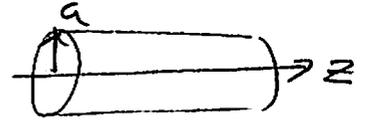


Differential balance

Summary) shall balance of pipe flow

momentum conservation eqn: $\frac{dP}{dz} - \frac{2}{r} \tau(z) = 0$ Newtonian constitutive eqn: $\tau = \mu \frac{du}{dr}$

$$u = -\frac{1}{4\mu} \frac{dP}{dz} (a^2 - r^2) = -\frac{1}{4\mu} \frac{\Delta P}{L} (a^2 - r^2)$$

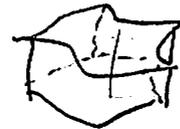
velocity profile

from $u(z)$

$$\textcircled{1} Q = \int_A u \, dA = \frac{\pi a^4}{8\mu} \left(-\frac{\Delta P}{L} \right) \quad \text{Hagen-Poiseuille law}$$

$$\textcircled{2} f_F = \frac{16}{Re} \quad \text{Fanning's friction factor for laminar flow}$$

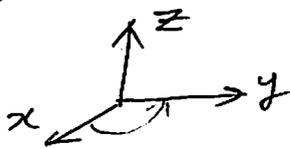
In this lecture, we will try to derive general mass & momentum balance equations for differential volume. dV_{cub}



→ Partial difference equations

comments)

For simplicity, we choose Cartesian coordinate which has 3 mutually perpendicular basis vectors or coordinate

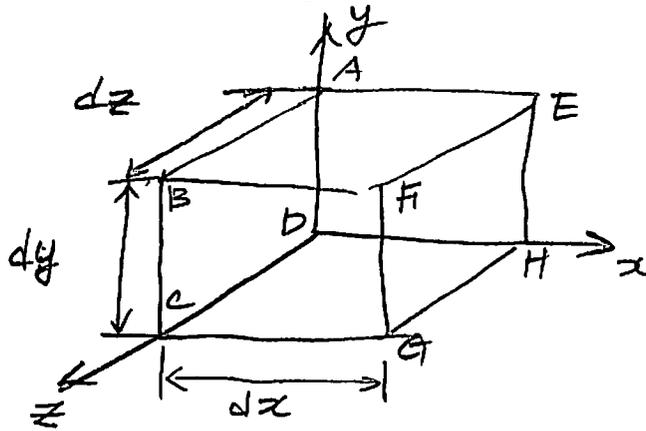


→ right handedness

In Cartesian Coordinate

the differential volume becomes

rectangular parallelepiped element:

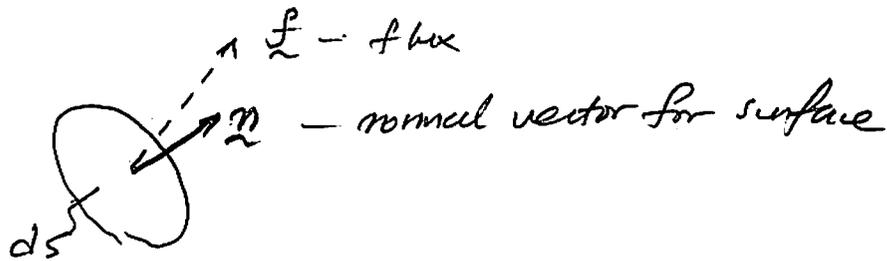


→ $dx dy dz$
stands for
this element.

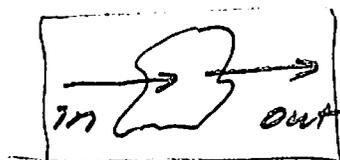
We try to set up mass & momentum balances

(We need to consider)

• We need to define flux across a given surface



This flux is required for set up balance or accounting



$$\text{flux} = \frac{\text{quantity}}{\text{Area} \cdot \text{time}}$$

e.x)

- concentration
- volume
- mass
- momentum

• Generation inside the element

(I) Mass balance

unit

$$\textcircled{1} \text{ define volume flux} = \frac{\text{volume}}{\text{area} \cdot \text{time}} = \frac{[L^3]}{[L^2][T]} = \left[\frac{L}{T} \right]$$

$$\textcircled{1} \text{ define mass flux} = \frac{\text{mass}}{\text{area} \cdot \text{time}} = \frac{[M]}{[L^2][T]}$$

when the density of the given material does not change

i.e. $\rho = \text{const}$

$$\rho = \frac{m}{V} \rightarrow m = \rho V \rightarrow \text{mass} \sim \text{volume}$$

So mass balance becomes volume balance.

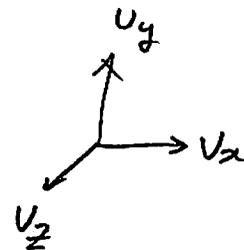
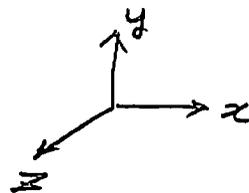
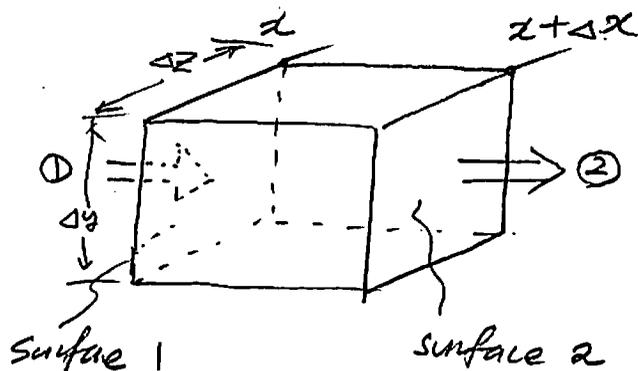
velocity = flux of volume or volume flux

Note) 1) $\rho = \text{const} \rightarrow$ incompressible flow (mostly liq)

2) Conservation of volume $\rightarrow \rho = \text{const}$
 \rightarrow isochoric flow
 (compressible gas can conserve volume)

3) isochoric flow $\xrightarrow{\quad}$ incompressible flow
 ($V = \text{const}$) $\xleftarrow{\quad}$ ($\rho = \text{const}$)

4) For simplicity, we consider incompressible flow.



• total volume flux at surface 1 : $U_x (dy dz)$

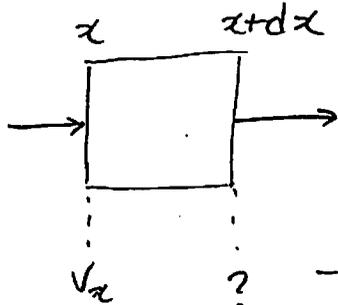
(flux) x (area)



• total volume flux at surface 2 : (flux) x (area) = $(U_x + \frac{\partial U_x}{\partial x} dx) dy dz$

Volume flux at surface 2 ?

$$\left[U_x + \frac{\partial U_x}{\partial x} dx \right]_{y,z}$$



→ Taylor series expansion around x

$$f(x + \delta x) = f(x) + \frac{df}{dx} \delta x + \frac{d^2 f}{dx^2} (\delta x)^2 + \dots$$

H.o.T

$$\delta x \rightarrow 0$$

$$f(x + \delta x) \approx f(x) + \frac{df}{dx} \delta x$$

U_x

- Net rate of outflow across surface ① & ② :
(along x-direction)

$$\underbrace{\left(U_x + \frac{\partial U_x}{\partial x} dx \right) dy dz}_{\text{"Out" from surface ②}} - \underbrace{\left(U_x \right) dy dz}_{\text{"in" from surface ①}} = \frac{\partial U_x}{\partial x} dx dy dz \dots (i)$$

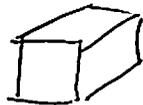
"Out" from surface ② "in" from surface ①

- Evaluate net rate of outflow along y-direction

y: $\frac{\partial U_y}{\partial y} dx dy dz \dots (ii)$

z: $\frac{\partial U_z}{\partial z} dx dy dz \dots (iii)$

Total net rate of outflow of the element $(dx dy dz)$



$$\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \right) dx dy dz = 0$$

Positive

Called "Continuity" eqn.

Called "Divergence"

There is no gen or consumption of volume!

$$\nabla \cdot \underline{U} = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} = 0$$

- Coord. dependent representation : → P 265
- " independent " : Vector & Tensor notation

* No constitutive eqn for this case

If $\rho \neq \text{const}$? & material moving w/ fluid

We have to consider

Convective derivative or material derivative

$\frac{\partial X}{\partial t}$: partial derivative $\frac{\partial X}{\partial t} \Big|_{x, y, z}$
 \square \square
 fix position

$\frac{DX}{Dt}$: material derivative (D/Dt) X
 \Rightarrow rate of change of X w.r. + time,
following the path taken by the fluid



In general, density depend on time & position
 t x, y, z

$$\rho = \rho(t, x, y, z)$$

total differential of ρ :

$$d\rho = \frac{\partial \rho}{\partial t} dt + \frac{\partial \rho}{\partial x} dx + \frac{\partial \rho}{\partial y} dy + \frac{\partial \rho}{\partial z} dz$$

over the time increment dt ,

$$\frac{dx}{dt} = u_x dt, \quad dy = u_y dt, \quad dz = u_z dt$$

correspond to the distance traveled by fluid ptcle.



$$\left(\frac{d\rho}{dt}\right)_{\text{moving w/ fluid}} = \frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\frac{\partial\rho}{\partial x} + \frac{dy}{dt}\frac{\partial\rho}{\partial y} + \frac{dz}{dt}\frac{\partial\rho}{\partial z}$$

$$= \frac{\partial\rho}{\partial t} + u_x\frac{\partial\rho}{\partial x} + u_y\frac{\partial\rho}{\partial y} + u_z\frac{\partial\rho}{\partial z}$$

In vector notation

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \underline{u} \cdot \nabla\rho$$

So the mass balance becomes

Volume expansion

$$\frac{1}{\rho} \frac{D\rho}{Dt} = (\text{volume change w/ moving fluid}) = \ominus \nabla \cdot \underline{u}$$

$$\downarrow \quad m = \rho V \quad (= \text{conserved}) \quad \rho \uparrow \quad V \downarrow$$

$$\rho = \frac{m}{V} \quad \frac{1}{\rho} \frac{D\rho}{Dt} = \frac{V}{m} \frac{D\left(\frac{m}{V}\right)}{Dt} = \ominus \frac{1}{V} \frac{DV}{Dt} = \dots -\nabla \cdot \underline{u}$$

Differential momentum balance

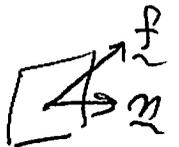
• two types of forces

- body force : e.g. gravity or electromagnetic force
- surface force :
 - viscous traction
 - pressure $\rightarrow \text{Stress} = \frac{\text{traction}}{\text{area}}$

• $\text{Stress} = \frac{\text{force}}{\text{area}}$

\nearrow direction of force
 \searrow direction of surface (Surface normal)

(tensorial quantity)



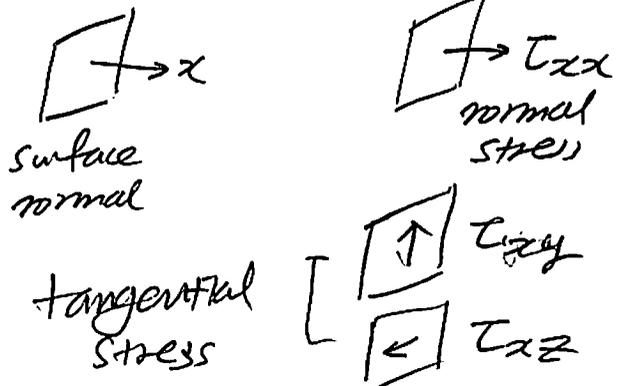
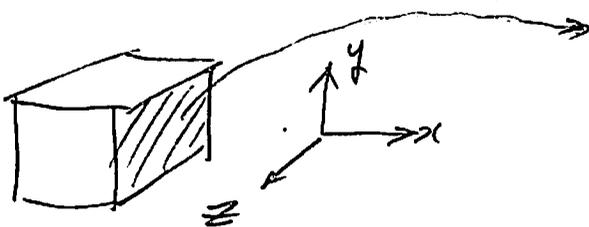
• At a given surface,

there are two types stresses:

normal stress & tangential stress

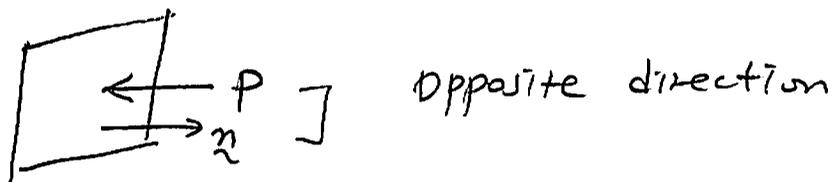


• For Cartesian coordinate
 viscous traction for
 e.g. differential element



Note) Pressure is "isotropic" stress

it always act normal to a given surface



So total normal stress

$$\sigma_n = -p + \tau_n$$

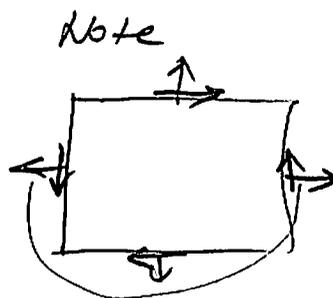
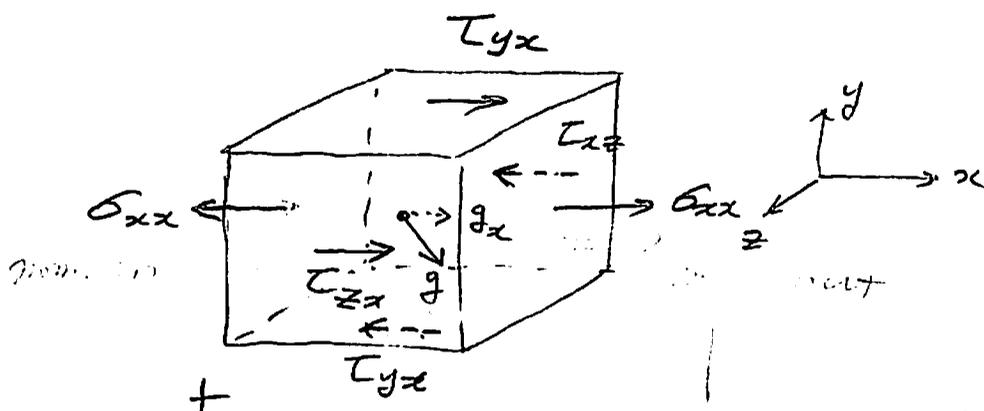
In previous example

$$\sigma_{xx} = -p + \tau_{xx}$$

Momentum balance for a differential element

$$dV = dx dy dz$$

x-direction



outward normal

read p271 for notation & directions

momentum changes inside dV

→ taken care by considering an element of fixed mass
($\rho dV = \rho dx dy dz$)
that is moving w/ fluid

$$\rho dV \frac{DU_x}{Dt} = \rho dV \frac{DU_x}{Dt} = \rho dx dy dz \frac{DU_x}{Dt}$$

① Set up momentum balance for x direction

$$\underbrace{\rho dx dy dz \frac{DU_x}{Dt}}_{\text{Rate of increase of } x \text{ momentum}} = \underbrace{\left(\frac{\partial \sigma_{xx}}{\partial x} dx \right) dy dz + \left(\frac{\partial \tau_{yx}}{\partial y} dy \right) dz dx}_{\text{Net surface force in } x \text{ dir}} + \underbrace{\rho g_x dx dy dz}_{\text{Body force}}$$

/ $dx dy dz$

$$\begin{aligned} \rightarrow \rho \frac{DU_x}{Dt} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x \\ \text{Similarly for } y, z & \\ \rho \frac{DU_y}{Dt} &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y \quad \dots \text{ (*)} \\ \rho \frac{DU_z}{Dt} &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z \end{aligned}$$

→ P273

Note)

- Cauchy equation of motion in Cartesian coordinate.

- This is ^{the} conservation equation for momentum

① Consider constitutive relationship
btw motion of fluid & stress

(see ④)

We have total $3 + 9 = 12$ unknowns
(velocity) (stress)

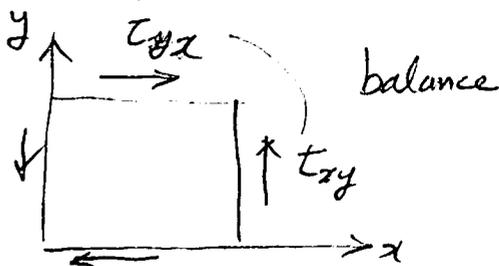
but we have only $3 + 1 = 4$ eqn.
(mom. balance) (mass balance)

→ we need 8 more equations!

Let's consider the simplest fluid: Newtonian fluid
(like ideal gas)

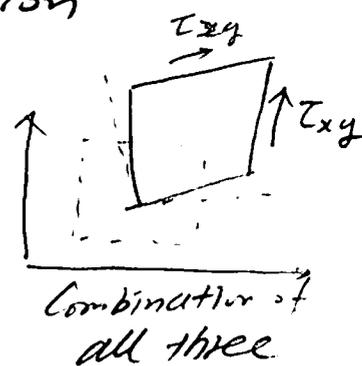
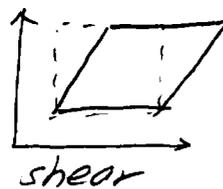
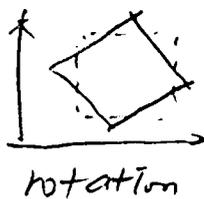
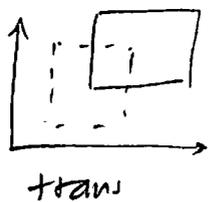
① No net rotational force (nonpolar fluid: no net torque, symmetric stress)

$$\tau_{yx} = \tau_{xy}, \quad \tau_{zy} = \tau_{yz}, \quad \tau_{xz} = \tau_{zx}$$

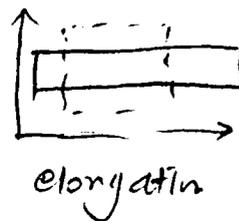


→ see p274 of Wilke

② Stress are only function of fluid motion
Three basic fluid motion:



p275 fig 5.15



No contribution in stress

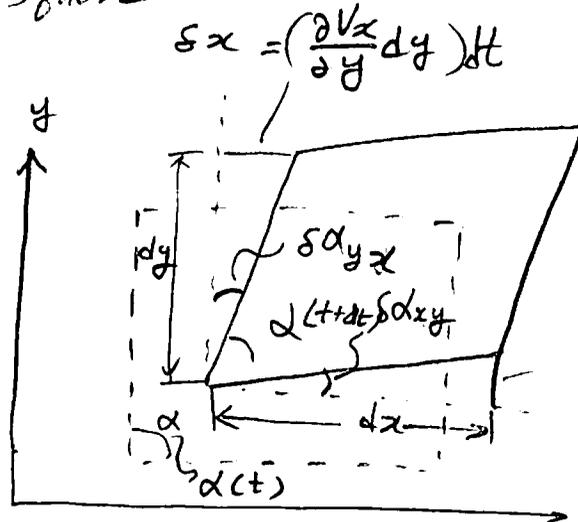
• Translation: the element moves to a new location w/o changing its shape

$\frac{dv_z}{dy}$ • Rotation: the element turns w/o moving its center of gravity No contrib ut'n

$\frac{dv_z}{dx}$ • Shear: the element deforms into parallelogram shear stress

$\frac{dv_x}{dx}$ • Elongation: the element deforms into rectangle Normal stress

• Shear stress relation



$$\delta x = \left(\frac{\partial v_x}{\partial y} dy \right) dt$$

$$\delta y = \left(\frac{\partial v_y}{\partial x} dx \right) dt$$

$$\delta \alpha_{yx} = \tan^{-1} \left[\frac{\delta x}{dy} \right] \sim \frac{\delta x}{dy} = \frac{\partial v_x}{\partial y} dt$$

$$\delta \alpha_{xy} = \tan^{-1} \left[\frac{\delta y}{dx} \right] \sim \frac{\delta y}{dx} = \frac{\partial v_y}{\partial x} dt$$

$$\alpha(t+dt) = \alpha(t) - \delta \alpha_{yx} - \delta \alpha_{xy}$$

$$\rightarrow \frac{d\alpha}{dt} = \lim_{dt \rightarrow 0} \frac{\alpha(t+dt) - \alpha(t)}{dt} = - \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

↳ rate of shear deformation
or strain rate

Newtonian liq. follow

$$\tau_{xy} (= \tau_{yx}) = -\mu \frac{d\alpha}{dt} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

↳ this proportionality constant is viscosity.

• Normal stress (← elongation)

$$\frac{\partial v_x}{\partial x} \longrightarrow \tau_{xx}$$

w/o derivation

$$\tau_{xx} = - \left(\lambda + \frac{2\mu}{3} \right) (\nabla \cdot \underline{v}) + 2\mu \frac{\partial v_x}{\partial x}$$

= coeff of bulk viscosity \uparrow = 0 for incompressible fluid

↳ important for compressible fluid.

(Stokes's postulation

$$\lambda + \frac{2\mu}{3} = 0)$$

∴ For incompressible flow

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x}$$

$$\sigma_{xx} = -p + 2\mu \frac{\partial v_x}{\partial x}$$

Very diff to verify experimentally

show table 5.7 P 277

→ we have 6 more eqns & one more unknown (pressure)

For Newtonian fluid

Ⓐ mass & momentum balance = 1 + 3 = 4

Ⓑ Newtonian constitute eqn = 3 + 6 = 9

+ | Symmetry stress & rate of strain relationship

total 13 Eqn

check unknowns: velocity, pressure, stress component
 3 + 1 + 9
 = 13 unknowns

→ we can solve the system!

We can get rid of stress components from mass & momentum balance in Newtonian fluid: plugging Ⓑ into Ⓐ

$$\rho \frac{Dv_k}{Dt} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_k}{\partial x^2} + \frac{\partial^2 v_k}{\partial y^2} + \frac{\partial^2 v_k}{\partial z^2} \right) + \rho g$$

$k = x, y, z$

$$0 = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

→ This is famous Navier - Stokes Eqn.