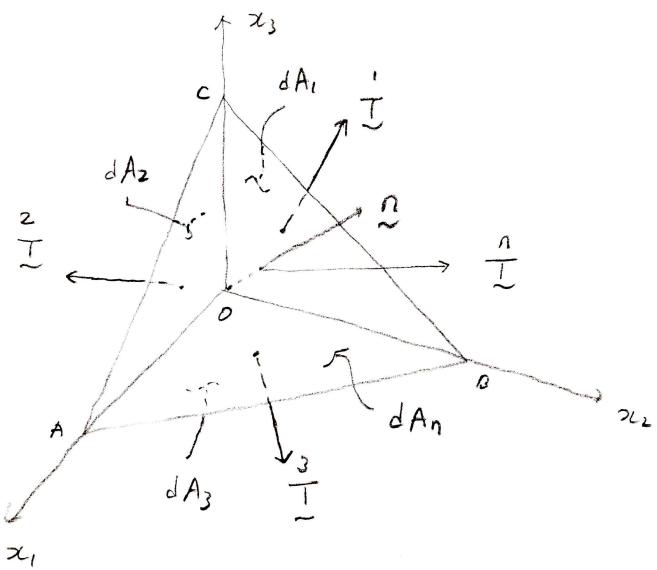
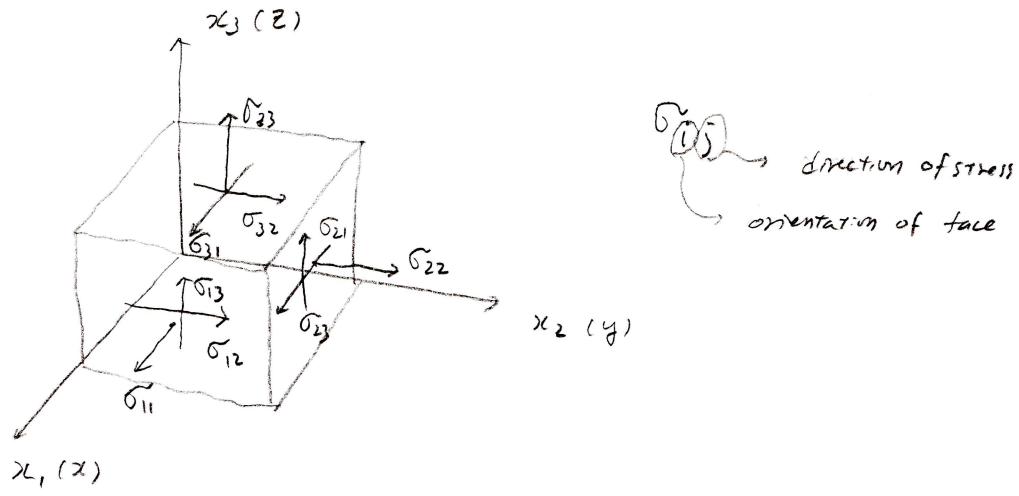


Chapter 4. General Solids



$$\underline{n} = n_1 \underline{i} + n_2 \underline{j} + n_3 \underline{k}$$

$$\underline{n}_1 = \underline{i}$$

$$\underline{n}_2 = \underline{j}$$

$$\underline{n}_3 = \underline{k}$$

$$\underline{T} = \sigma_{11} \underline{i} + \sigma_{12} \underline{j} + \sigma_{13} \underline{k}$$

\underline{T} : surface traction

$$\underline{T} = \sigma_{21} \underline{i} + \sigma_{22} \underline{j} + \sigma_{23} \underline{k}$$

$$\underline{T} = \sigma_{31} \underline{i} + \sigma_{32} \underline{j} + \sigma_{33} \underline{k}$$

$$\underline{T} = \sigma_{ij}$$

$$\text{Area of } ABC = dA_n$$

$$OBC = dA_1 = dA_n \cdot \underline{\Omega} \circ \underline{e}_1 = dA_n n_1$$

$$OCA = dA_2 = dA_n \cdot \underline{\Omega} \circ \underline{e}_2 = dA_n n_2$$

$$OAB = dA_3 = dA_n \cdot \underline{\Omega} \circ \underline{e}_3 = dA_n n_3$$

or

$$\frac{dA_1}{dA_n} = n_1, \quad \frac{dA_2}{dA_n} = n_2, \quad \frac{dA_3}{dA_n} = n_3$$

Force Equilibrium

$$\frac{n}{\underline{\Omega}} dA_n = \frac{1}{\underline{\Omega}} dA_1 + \frac{2}{\underline{\Omega}} dA_2 + \frac{3}{\underline{\Omega}} dA_3$$

$$\begin{aligned} \frac{n}{\underline{\Omega}} &= \frac{1}{\underline{\Omega}} \frac{dA_1}{dA_n} + \frac{2}{\underline{\Omega}} \frac{dA_2}{dA_n} + \frac{3}{\underline{\Omega}} \frac{dA_3}{dA_n} \\ &= \frac{1}{\underline{\Omega}} n_1 + \frac{2}{\underline{\Omega}} n_2 + \frac{3}{\underline{\Omega}} n_3 \\ &= \frac{1}{\underline{\Omega}} n_j \end{aligned}$$

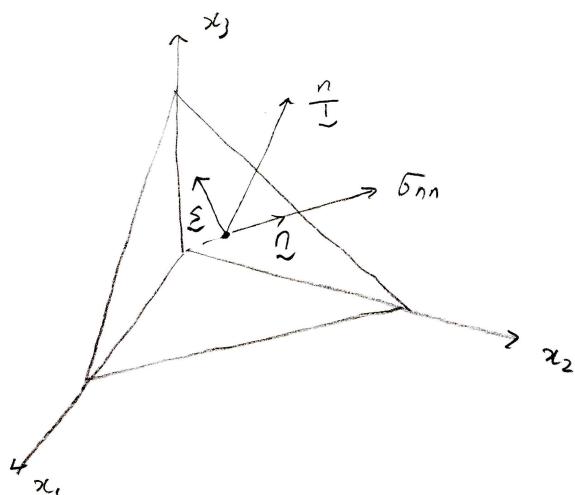
or

$$\begin{aligned} \frac{n}{T_i} dA_n &= \frac{1}{T_i} dA_1 + \frac{2}{T_i} dA_2 + \frac{3}{T_i} dA_3 \\ &= \sigma_{1i} dA_1 + \sigma_{2i} dA_2 + \sigma_{3i} dA_3 \\ \frac{n}{T_i} &= \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3 \\ &= \sigma_{ji} n_j \end{aligned}$$

$$\begin{aligned} \text{Generally } \frac{n}{T_i} &= \delta_{ji} n_j \quad (\text{Cauchy's formula}) \\ &= \sigma_{ij} n_j \quad (\sigma_{ij} = \delta_{ji}) \end{aligned}$$

$$\begin{cases} \frac{n}{T_1} = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\ \frac{n}{T_2} = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\ \frac{n}{T_3} = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \end{cases}$$

$$\begin{aligned} \frac{n}{\tilde{T}} &= \begin{bmatrix} \frac{n}{T_1} \\ \frac{n}{T_2} \\ \frac{n}{T_3} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \end{aligned}$$



normal stress

$$\sigma_{nn} = \underline{\tau} \cdot \underline{n} = \underline{n} \cdot \frac{\underline{\sigma}}{\underline{\tau}} = \underline{n} \cdot (\sigma_i n_j)$$

$$= [n_1 \ n_2 \ n_3] \begin{bmatrix} \frac{n}{\tau_1} \\ \frac{n}{\tau_2} \\ \frac{n}{\tau_3} \end{bmatrix}$$

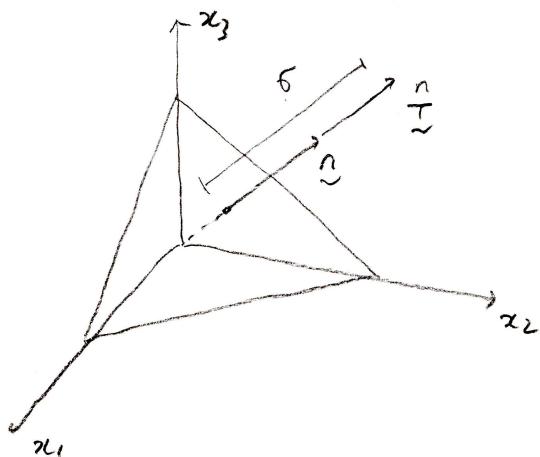
$$= [n_1 \ n_2 \ n_3] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

shear stress

$$\sigma_{ns} = \underline{\tau} \cdot \underline{s}$$

$$= [s_1 \ s_2 \ s_3] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

principal stress



$$\frac{n}{T_i} = \sigma_{ij} n_j \quad \left[\begin{array}{l} \\ \end{array} \right] \rightarrow \sigma n_i = \sigma_{ij} n_j$$

$$\left[\begin{array}{l} \sigma n_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ \sigma n_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ \sigma n_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{array} \right]$$

$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\sigma \underline{\underline{I}} \underline{\underline{n}} = \underline{\underline{\sigma}} \underline{\underline{n}}$$

$$[\underline{\underline{\sigma}} - \sigma \underline{\underline{I}}] \underline{\underline{n}} = \underline{\underline{0}}$$

$$\text{or } (\sigma_{ij} - \sigma \delta_{ij}) n_j = 0$$

for non-trivial solution,

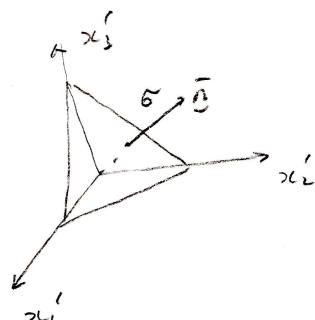
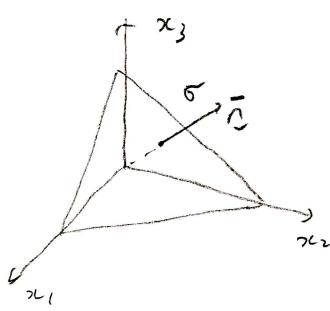
$$|\underline{\underline{\sigma}} - \sigma \underline{\underline{I}}| = 0 \Rightarrow \text{Eigenvalue Problem}$$

$$\sigma^3 + I_1 \sigma^2 + I_2 \sigma + I_3 = 0$$

$\sigma_1, \sigma_2, \sigma_3$ = principal stresses

$$I_1, I_2, I_3 = \text{stress invariants}$$

$\bar{x}_1, \bar{x}_2, \bar{x}_3$ = P.S axes



same principal stresses

$\Rightarrow I_1, I_2, I_3$ = invariants

= invariants

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{KK}$$

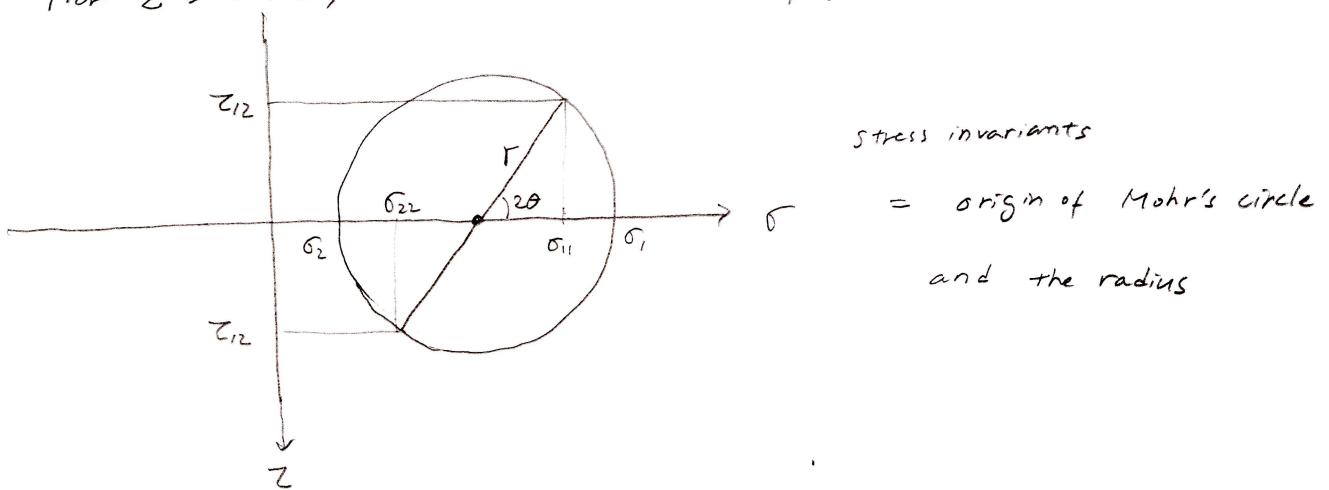
$$\begin{aligned} I_2 &= \left| \begin{array}{cc} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{array} \right| + \left| \begin{array}{cc} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{array} \right| + \left| \begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{array} \right| \\ &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\ &= \frac{1}{2} (\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \end{aligned}$$

$$I_3 = \det(\sigma_{ij})$$

$$= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{31}^2\sigma_{22}$$

For 2-D cases,

Mohr's circle



$$\text{stress invariants} : \frac{1}{2} (\sigma_{11} + \sigma_{22}) \text{ or } \frac{1}{2} [\sigma_1 + \sigma_2]$$

$$r^2 = \left[\frac{1}{2} (\sigma_{11} - \sigma_{22}) \right]^2 + \tau_{12}^2$$

$$= \frac{1}{4} [\sigma_{11}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2] + \tau_{12}^2$$

$$= \frac{1}{4} (\sigma_{11} + \sigma_{22})^2 - \sigma_{11}\sigma_{22} + \tau_{12}^2$$

$$= \frac{1}{4} I_1^2 - I_2$$

Material Strength Criteria

* J_2 theory - von Mises criterion (For metals)

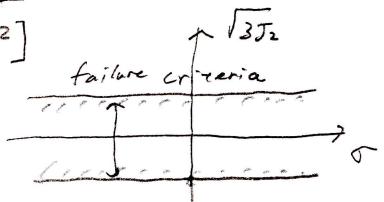
$$\sigma^3 + J_1 \sigma^2 + J_2 \sigma + J_3 = 0 \quad \sigma = \sqrt{\frac{1}{3} I_1}, \quad J_1 = 0$$

$$\sqrt{3J_2} = A_1, \quad A_1 \text{ is determined from material tests}$$

$$\sqrt{3J_2} = \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 6(\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2)]}$$

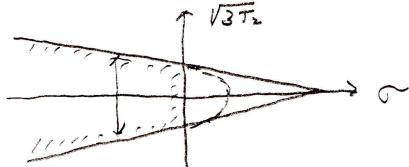
$$= \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}$$

$$\text{For 1-D case, } \sqrt{3J_2} = \sigma_1 = \sigma_y$$



* Elastic-hardening plasticity model (For concrete)

$$A_1 \sigma + A_2 \sqrt{3J_2} + A_3 = 0 \quad \sigma = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) / I_1$$



Why are the existing material failure criteria

defined with the stress-invariants?

⇒ For isotropic materials and similarities,

the material failure criteria should not be dependent on the definition of coordinate system.

$$|\underline{\sigma} - \sigma \underline{I}| = 0$$

solutions = $\sigma_1, \sigma_2, \sigma_3$ Eigenvalues \Rightarrow principal stresses

corresponding vectors $\underline{\hat{n}}_1, \underline{\hat{n}}_2, \underline{\hat{n}}_3$ Eigenvalues
 \Rightarrow orientations of principal axes

$$\underline{\hat{N}} = [\underline{\hat{n}}_1 \underline{\hat{n}}_2 \underline{\hat{n}}_3]$$

$$= \begin{bmatrix} \bar{n}_{11} & \bar{n}_{21} & \bar{n}_{31} \\ \bar{n}_{12} & \bar{n}_{22} & \bar{n}_{32} \\ \bar{n}_{13} & \bar{n}_{23} & \bar{n}_{33} \end{bmatrix}$$

$$\sigma_1 \underline{\hat{n}}_1 = \underline{\sigma} \underline{\hat{n}}_1 \quad \sigma_1 \underline{\hat{n}}_1^T \underline{\hat{n}}_1 = \underline{\hat{n}}_1^T \underline{\sigma} \underline{\hat{n}}_1$$

$$\sigma_2 \underline{\hat{n}}_2 = \underline{\sigma} \underline{\hat{n}}_2 \quad \underline{\hat{n}}_2^T \underline{\hat{n}}_1 = 1$$

$$\sigma_3 \underline{\hat{n}}_3 = \underline{\sigma} \underline{\hat{n}}_3 \quad \underline{\hat{n}}_3^T \underline{\hat{n}}_1 = 0$$

$$\left\{ \begin{array}{l} \sigma_1 = \underline{\hat{n}}_1^T \underline{\sigma} \underline{\hat{N}} \quad \underline{\hat{n}}_1^T \underline{\hat{n}}_2 = 0 \\ \sigma_2 = \underline{\hat{n}}_2^T \underline{\sigma} \underline{\hat{N}} \\ \sigma_3 = \underline{\hat{n}}_3^T \underline{\sigma} \underline{\hat{N}} \end{array} \right.$$

$$\underline{\Sigma} = \underline{\hat{N}}^T \underline{\sigma} \underline{\hat{N}}$$

$\underline{\hat{N}}$: orthogonal matrix

$$\underline{\hat{N}}^T \underline{\hat{N}} = \underline{\hat{I}}$$

Orthogonality

$$\underline{\Sigma} \underline{n}_i = \sigma_i \underline{\bar{n}}_i - \textcircled{1} \quad (i \neq j)$$

$$\underline{\Sigma} \underline{n}_j = \sigma_j \underline{\bar{n}}_j - \textcircled{2}$$

$$\textcircled{1} - \underline{\bar{n}}_j^T \underline{\Sigma} \underline{\bar{n}}_i = \sigma_i \underline{\bar{n}}_j^T \underline{\bar{n}}_i$$

$$\textcircled{2} - \underline{\bar{n}}_i^T \underline{\Sigma} \underline{\bar{n}}_j = \sigma_j \underline{\bar{n}}_i^T \underline{\bar{n}}_j - \textcircled{2}'$$

$$\textcircled{2}' - \underline{\bar{n}}_j^T \underline{\Sigma} \underline{\bar{n}}_i = \sigma_j \underline{\bar{n}}_j^T \underline{\bar{n}}_i$$

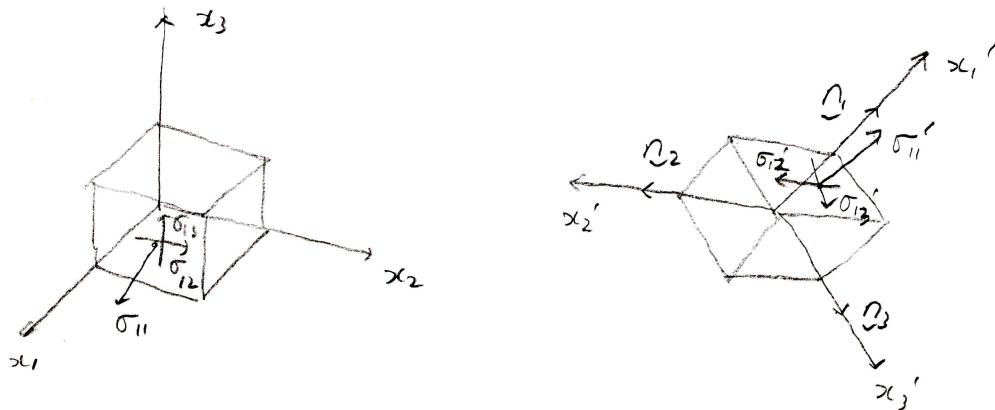
from $\textcircled{1}$ and $\textcircled{2}'$, $\sigma_i \underline{\bar{n}}_j^T \underline{\bar{n}}_i = \sigma_j \underline{\bar{n}}_j^T \underline{\bar{n}}_i$

$$(\sigma_i - \sigma_j) \underline{\bar{n}}_j^T \underline{\bar{n}}_i = 0$$

since $\sigma_i \neq \sigma_j$, $\underline{\bar{n}}_j^T \underline{\bar{n}}_i = 0 \quad (i \neq j)$

$\Rightarrow \underline{\bar{n}}_j \perp \underline{\bar{n}}_i \quad (\text{orthogonality})$

Axis - transformation of stresses



$$\underline{n}_1 = \begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix} \quad \underline{n}_2 = \begin{bmatrix} n_{21} \\ n_{22} \\ n_{23} \end{bmatrix} \quad \underline{n}_3 = \begin{bmatrix} n_{31} \\ n_{32} \\ n_{33} \end{bmatrix}$$

$$\begin{aligned} \tilde{\sigma}_{11}' &= \tilde{\sigma}_{n_1 n_1} = \underbrace{[n_{11} \ n_{12} \ n_{13}]}_{\underline{n}_1^T} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \underbrace{\begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix}}_{\underline{n}_1} \\ &= n_{11} \sigma_{11} + n_{12} \sigma_{12} + n_{13} \sigma_{13} \end{aligned}$$

$$\begin{aligned} \tilde{\sigma}_{12}' &= \tilde{\sigma}_{n_1 n_2} = \underbrace{[n_{21} \ n_{22} \ n_{23}]}_{(\underline{n}_2)^T} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \underbrace{\begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix}}_{\underline{n}_1} \\ &= n_{21} \sigma_{11} + n_{22} \sigma_{12} + n_{23} \sigma_{13} \end{aligned}$$

$$\begin{aligned} \text{Generally } \tilde{\sigma}_{ji}' &= n_{ij} \sigma_{kk} n_{jh} \\ &= \tilde{\sigma}_{ij}' \end{aligned}$$

$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} = \underbrace{\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}}_{\underline{\Sigma}} \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}_{\underline{\Sigma}} \underbrace{\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{12} & n_{22} & n_{23} \\ n_{13} & n_{23} & n_{33} \end{bmatrix}}_{\underline{R}}$$

$$\underline{\Sigma}' = \underline{R}^T \underline{\Sigma} \underline{R}$$

In text book, $\underline{R} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

$$\underline{\Sigma}' = \underline{T}_{\underline{\sigma}} \underline{\Sigma}$$

$$\begin{bmatrix} \sigma_{11}' \\ \sigma_{21}' \\ \sigma_{31}' \\ \sigma_{12}' \\ \sigma_{23}' \\ \sigma_{31}' \end{bmatrix} = \begin{bmatrix} l_1^2 m_1^2 n_1^2 & 2l_1 m_1 & 2m_1 n_1 & 2n_1 l_1 \\ l_2^2 m_2^2 n_2^2 & 2l_2 m_2 & 2m_2 n_2 & 2n_2 l_2 \\ l_3^2 m_3^2 n_3^2 & 2l_3 m_3 & 2m_3 n_3 & 2n_3 l_3 \\ l_1 l_2 m_1 m_2 n_1 n_2 & (l_1 m_2 + l_2 m_1) & (m_1 n_2 + m_2 n_1) & (n_1 l_2 + n_2 l_1) \\ l_2 l_3 m_2 m_3 n_2 n_3 & (l_2 m_3 + l_3 m_2) & (m_2 n_3 + m_3 n_2) & (n_2 l_3 + n_3 l_2) \\ l_3 l_1 m_3 m_1 n_3 n_1 & (l_3 m_1 + l_1 m_3) & (m_3 n_1 + m_1 n_3) & (n_3 l_1 + n_1 l_3) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\underline{\underline{\Sigma}} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} \quad \text{where } \Sigma_{12} = \frac{1}{2} \gamma_{12}, \Sigma_{13} = \frac{1}{2} \gamma_{13}, \dots$$

$$\underline{\underline{\Sigma}}' = \underbrace{R^T}_{3 \times 3} \underline{\underline{\Sigma}} R$$

$$\text{also, } \underline{\underline{\Sigma}}' = \underbrace{I_6}_{6 \times 1} \underline{\underline{\Sigma}}$$

$$\text{When we use, } \underline{\underline{\Sigma}} = [\Sigma_{11} \quad \Sigma_{22} \quad \Sigma_{33} \quad \gamma_{12} \quad \gamma_{23} \quad \gamma_{31}]^T$$

$$\underline{\underline{\Sigma}}' = \underbrace{I_6}_{\text{sym}} \underline{\underline{\Sigma}} \quad \underbrace{I_6}_{\text{not sym}} \neq \underbrace{I_6}_{\text{not sym}}$$

using virtual strain energy density

$$(\delta \underline{\underline{\Sigma}}')^T \underline{\underline{\Omega}}' = \delta \underline{\underline{\Sigma}}^T \underline{\underline{\Omega}}$$

$$\delta \underline{\underline{\Sigma}}^T \underbrace{\underline{\underline{\Omega}}^T}_{\text{sym}} \underline{\underline{\Omega}}' = \delta \underline{\underline{\Sigma}}^T \underline{\underline{\Omega}}$$

$$\underbrace{\underline{\underline{\Omega}}^T}_{\text{sym}} \underline{\underline{\Omega}}' = \underline{\underline{\Omega}}$$

$$\underbrace{\underline{\underline{\Omega}}}_{\text{sym}} = \underbrace{\underline{\underline{\Omega}}^{-T}}_{\text{sym}} \quad \text{or} \quad \underline{\underline{\Omega}}^{-1} = \underbrace{\underline{\underline{\Omega}}^T}_{\text{sym}}$$

$A \times 3$ - transformation of constitutive matrix $\underline{\underline{\Sigma}}$

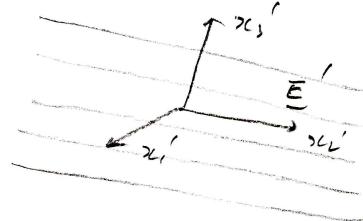
$$\underline{\underline{\Sigma}}' = \underline{\underline{E}}' \underline{\underline{\Sigma}}'$$

$$\underline{T}_\varepsilon \underline{\underline{\Sigma}} = \underline{\underline{E}}' \underline{T}_\varepsilon \underline{\underline{\Sigma}}$$

$$\underline{\underline{\Sigma}} = \underline{T}_\varepsilon^{-1} \underline{\underline{E}}' \underline{T}_\varepsilon \underline{\underline{\Sigma}}$$

$$= \underline{T}_\varepsilon^T \underline{\underline{E}}' \underline{T}_\varepsilon \underline{\underline{\Sigma}}$$

$$= \underline{\underline{E}} \underline{\underline{\Sigma}}$$



$$\text{Therefore, } \underline{\underline{E}} = \underline{T}_\varepsilon^T \underline{\underline{E}}' \underline{T}_\varepsilon$$

$$\text{However, for isotropic material, } \underline{\underline{E}} = \underline{\underline{E}}'$$

Stress - Strain Relationship

$$\underline{\underline{\Sigma}} = \underline{\underline{E}} \underline{\underline{\Sigma}}$$

for isotropic material

$$\underline{\underline{E}} = \frac{E}{(1+\nu)e_2} \begin{bmatrix} e_1 & \nu & \nu \\ e_1 & e_1 & \nu \\ e_1 & \nu & e_1 \\ \text{sym.} & & e_3 \\ & & e_3 \\ & & e_3 \end{bmatrix}$$

$$e_1 = 1 - \nu, \quad e_2 = 1 - 2\nu, \quad e_3 = \frac{e_2}{2}$$

$\underline{\underline{E}}$ can be defined with 2 independent constants E and ν

why?

Anisotropic Material

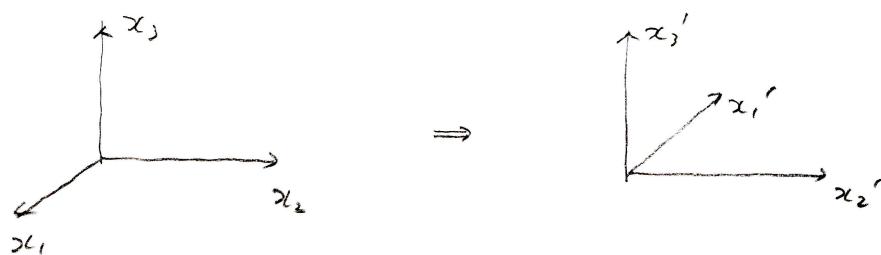
$$\sigma_i = C_{ij} \epsilon_j$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

Sym.

21 constants

Elastic symmetry with respect to one plane (Monoclinic Material)



Transformation matrix

$$\underline{R} = \begin{bmatrix} \eta_{11} & \eta_{21} & \eta_{31} \\ \eta_{12} & \eta_{22} & \eta_{32} \\ \eta_{13} & \eta_{23} & \eta_{33} \end{bmatrix} = \begin{bmatrix} x'_1 & x'_2 & x'_3 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

{ seek condition of $\underline{\sigma}$ to satisfy $\underline{\sigma} = \underline{\sigma} \underline{\epsilon}$ and $\underline{\sigma}' = \underline{\sigma}' \underline{\epsilon}'$, }
 where $\underline{\sigma}' = \underline{R}^T \underline{\sigma} \underline{R}$ and $\underline{\epsilon}' = \underline{R}^T \underline{\epsilon} \underline{R}$.

$$\underline{\sigma}' = \underline{R}^T \underline{\sigma} \underline{R}$$

$$\Rightarrow \sigma'_{11} = \sigma_{11}, \quad \sigma'_{22} = \sigma_{22}, \quad \sigma'_{33} = \sigma_{33}$$

$$\sigma'_{12} = -\sigma_{12}, \quad \sigma'_{23} = \sigma_{23}, \quad \sigma'_{31} = -\sigma_{31}$$

Also $\underline{\epsilon}' = \underline{R}^T \underline{\epsilon} \underline{R}$

$$\Rightarrow \epsilon'_{11} = \epsilon_{11}, \quad \epsilon'_{22} = \epsilon_{22}, \quad \epsilon'_{33} = \epsilon_{33}$$

$$\epsilon'_{12} = -\epsilon_{12}, \quad \epsilon'_{23} = \epsilon_{23}, \quad \epsilon'_{31} = -\epsilon_{31}$$

from $\underline{\sigma} = \underline{C} \underline{\epsilon}$

$$\sigma_{11} = C_{11} \epsilon_{11} + C_{12} \epsilon_{22} + C_{13} \epsilon_{33} + C_{14} \epsilon_{12} + C_{15} \epsilon_{23} + C_{16} \epsilon_{31}$$

from $\underline{\sigma}' = \underline{C}' \underline{\epsilon}'$

$$\sigma'_{11} = C_{11}' \epsilon'_{11} + C_{12}' \epsilon'_{22} + C_{13}' \epsilon'_{33} + C_{14}' \epsilon'_{12} + C_{15}' \epsilon'_{23} + C_{16}' \epsilon'_{31}$$

$$= C_{11} \epsilon_{11} + C_{12} \epsilon_{22} + C_{13} \epsilon_{33} - C_{14} \epsilon_{12} + C_{15} \epsilon_{23} - C_{16} \epsilon_{31}$$

from $\sigma_{11} = \sigma'_{11}$, $C_{14} = C_{16} = 0$

Similarly, $C_{24} = C_{26} = C_{34} = C_{36} = C_{45} = C_{56} = 0$

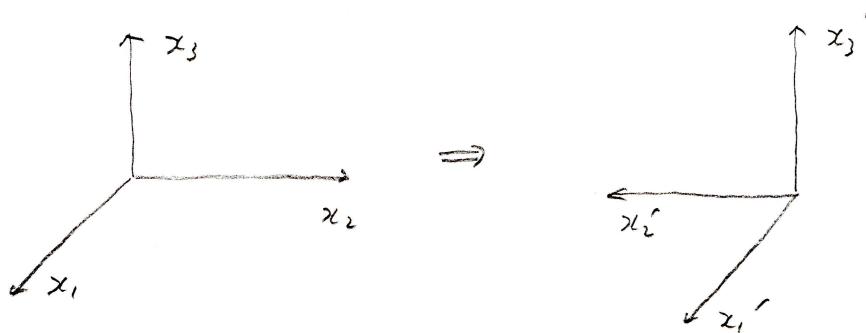
Hence,

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{21} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{31} & C_{32} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{45} \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

13 independent
constants
for monoclinic
material

sym.

Elastic Symmetry w.r.t. two orthogonal planes
(Orthotropic material)



$$\underline{R} = \begin{bmatrix} x_1' & x_2' & x_3' \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

from $\underline{\epsilon}' = \underline{\epsilon}$, $\sigma_{ii}' = \sigma_{ii}$, $\sigma_{12}' = -\sigma_{12}$,
 $\sigma_{23}' = -\sigma_{23}$, $\sigma_{31}' = \sigma_{31}$

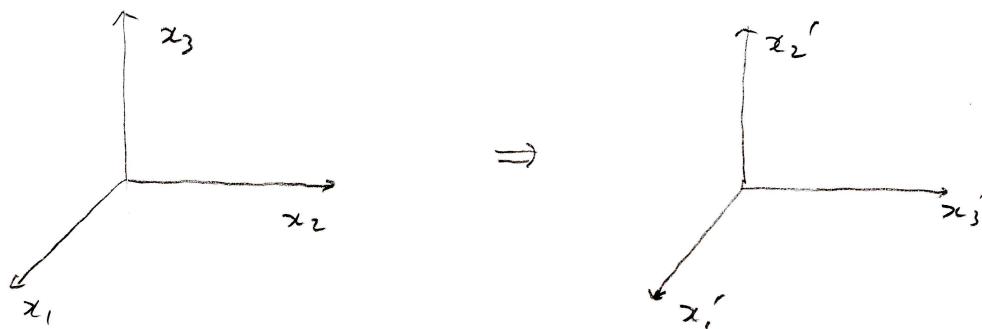
$$\epsilon_{ij} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$

sym.

9 constants
for orthotropic
material

No coupling between (normal stress - shear strains
normal strain - shear stresses)

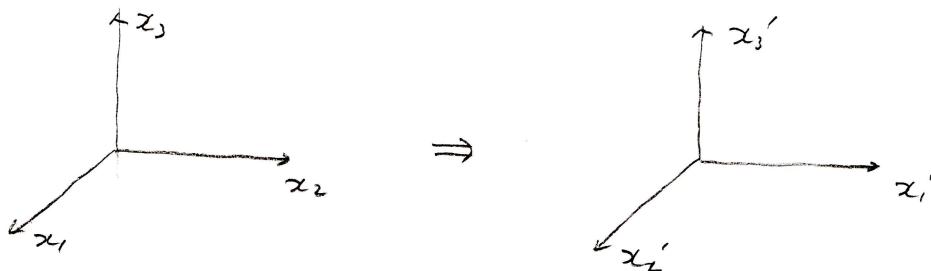
Directional Independence (Interchange Axes)
(Cubic material)



$$R = \begin{bmatrix} x_1' & x_2' & x_3' \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

from $\underline{\underline{C}}' = \underline{\underline{C}}$, $\sigma_{11}' = \sigma_{11}$, $\sigma_{22}' = \sigma_{33}$, $\sigma_{33}' = \sigma_{22}$
 $\sigma_{12}' = \sigma_{13}$, $\sigma_{23}' = \sigma_{32}$, $\sigma_{13}' = \sigma_{12}$

$\Rightarrow C_{22} = C_{33}$, $C_{44} = C_{66}$, $C_{12} = C_{13}$,



$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

from $\underline{\underline{C}}' = \underline{\underline{C}}$, $\sigma_{11}' = \sigma_{33}$, $\sigma_{22}' = \sigma_{22}$, $\sigma_{33}' = \sigma_{11}$
 $\sigma_{12}' = \sigma_{12}$, $\sigma_{23}' = \sigma_{31}$, $\sigma_{13}' = \sigma_{32}$

$\Rightarrow C_{11} = C_{33}$, $C_{55} = C_{66}$, $C_{12} = C_{23}$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{11} & 0 & 0 & 0 & 0 & 0 \\ C_{44} & 0 & 0 & 0 & 0 & 0 \\ \text{Sym.} & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \end{bmatrix}$$

3 independent constants
for a cubic material

Rotational Independence (Isotropic material)

$$\underline{R} = \begin{bmatrix} x_1' & x_2' & x_3' \\ 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

from $\underline{\Sigma}' = \underline{\Sigma}$, $\underline{\Omega} = \underline{R}^T \underline{\Omega}' \underline{R}$, $\underline{\varepsilon} = \underline{R}^T \underline{\varepsilon}' \underline{R}$
 $\underline{\Omega} = \underline{\Sigma} \underline{\varepsilon}$, $\underline{\Omega}' = \underline{\Sigma}' \underline{\varepsilon}'$

$$\Rightarrow C_{11} - C_{12} = C_{44}$$

$$C_{ij} = \begin{bmatrix} C_{12} + C_{44} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} + C_{44} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} + C_{44} & 0 & 0 & 0 & 0 & 0 \\ C_{44} & 0 & 0 & 0 & 0 & 0 \\ \text{Sym.} & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \end{bmatrix}$$

set $C_{12} \rightarrow \lambda$, $C_{44} \rightarrow 2\mu$

$$C_{ij} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{bmatrix}$$

λ
 2μ
 2μ

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \mu = G = \frac{E}{2(1+\nu)}$$

λ, μ = Lame' constants

$$\sigma_{ij} = \lambda \delta_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

If we define the strains as

$$\underline{\epsilon} = \underline{\Sigma} \underline{\epsilon}$$

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

$$\underline{\Sigma} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & \mu & \mu \\ 0 & 0 & 0 & \mu & \mu & \mu \\ 0 & 0 & 0 & \mu & \mu & \mu \end{bmatrix}$$

Strain - Displacement Relationship

$$\underline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \underline{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

$$\epsilon_{11} = \frac{\partial u}{\partial x}, \quad \epsilon_{22} = \frac{\partial v}{\partial y}, \quad \epsilon_{33} = \frac{\partial w}{\partial z}$$

$$\gamma_{12} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad \gamma_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{23} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \quad \gamma_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

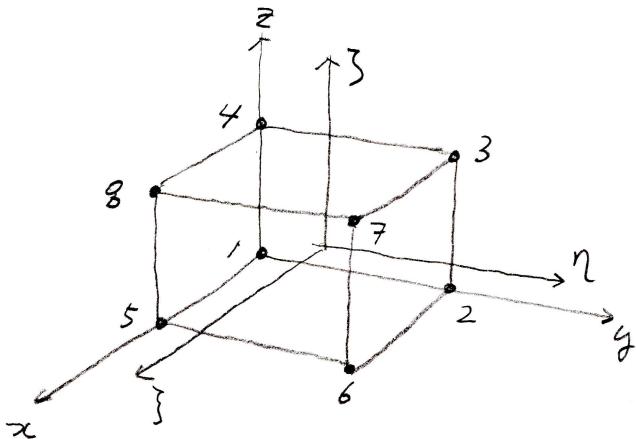
$$\gamma_{31} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \quad \gamma_{31} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\underline{\epsilon} = \underline{d} \underline{u}$$

General Solids

8-node Solid Element (Isoparametric Element)



$$\left\{ \begin{array}{l} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 + \dots + f_8 u_8 = \sum_i f_i u_i \\ v = \sum_i f_i v_i \\ w = \sum_i f_i w_i \\ x = \sum_i f_i x_i \\ y = \sum_i f_i y_i \\ z = \sum_i f_i z_i \end{array} \right.$$

$$\text{node 1 } (-1 -1 -1) \quad f_1 = \frac{1}{8} (1-\xi)(1-\eta)(1-\zeta)$$

$$\text{node 2 } (-1 1 -1) \quad f_2 = \frac{1}{8} (1-\xi)(1+\eta)(1-\zeta)$$

$$\text{node 3 } (1 1 1) \quad f_3 = \frac{1}{8} (1-\xi)(1+\eta)(1+\zeta)$$

$$\vdots$$

$$\text{node 8 } (1 -1 1) \quad f_8 = \frac{1}{8} (1+\xi)(1-\eta)(1+\zeta)$$

$$\Rightarrow f_i = \frac{1}{8} (1+\xi_i)(1+\eta_i)(1+\zeta_i)$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f_1 & 0 & 0 \\ 0 & f_1 & 0 \\ 0 & 0 & f_1 \end{bmatrix} \begin{bmatrix} f_2 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_2 \end{bmatrix} \cdots \begin{bmatrix} f_8 & 0 & 0 \\ 0 & f_8 & 0 \\ 0 & 0 & f_8 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_8 \\ v_8 \\ w_8 \end{bmatrix}$$

$$u = f \frac{1}{8}$$

$$\Sigma = \underline{\beta} \underline{g}$$

$$\underline{\beta} = \underline{d} f$$

$(6 \times 24) (6 \times 3) (3 \times 24)$

$$\underline{\beta} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & 0 & 0 & | & \frac{\partial f_8}{\partial x} & 0 & 0 \\ 0 & \frac{\partial f_1}{\partial y} & 0 & | & 0 & \frac{\partial f_8}{\partial y} & 0 \\ 0 & 0 & \frac{\partial f_1}{\partial z} & | & - & 0 & 0 & \frac{\partial f_8}{\partial z} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & 0 & | & \frac{\partial f_8}{\partial y} & \frac{\partial f_8}{\partial z} & 0 \\ 0 & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial y} & | & 0 & \frac{\partial f_8}{\partial z} & \frac{\partial f_8}{\partial y} \\ \frac{\partial f_1}{\partial z} & 0 & \frac{\partial f_1}{\partial y} & | & \frac{\partial f_8}{\partial z} & 0 & \frac{\partial f_8}{\partial x} \end{bmatrix}$$

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \gamma} & \frac{\partial y}{\partial \gamma} & \frac{\partial z}{\partial \gamma} \end{bmatrix}$$

$$\frac{\partial x}{\partial \beta} = \sum_i \frac{\partial f_i}{\partial \beta} x_i \quad \frac{\partial x}{\partial \eta} = \sum \frac{\partial f_i}{\partial \eta} x_i \quad \frac{\partial x}{\partial \gamma} = \sum \frac{\partial f_i}{\partial \gamma} x_i$$

$$\frac{\partial y}{\partial \beta} = \sum \frac{\partial f_i}{\partial \beta} y_i \quad \frac{\partial y}{\partial \eta} = \sum \frac{\partial f_i}{\partial \eta} y_i \quad \frac{\partial y}{\partial \gamma} = \sum \frac{\partial f_i}{\partial \gamma} y_i$$

$$\frac{\partial z}{\partial \beta} = \sum \frac{\partial f_i}{\partial \beta} z_i \quad \frac{\partial z}{\partial \eta} = \sum \frac{\partial f_i}{\partial \eta} z_i \quad \frac{\partial z}{\partial \gamma} = \sum \frac{\partial f_i}{\partial \gamma} z_i$$

$$\underline{J}^* = \underline{J}^{-1} = \begin{bmatrix} \frac{\partial \underline{\eta}}{\partial x} & \frac{\partial \underline{\eta}}{\partial x} & \frac{\partial \underline{\eta}}{\partial x} \\ \frac{\partial \underline{\eta}}{\partial y} & \frac{\partial \underline{\eta}}{\partial y} & \frac{\partial \underline{\eta}}{\partial y} \\ \frac{\partial \underline{\eta}}{\partial z} & \frac{\partial \underline{\eta}}{\partial z} & \frac{\partial \underline{\eta}}{\partial z} \end{bmatrix}$$

$$\frac{\partial f_i}{\partial x} = \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial x} + \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial x} + \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial x}$$

$$\frac{\partial f_i}{\partial y} = \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial y} + \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial y} + \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial y}$$

$$\frac{\partial f_i}{\partial z} = \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial z} + \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial z} + \frac{\partial f_i}{\partial \underline{\eta}} \frac{\partial \underline{\eta}}{\partial z}$$

$$\underline{K} = \int \underline{B}^T \underline{E} \underline{B} dV$$

$$= \iiint_{-1}^1 \underline{B}^T \underline{E} \underline{B} |\underline{J}| d\underline{\eta} d\underline{\eta} d\underline{\eta}$$

using Gaussian Quadrature

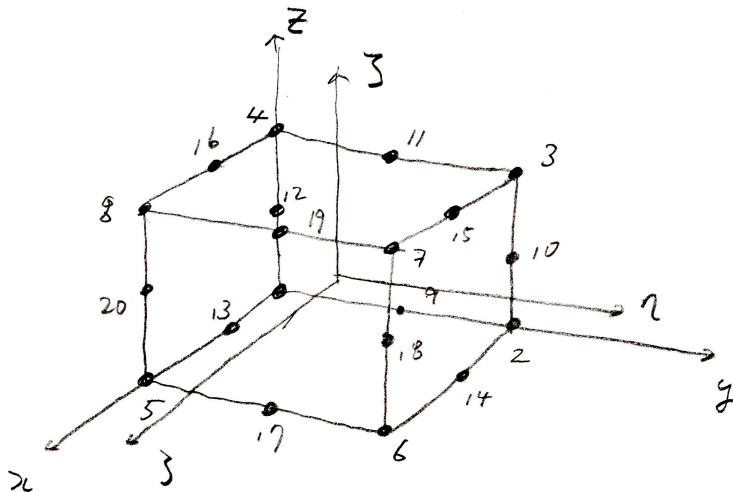
$$\underline{K} = \sum_i^l \sum_j^m \sum_k^n R_i R_j R_k f(\underline{\eta}_i, \underline{\eta}_j, \underline{\eta}_k)$$

Equivalent nodal forces

$$\underline{P_b} = \iiint \underline{f}^T \underline{b} |\underline{J}| d\underline{\eta} d\underline{\eta} d\underline{\eta}$$

$$\underline{P_o} = \iiint \underline{B}^T \underline{E} \underline{\Sigma} |\underline{J}| d\underline{\eta} d\underline{\eta} d\underline{\eta}$$

20 nodes Solid Element (Isoparametric Element)



see p160 and p161

$$\left\{ \begin{array}{l} u = \sum f_i u_i \\ v = \sum f_i v_i \\ w = \sum f_i w_i \end{array} \right. \quad \left\{ \begin{array}{l} x = \sum f_i x_i \\ y = \sum f_i y_i \\ z = \sum f_i z_i \end{array} \right.$$

$$f_i = \frac{1}{8} (1 + \beta_i) (1 + \eta_i) (1 + \zeta_i) (\beta_i + \eta_i + \zeta_i - 2) \quad (i=1, 2, \dots, 8)$$

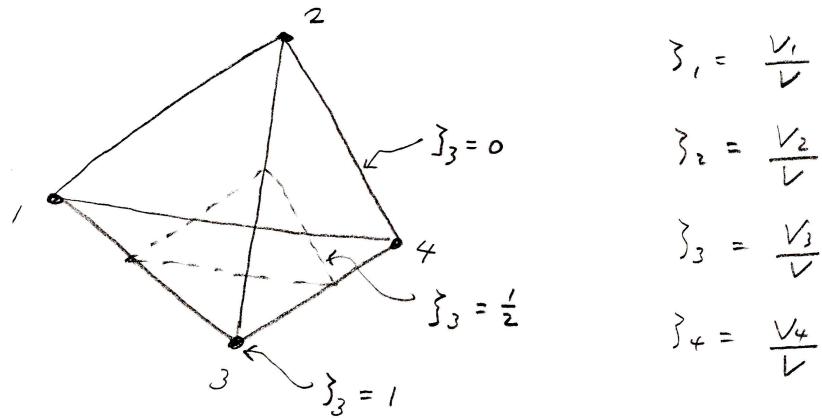
$$f_i = \frac{1}{4} (1 - \beta_i^2) (1 + \eta_i) (1 + \zeta_i) \quad i=9, 11, 17, 19$$

$$f_i = \frac{1}{4} (1 - \eta_i^2) (1 + \beta_i) (1 + \zeta_i) \quad i=10, 12, 18, 20$$

$$f_i = \frac{1}{4} (1 - \zeta_i^2) (1 + \beta_i) (1 + \eta_i) \quad i=13, 14, 15, 16$$

4-node Tetrahedron Element

Volume coordinates



$$\beta_1 = \frac{V_1}{V}$$

$$\beta_2 = \frac{V_2}{V}$$

$$\beta_3 = \frac{V_3}{V}$$

$$\beta_4 = \frac{V_4}{V}$$

$$V_1 + V_2 + V_3 + V_4 = V$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$$

$$\left\{ \begin{array}{l} x = \sum f_i x_i = \sum \beta_i x_i \\ y = \sum \beta_i y_i \\ z = \sum \beta_i z_i \end{array} \right. \quad f_i = \beta_i$$

$$\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

$$\underline{x} = \underline{H} \underline{\beta}$$

$$\underline{\underline{\beta}} = H^{-1} \underline{\underline{x}}$$

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \frac{1}{|H|} \begin{bmatrix} H_{11}^* & H_{12}^* & H_{13}^* & H_{14}^* \\ H_{21}^* & H_{22}^* & H_{23}^* & H_{24}^* \\ H_{31}^* & H_{32}^* & H_{33}^* & H_{34}^* \\ H_{41}^* & H_{42}^* & H_{43}^* & H_{44}^* \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$$

$$\left\{ \begin{array}{l} u = \sum f_i u_i = \sum \beta_i u_i \\ v = \sum \beta_i v_i \\ w = \sum \beta_i w_i \end{array} \right.$$

$$\underline{\underline{\Sigma}} = B \underline{\underline{\beta}}$$

$$\underline{\underline{B}} = \begin{pmatrix} f_{1,x} & 0 & 0 & \dots & f_{4,x} & 0 & 0 \\ 0 & f_{1,y} & 0 & \dots & 0 & f_{4,y} & 0 \\ 0 & 0 & f_{1,z} & \dots & 0 & 0 & f_{4,z} \\ f_{1,y} & f_{1,z} & 0 & \dots & f_{4,y} & f_{4,x} & 0 \\ 0 & f_{1,z} & f_{1,y} & \dots & 0 & f_{4,z} & f_{4,y} \\ f_{1,z} & 0 & f_{1,x} & \dots & f_{4,z} & 0 & f_{4,x} \end{pmatrix}$$

$$\begin{aligned} \frac{\partial f_i}{\partial x} &= \frac{\partial f_i}{\partial \beta_1} \frac{\partial \beta_1}{\partial x} + \frac{\partial f_i}{\partial \beta_2} \frac{\partial \beta_2}{\partial x} + \frac{\partial f_i}{\partial \beta_3} \frac{\partial \beta_3}{\partial x} + \frac{\partial f_i}{\partial \beta_4} \frac{\partial \beta_4}{\partial x} \\ &= \frac{\partial f_i}{\partial \beta_1} \frac{1}{|H|} H_{12}^* + \frac{\partial f_i}{\partial \beta_2} \frac{1}{|H|} H_{22}^* + \frac{\partial f_i}{\partial \beta_3} \frac{1}{|H|} H_{32}^* + \frac{\partial f_i}{\partial \beta_4} \frac{1}{|H|} H_{42}^* \end{aligned}$$

for 4-node tetrahedron element. $f_i \approx \beta_i$

$$\frac{\partial f_i}{\partial x} = \frac{\partial \beta_i}{\partial \beta_i} \frac{1}{|H|} H_{i2}^* = \frac{1}{|H|} H_{i2}^*$$

$$\frac{\partial f_i}{\partial y} = \frac{1}{(H_1)} H_{i3}^*, \quad \frac{\partial f_i}{\partial z} = \frac{1}{(H_1)} H_{i4}^*$$

$$K = \int \underline{B}^T \underline{E} \underline{B} dV = \underline{B}^T \underline{E} \underline{B} V \quad (\text{constant strain})$$

Generally

$$K = \int_V \{ \}_1^a \{ \}_2^b \{ \}_3^c \{ \}_4^d dV = \frac{a! b! c! d!}{(a+b+c+d+3)} (6V)$$

or

$$K = \frac{1}{6} \sum_{j=1}^6 w_j f(\{ \}_1, \{ \}_2, \{ \}_3, \{ \}_4) |J(\{ \}_1, \{ \}_2, \{ \}_3, \{ \}_4)|$$

For numerical integration of
Tetrahedron element,
see Table 4.1 (P 156)

$$\underline{J} = \begin{bmatrix} x_{1,1} & y_{1,1} & z_{1,1} \\ x_{1,2} & y_{1,2} & z_{1,2} \\ x_{1,3} & y_{1,3} & z_{1,3} \end{bmatrix}$$

$$x_{1,1} = \frac{\partial x}{\partial \{ \}_1} = \sum \frac{\partial f_i}{\partial \{ \}_1} x_i \dots \dots$$

$$\begin{aligned} x &= \{ \}_1 x_1 + \{ \}_2 x_2 + \{ \}_3 x_3 + \{ \}_4 x_4 \\ &= \{ \}_1 x_1 + \{ \}_2 x_2 + \{ \}_3 x_3 + (1 - \{ \}_1 - \{ \}_2 - \{ \}_3) x_4 \end{aligned}$$

$$x_{1,1} = x_1 - x_4 \Rightarrow J \Rightarrow \text{constants}$$

$$|J| = 6V$$

Tetrahedron 10 node Element

→ see P. 164

