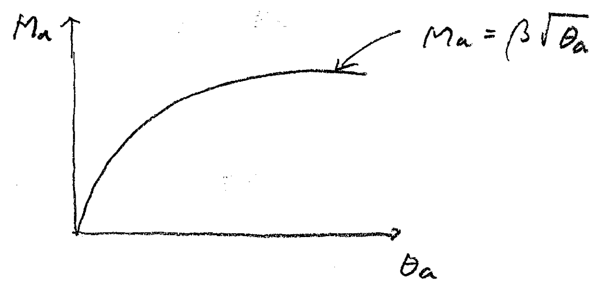


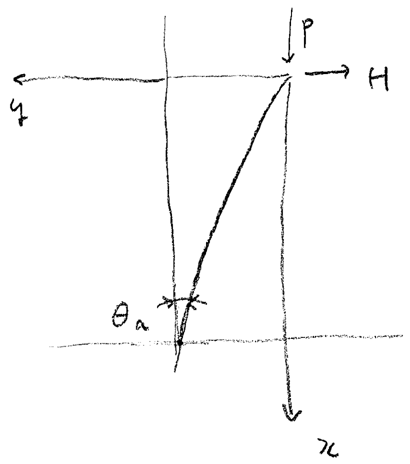


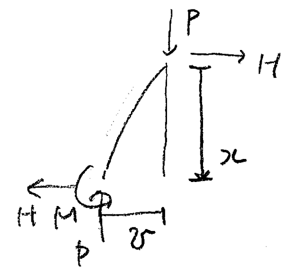


Example 8.7 Geometric + material nonlinearity









$$M = H \cdot x + Pv$$

$$EIv'' + Pv = -Hx$$

$$EIv_h'' + Pv_h = 0 \qquad EIv_p'' + Pv_p = -Hx \qquad v_p = -\frac{Hx}{P}$$

$$v_h = C_1 \sin kx + C_2 \cos kx \qquad k = \sqrt{\frac{P}{EI}}$$

$$v = v_h + v_p = C_1 \sin kx + C_2 \cos kx - \frac{Hx}{P}$$

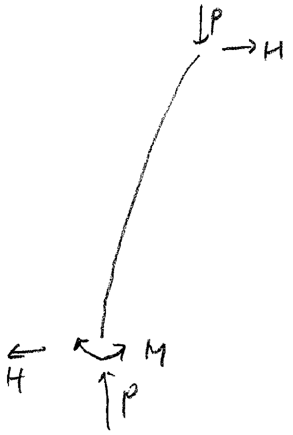
$x = 0,\ v = 0,\ \Rightarrow\ C_2 = 0$

$x = L \quad \frac{dv}{dx} = \theta_a \qquad C_1 = \dfrac{H/P + \theta_a}{k \cos kL}$

$$\therefore\ v = \frac{(H/P + \theta_a)\sin kx}{k \cos kL} - \frac{Hx}{P}$$

$x = L,\ \ v(L) = 0$

$$v(L) = \frac{(H/P + \theta_a)\sin kL}{k\cos kL} - \frac{HL}{P} = \Delta$$



$$M = HL + P\Delta$$

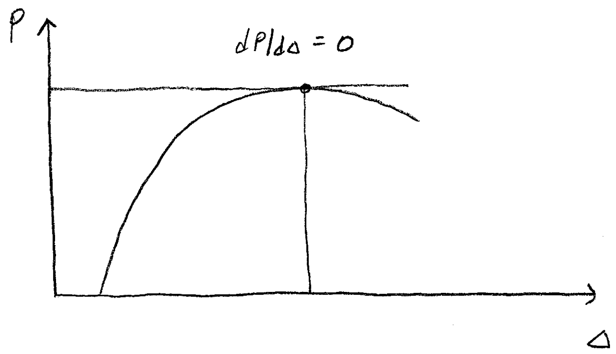
$$\beta\sqrt{\theta_a} = HL + P\cdot\Delta$$

$$= HL + \frac{(H + P\theta_a)\sin kL}{k\cos kL} - \frac{HL}{P} \times P$$

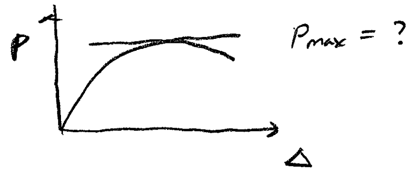
$$\left(\frac{P}{k}\tan kL\right)\theta_a - \beta\sqrt{\theta_a} + HL + \frac{H}{k}\tan kL - \frac{HL}{P}\times P = 0$$

solve θ_a, then $v(x)$

See p 230



8.1.4 A commentary on stability



Principle of stationary total potential energy

$$\pi = U + V$$

π = total potential energy

U = strain energy

V = changes in potential of the applied load

First theorem

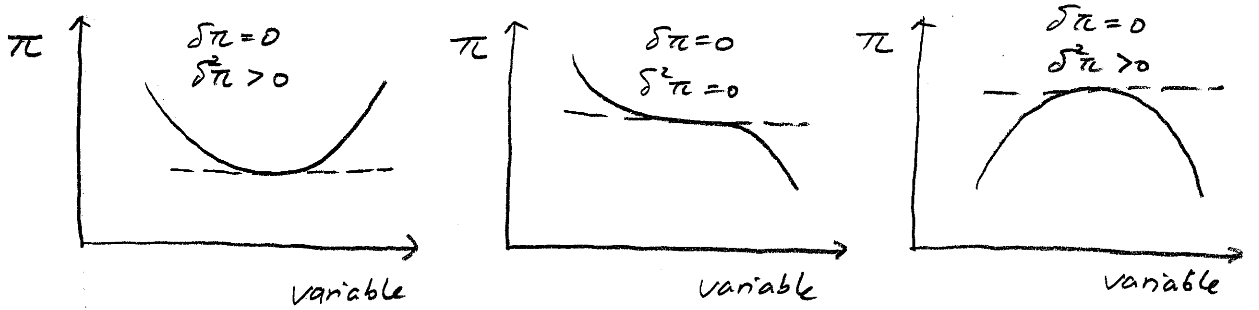
The force-equilibrium that we are seeking can be found under the condition that 1st variation of π becomes zero.

$$\delta\pi = \delta U + \delta V = 0 \Rightarrow \text{produce equilibrium condition}$$

$$\Rightarrow \delta U = -\delta V$$

$$\Rightarrow \delta W_{int} = \delta W_{ext} \quad (= \text{principle of virtual displacement})$$

But, characteristic of the equilibrium has not been determined yet.



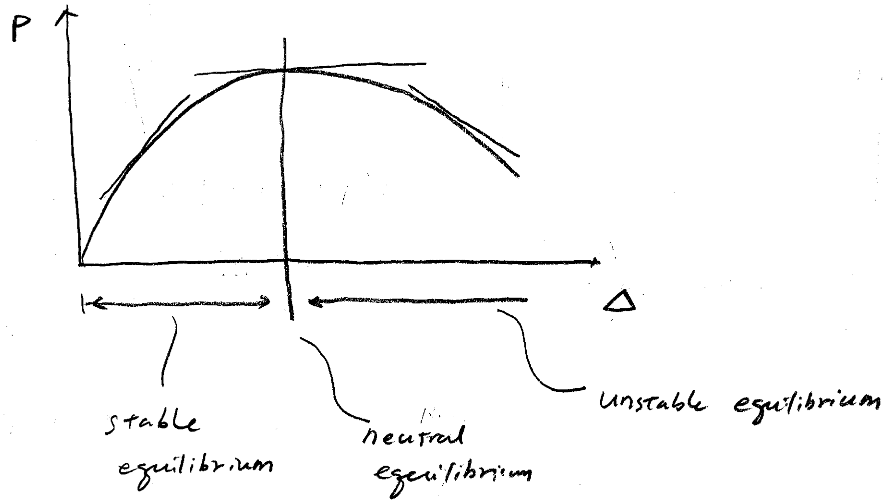
stable equilibrium   neutral equilibrium   unstable equilibrium

Second theorem

$\delta^2 \pi > 0 \Rightarrow$ stable equilibrium

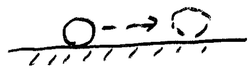
$= 0 \Rightarrow$ neutral equilibrium

$< 0 \Rightarrow$ unstable equilibrium

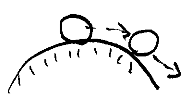


stable equilibrium : with small variations of variables, the energy level should be increased.

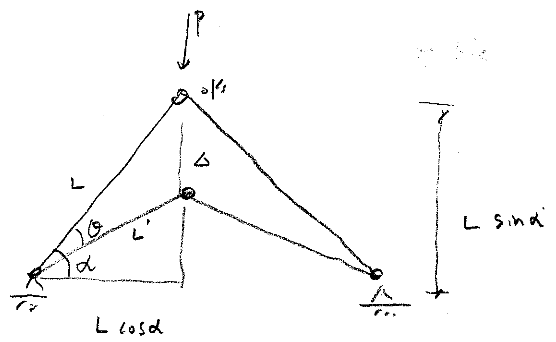
neutral equilibrium : with variations of variables the energy level is not changed



unstable equilibrium : with small changes of variables the energy level is quickly dropped



Example 8.8



$$\begin{cases} L' = L\cos\alpha \times \dfrac{1}{\cos(\alpha-\theta)} \\ L - L' = L\left(1 - \dfrac{\cos\alpha}{\cos(\alpha-\theta)}\right) \end{cases}$$

$$\begin{cases} \Delta = L\sin\alpha - L'\sin(\alpha-\theta) \\ \quad = L\sin\alpha - L\cos\alpha\tan(\alpha-\theta) \end{cases}$$

$$\pi = U + V$$

$$= 2\cdot\left(\frac{1}{2}k(\Delta L_{ab}^2)\right) - P\Delta \qquad k = \frac{EA}{L}$$

$$= kL^2\left[1 - \frac{\cos\alpha}{\cos(\alpha-\theta)}\right]^2 - PL\left[\sin\alpha - \cos\alpha\tan(\alpha-\theta)\right]$$

$$\delta\pi = \frac{d\pi}{d\theta} = 0$$

$$\Rightarrow P = 2kL\left[\sin(\alpha-\theta) - \cos\alpha\tan(\alpha-\theta)\right]$$

$$\delta^2\pi = \frac{dP}{d\theta} = 0$$

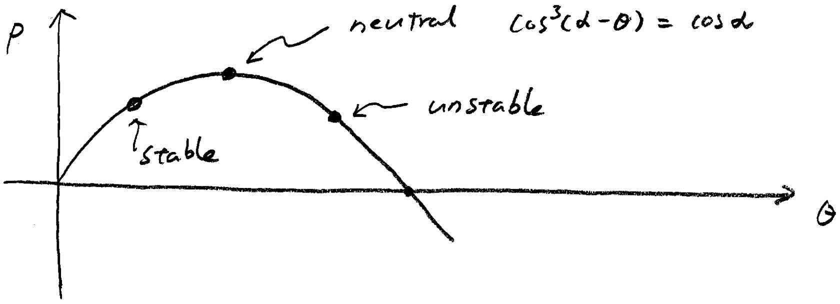
$$\Rightarrow \delta^2\pi = \frac{2kL^2\left[\cos\alpha - \cos^3(\alpha-\theta)\right]}{\cos^4(\alpha-\theta)}$$

for $|\alpha - \theta| < \pi/2$

$$\delta^2 \pi = \frac{d^2\pi}{d\theta^2} > 0 \quad \text{for} \quad \cos^3(\alpha-\theta) < \cos\alpha \quad : \text{stable}$$

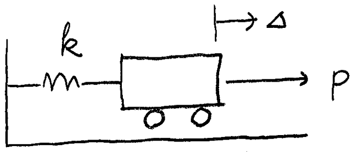
$$\delta^2 \pi = \frac{d^2\pi}{d\theta^2} = 0 \quad \text{for} \quad \cos^3(\alpha-\theta) = \cos\alpha \quad : \text{neutral}$$

$$\delta^2 \pi = \frac{d^2\pi}{d\theta^2} < 0 \quad \text{for} \quad \cos^3(\alpha-\theta) > \cos\alpha \quad : \text{unstable}$$



Conditions of tangent stiffness (k_t) regarding to equilibrium

one-degree of freedom system



the principle of stationary total potential energy is assumed to be applicable to incremental force-displacement relationship

P_t = incremental force

Δ_t = incremental displacement

$$\pi = \frac{1}{2} k_t \Delta_t^2 - P_t \Delta_t$$

$$\delta\pi = k_t \Delta_t \delta\Delta_t - P_t \delta\Delta_t = 0$$

$$\Rightarrow \quad k_t \Delta_t = P_t \quad : \text{equilibrium equation}$$

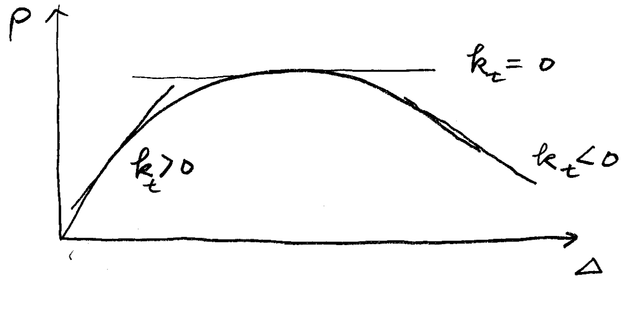
assuming k_t (tangent stiffness) is constant.

$$\delta\pi^2 = k_t \delta\Delta_t \delta\Delta_t$$

$\delta\pi^2 > 0 \Rightarrow k_t > 0$ stable equilibrium

$= 0 \Rightarrow k_t = 0$ neutral equilibrium

$< 0 \Rightarrow k_t < 0$ unstable equilibrium



multi-degree of freedom system

$$\pi = \frac{1}{2} \underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \underset{\sim}{\Delta}_t - \underset{\sim}{\Delta}_t^T \underset{\sim}{P}_t$$

$$\delta \pi = \delta \underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \underset{\sim}{\Delta}_t - \delta \underset{\sim}{\Delta}_t^T \underset{\sim}{P}_t$$

$$\delta \pi = 0 \Rightarrow \underset{\sim}{k}_t \underset{\sim}{\Delta}_t = \underset{\sim}{P}_t$$: equilibrium equation

assuming $\underset{\sim}{k}_t$ = constant

$$\delta^2 \pi = \delta \underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \delta \underset{\sim}{\Delta}_t$$

$$\delta^2 \pi > 0 \Rightarrow \delta \underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \delta \underset{\sim}{\Delta}_t > 0$$ stable equilibrium

$\underset{\sim}{k}$ = symmetric . positive definite

$$\delta^2 \pi = 0 \Rightarrow \delta \underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \delta \underset{\sim}{\Delta}_t = 0$$ neutral equilibrium

$$\underset{\sim}{k}_t \delta \underset{\sim}{\Delta}_t = \underset{\sim}{0} \Rightarrow |\underset{\sim}{k}_t| = 0$$ for any $\delta \underset{\sim}{\Delta}$

eigenvalue problem

$$\delta^2 \pi < 0 \Rightarrow \delta \underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \delta \underset{\sim}{\Delta}_t < 0$$ unstable equilibrium

When the geometry is not significantly changed,

$\underset{\sim}{k}_t \approx \underset{\sim}{k}$, tangent stiffness = secant stiffness

$\underset{\sim}{k} \underset{\sim}{\Delta} = \underset{\sim}{P}$

$|\underset{\sim}{k}| = 0 \Rightarrow$ critical load analysis