Particle Dynamics

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Values of some important physical constants

Constant	Symbol	Value
Speed of light (in vacuum)	c	2.99792458×10^{8} m s^{-1}*
Electron charge	e	$1.6021766208(98) \times 10^{-19}$ C
Atomic mass constant	u	$1.660539040(20) \times 10^{-27}$ kg $(931.4940954(57)$ MeV$/c^2)$
Electron rest mass	m_e	$9.10938356(11) \times 10^{-31}$ kg $(0.5109989461(3)$ MeV$/c^2)$ $(5.48579909070(16) \times 10^{-4}$ u$)$
Proton rest mass	m_p	$1.672621898(21) \times 10^{-27}$ kg $(938.2720813(58)$ MeV$/c^2)$ $(1.007276466879(91)$ u$)$
Neutron rest mass	m_n	$1.674927471(21) \times 10^{-27}$ kg $(939.5654133(58)$ MeV$/c^2)$ $(1.00866491588(49)$ u$)$
Planck's constant	h	$6.626070040(81) \times 10^{-34}$ J s $(4.135667662(25) \times 10^{-15}$ eV s$)$
Avogadro's constant	N_a	$6.022140857(74) \times 10^{23}$ mol^{-1}
Boltzmann constant	k	$1.38064852(79) \times 10^{-23}$ J K^{-1} $(8.6173303(50) \times 10^{-5}$ eV K$^{-1})$
Ideal gas constant (STP)	R	$8.3144598(48)$ J mol^{-1} K^{-1}
Electric constant	ϵ_o	$8.854187817\ldots \times 10^{-12}$ F m^{-1}*
Magnetic constant	μ_o	$4\pi \times 10^{-7}$ N A^{-2}* $= 12.566370614\ldots \times 10^{-7}$ N A^{-2}

* indicates exact values.

Source: `http://physics.nist.gov/cuu/index.html`

Relativistic particle momentum

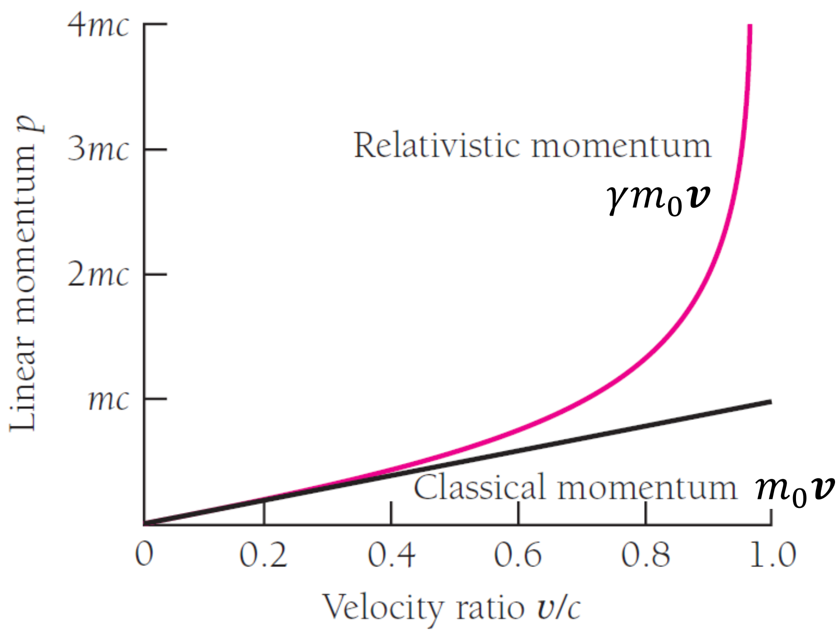
- The special theory of relativity states that the inertia of a particle observed in a frame of reference depends on the magnitude of its speed in that frame.

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

Lorentz factor

- The inertia of a particle is proportional to γ. The apparent mass is γm_0.
- The particle momentum, a vector quantity, equals $\boldsymbol{p} = \gamma m_0 \boldsymbol{v}$.
- The equation of motion:

$$\frac{d\boldsymbol{p}}{dt} = \frac{d(\gamma m_0 \boldsymbol{v})}{dt} = \boldsymbol{F}$$



Relativistic particle energy

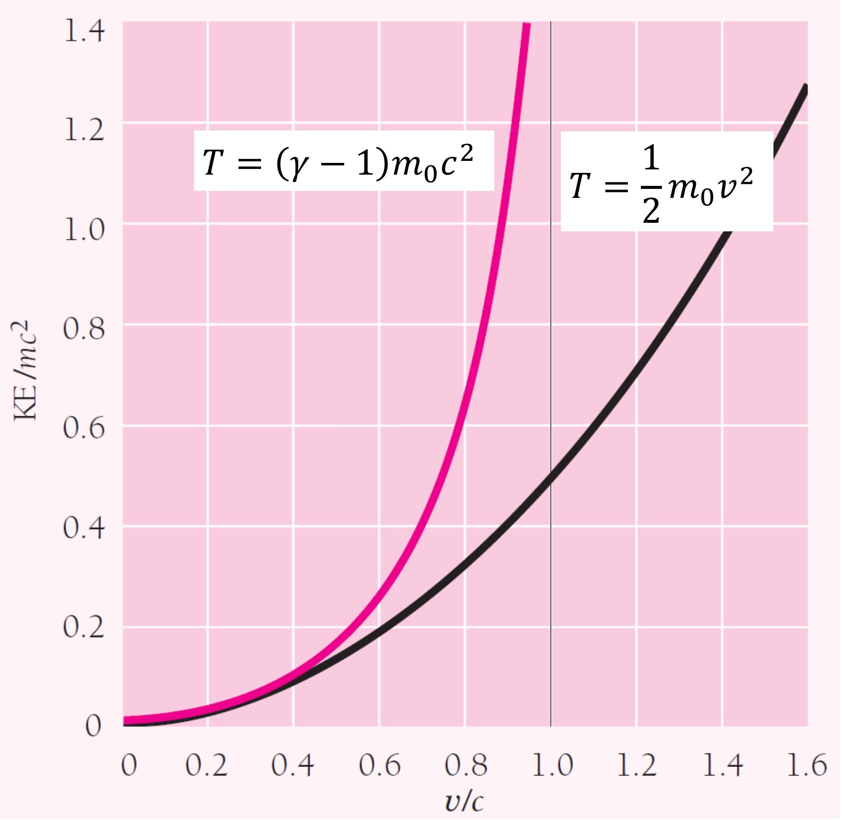
- The kinetic energy equals the total energy minus the rest energy:

$$E = \gamma m_0 c^2 \qquad\qquad T = (\gamma - 1) m_0 c^2$$

- Newtonian dynamics describes the motion of low-energy particles when $T \ll m_0 c^2$.

$$T = \frac{1}{2} m_0 v^2$$

$$(1 + x)^{-1/2} \approx 1 - \frac{1}{2} x + \cdots$$



- Energy and momentum

$$E^2 = (m_0 c^2)^2 + p^2 c^2$$

- For massless particles

$$E = pc$$

Charged particle properties

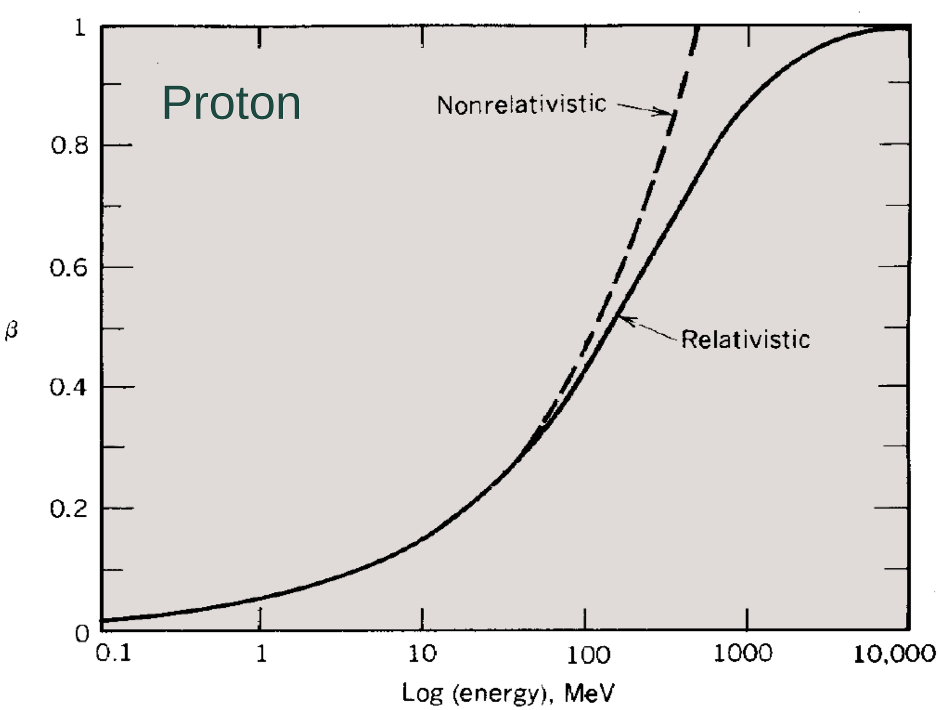
Particle	Charge (coulomb)	Mass (kg)	Rest Energy (MeV)	A	Z	Z^*
Electron (β particle)	-1.60×10^{-19}	9.11×10^{-31}	0.511	—	—	—
Proton	$+1.60 \times 10^{-19}$	1.67×10^{-27}	938	1	1	1
Deuteron	$+1.60 \times 10^{-19}$	3.34×10^{-27}	1875	2	1	1
Triton	$+1.60 \times 10^{-19}$	5.00×10^{-27}	2809	3	1	1
He^{+}	$+1.60 \times 10^{-19}$	6.64×10^{-27}	3728	4	2	1
He^{++} (α particle)	$+3.20 \times 10^{-19}$	6.64×10^{-27}	3728	4	2	2
C^{+}	$+1.6 \times 10^{-19}$	1.99×10^{-26}	1.12×10^{4}	12	6	1
U^{+}	$+1.6 \times 10^{-19}$	3.95×10^{-25}	2.22×10^{5}	238	92	1

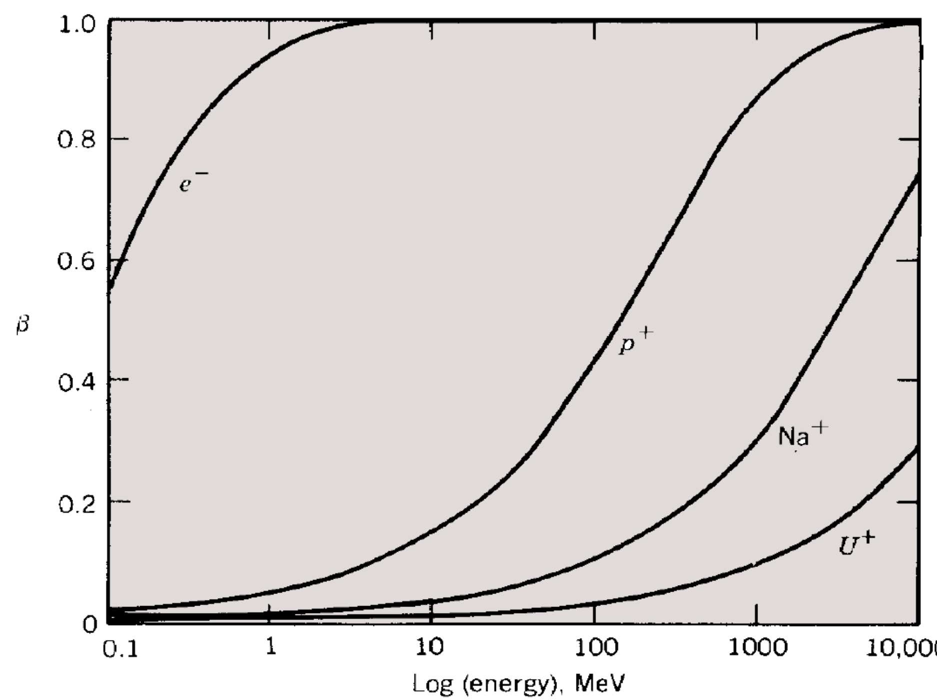
1 eV = 1.6×10^{-19} J = 1240 nm = 241.8 THz

β for particles as a function of kinetic energy

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

$$T = (\gamma - 1)m_0c^2$$





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Nonrelativistic approximation for transverse motion

- In the study of the transverse motions of charged particle beams, it is often possible to express the problem in the form of Newtonian equations with the rest mass replaced by the relativistic mass.
- This approximation is valid when the beam is well directed so that transverse velocity components are small compared to the axial velocity of beam particles ($v_x \ll v_z$).
- The equation of motion in transverse direction:

$$\frac{dp_x}{dt} = \frac{d(\gamma m_0 v_x)}{dt} = \gamma m_0 v_x \left(\frac{1}{\gamma} \frac{d\gamma}{dt} + \frac{1}{v_x} \frac{dv_x}{dt} \right) = F_x$$

- When $v_x \ll v_z$, relative changes in γ resulting from the transverse motion are small. Then, the equation of motion is approximately

$$\gamma m_0 \frac{dv_x}{dt} = F_x$$

Maxwell's equations

$$\boxed{\begin{gathered}\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\epsilon_0 \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \\ \boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \\ \boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{J}\end{gathered}}$$

- Gauss's law
- Gauss's law for magnetism
- Faraday's law of induction
- Ampere's law with Maxwell's addition

$$\boxed{\begin{gathered}\rho = e(Zn_i - n_e) \\ \boldsymbol{J} = e(Zn_i\boldsymbol{u}_i - n_e\boldsymbol{u}_e)\end{gathered}}$$

- Constitutive relation

$$\boxed{\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0}$$

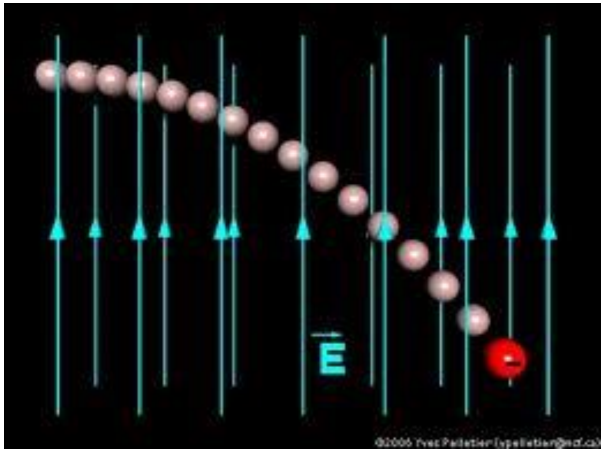
- Charge continuity equation

Motions of a charged particle in uniform electric field

- Equation of motion of a charged particle in fields

$$m\frac{d\boldsymbol{v}}{dt} = q[\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{r},t)], \qquad \frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}(t)$$

- Motion in constant electric field



✓ For a constant electric field $\boldsymbol{E} = \boldsymbol{E_0}$ with $\boldsymbol{B} = 0$,

$$\boldsymbol{r}(t) = \boldsymbol{r_0} + \boldsymbol{v_0}t + \frac{q\boldsymbol{E_0}}{\boldsymbol{2}m}t^2$$

✓ Electrons are easily accelerated by electric field due to their smaller mass than ions.

✓ Electrons (Ions) move against (along) the electric field direction.

✓ The charged particles get kinetic energies.

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Motions of a charged particle in uniform magnetic field

- Motion in constant magnetic field

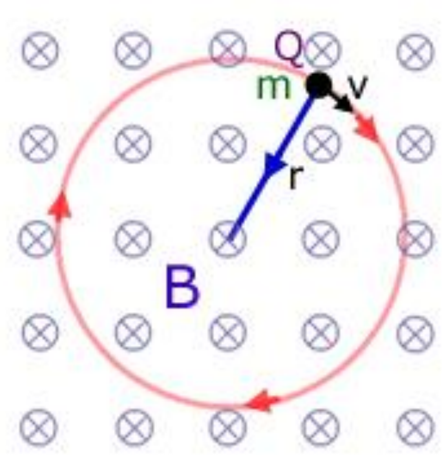
$$m\frac{d\boldsymbol{v}}{dt} = q\boldsymbol{v} \times \boldsymbol{B}$$

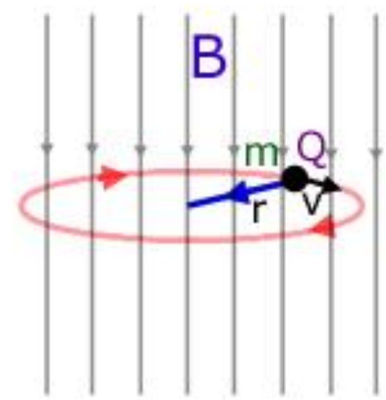
- For a constant magnetic field $\boldsymbol{B} = B_0\boldsymbol{z}$ with $\boldsymbol{E} = 0$,

$$m\frac{dv_x}{dt} = qB_0v_y$$

$$m\frac{dv_y}{dt} = -qB_0v_x$$

$$m\frac{dv_z}{dt} = 0$$





- Cyclotron (gyration) frequency

$$\frac{d^2v_x}{dt^2} = -\omega_c^2 v_x$$

$$\boxed{\omega_c = \frac{|q|B_0}{m}}$$

Motions of a charged particle in uniform magnetic field

- Particle velocity

$$v_x = v_\perp \cos(\omega_c t + \phi_0)$$
$$v_y = -v_\perp \sin(\omega_c t + \phi_0)$$
$$v_z = v_{z0}$$

- Particle position

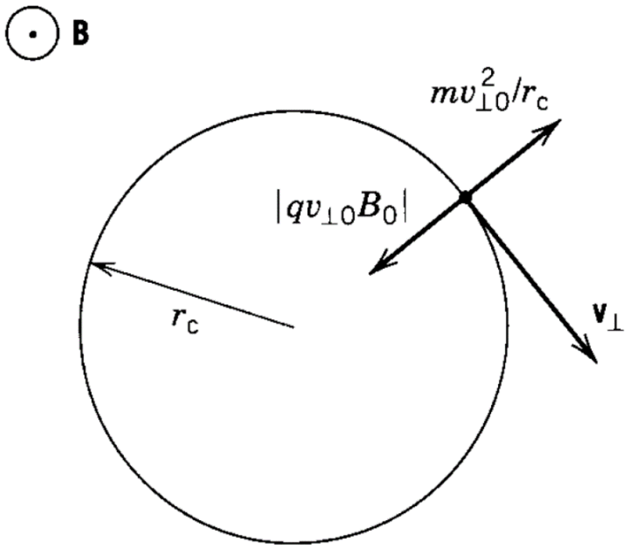
$$x = x_0 + r_c \sin(\omega_c t + \phi_0)$$
$$y = y_0 + r_c \cos(\omega_c t + \phi_0)$$
$$z = z_0 + v_{z0} t$$

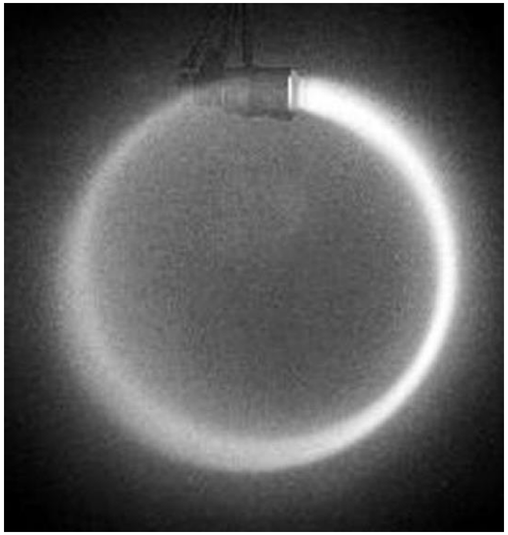
- Guiding center

$$(x_0, y_0, z_0 + v_{z0} t\ )$$

- Larmor (gyration) radius

$$\boxed{r_c = r_{\mathrm{L}} = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q| B_0}}$$





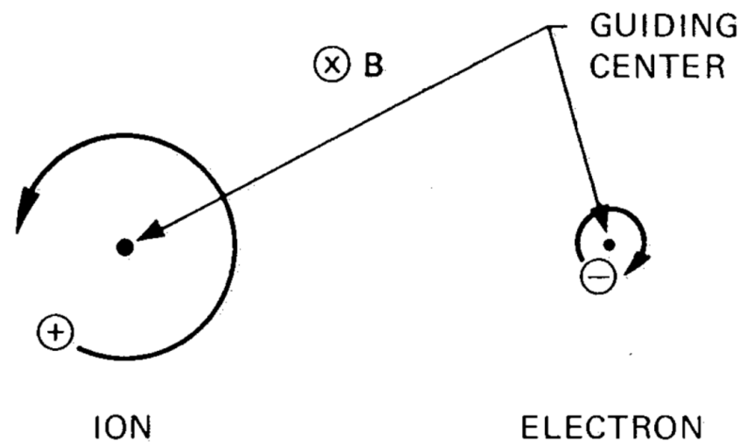
Gyro-frequency and gyro-radius

- The direction of gyration is always such that the magnetic field generated by the charged particle is opposite to the externally imposed field. → diamagnetic

- For electrons

$$f_{ce} = 2.80 \times 10^6\ B_0\ [\mathrm{Hz}] \quad (B_0 \text{ in gauss})$$

$$r_{ce} = \frac{3.37\sqrt{E}}{B_0}\ [\mathrm{cm}] \quad (E \text{ in volts})$$



- For singly charged ions

$$f_{ci} = 1.52 \times 10^3\ B_0/M_A\ [\mathrm{Hz}] \quad (B_0 \text{ in gauss})$$

$$r_{ci} = \frac{144\sqrt{EM_A}}{B_0}\ [\mathrm{cm}] \quad (E \text{ in volts}, M_A \text{ in amu})$$

- Energy gain?

Motions of a charged particle in uniform E and B fields

- Equation of motion

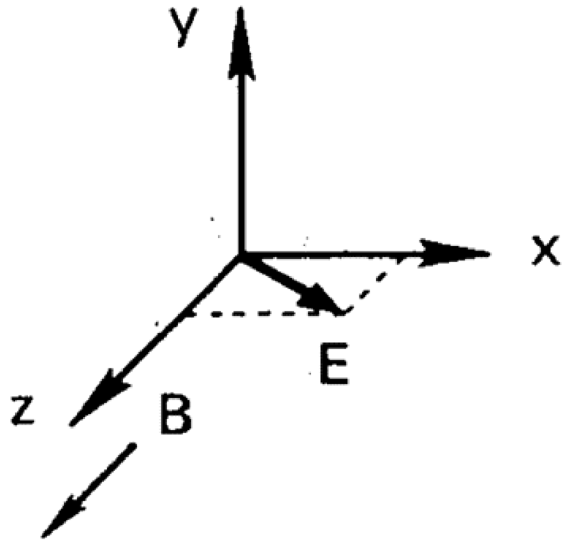
$$m\frac{d\boldsymbol{v}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

- Parallel motion: $\boldsymbol{B} = B_0\boldsymbol{z}$ and $\boldsymbol{E} = E_0\boldsymbol{z}$,

$$m\frac{dv_z}{dt} = qE_z$$

$$v_z = \frac{qE_z}{m}t + v_{z0}$$

→ Straightforward acceleration along B



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$\boldsymbol{E}\times\boldsymbol{B}$ drift

- Transverse motion: $\boldsymbol{B} = B_0\boldsymbol{z}$ and $\boldsymbol{E} = E_0\boldsymbol{x}$,

$$m\frac{dv_x}{dt} = qE_0 + qB_0v_y$$

$$m\frac{dv_y}{dt} = -qB_0v_x$$

- Differentiating,

$$\frac{d^2v_x}{dt^2} = -\omega_c^2 v_x$$

$$\frac{d^2v_y}{dt^2} = -\omega_c^2\left(\frac{E_0}{B_0} + v_y\right)$$

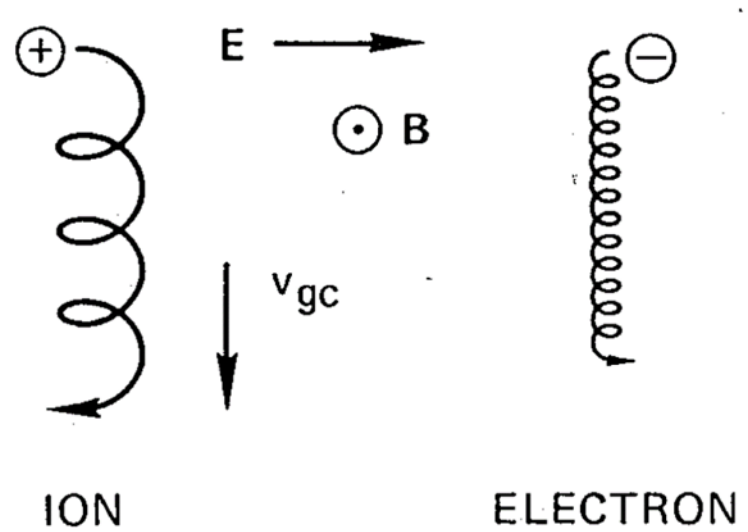
$$\boxed{\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}}$$

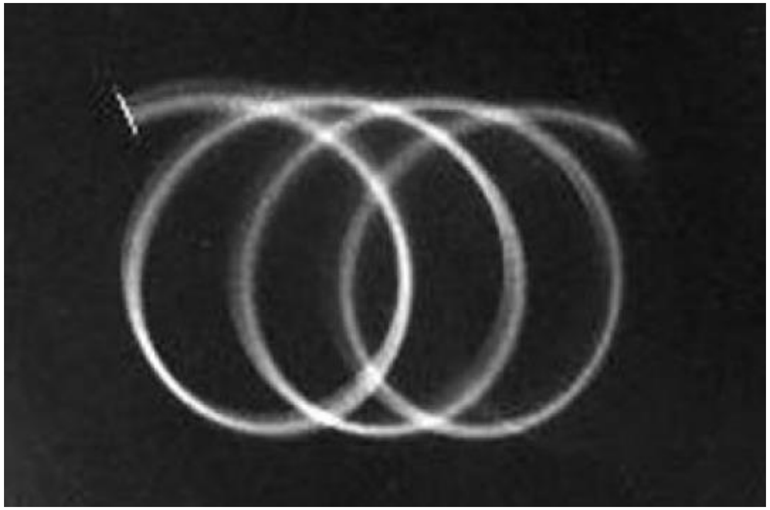
- Particle velocity

$$v_x = v_\perp \cos(\omega_c t + \phi_0)$$

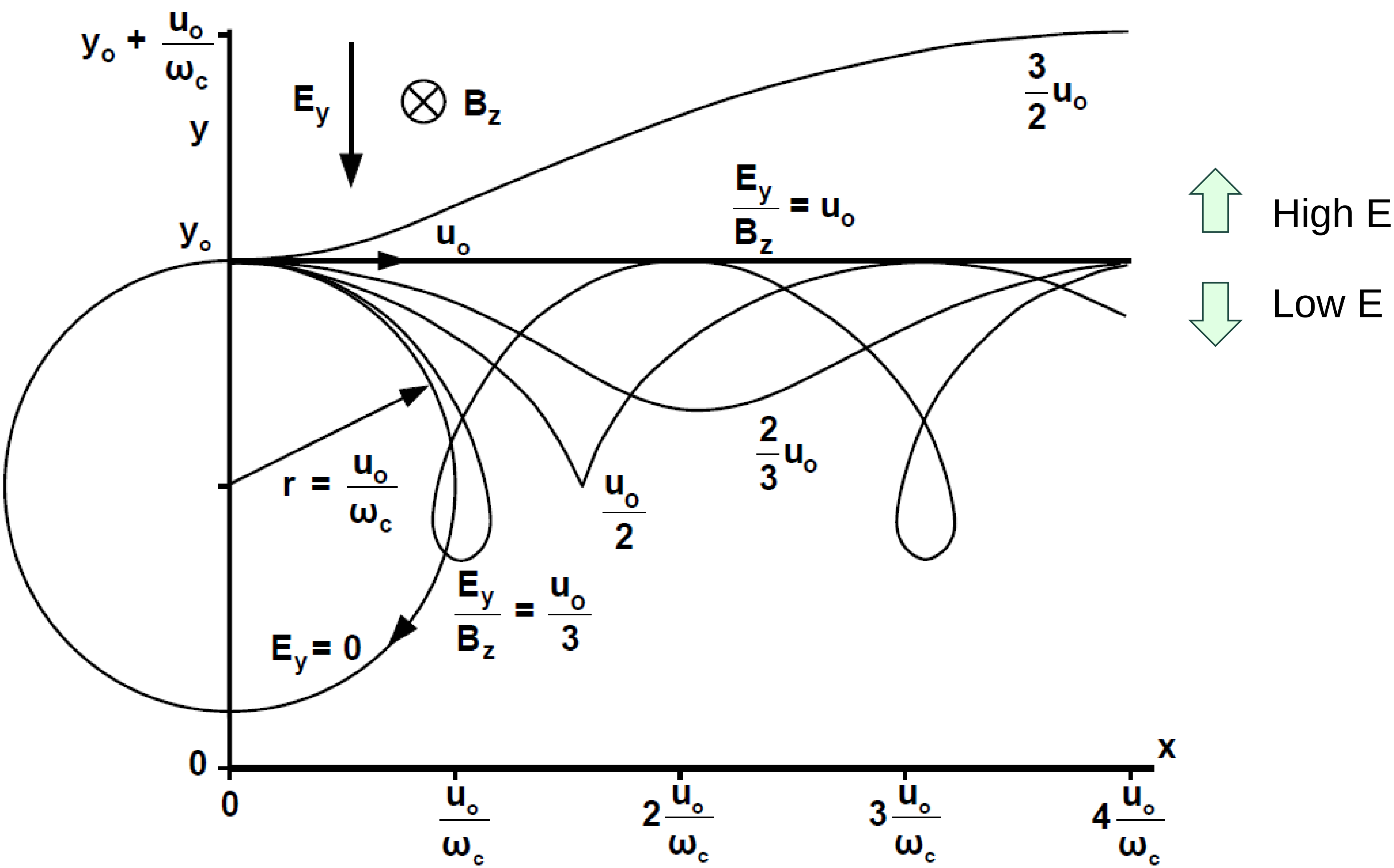
$$v_y = -v_\perp \sin(\omega_c t + \phi_0) - \frac{E_0}{B_0}$$

v_{gc}



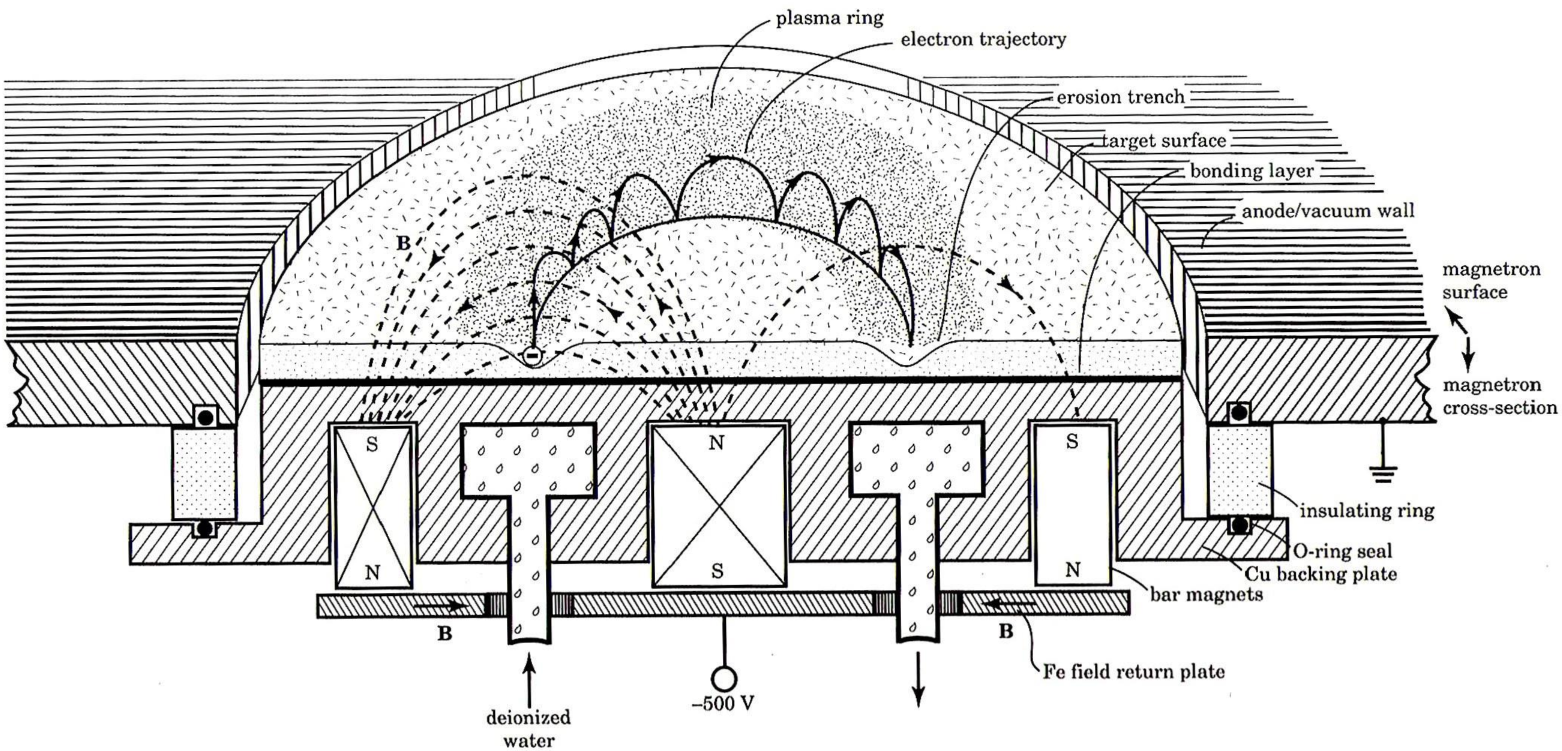


Electron trajectories for various electric field levels



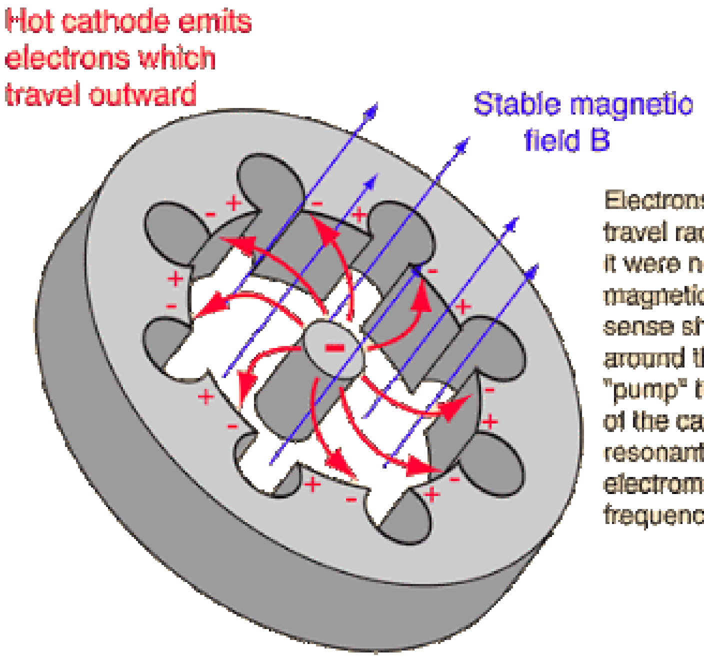
DC magnetron

- A magnetron which is widely used in the sputtering system uses the $\boldsymbol{E}\times\boldsymbol{B}$ drift motion for plasma confinement.

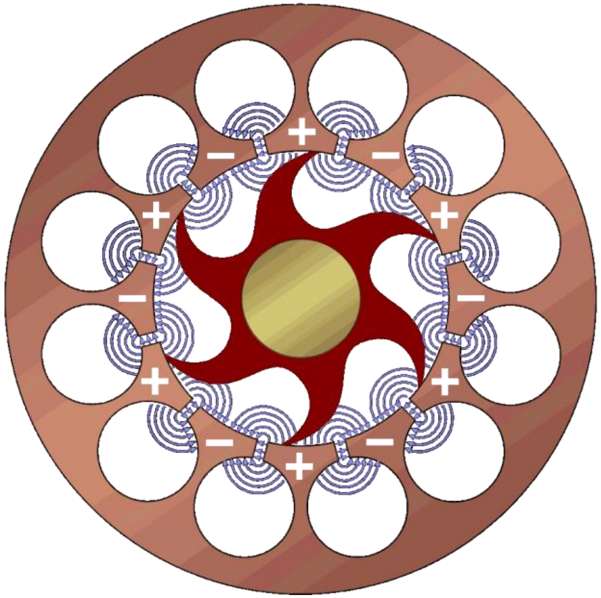


- What is the direction of $\boldsymbol{E}\times\boldsymbol{B}$ drift motion?

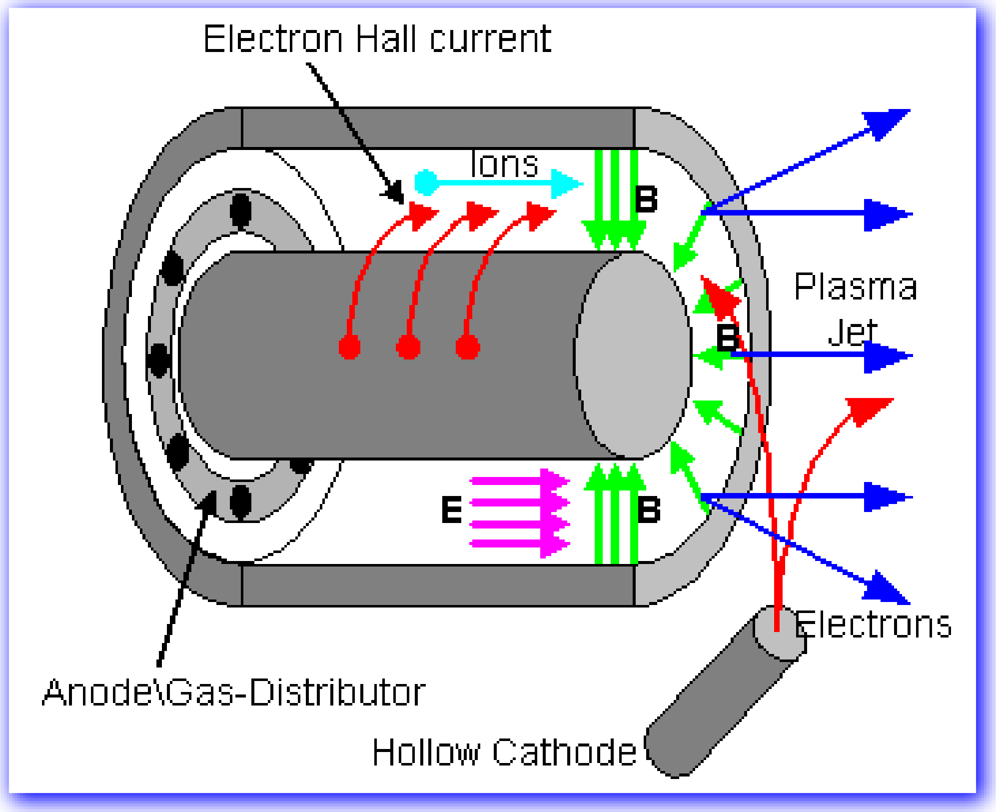
Magnetron (MW generator)



Electrons from a hot filament would travel radially to the outside ring if it were not for the magnetic field. The magnetic force deflects them in the sense shown and they tend to sweep around the circle. In so doing, they "pump" the natural resonant frequency of the cavities. The currents around the resonant cavities cause them to radiate electromagnetic energy at that resonant frequency.



Hall thruster



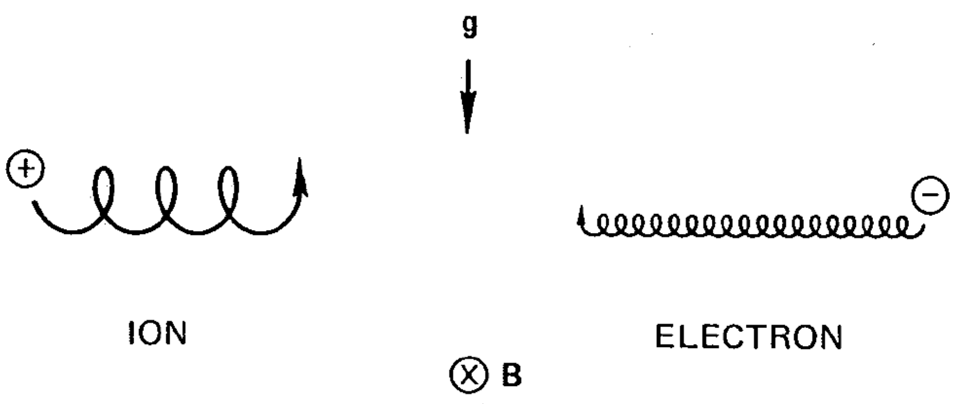
Motions of a charged particle in gravitational field

- Generally, the guiding center drift caused by general force $\boldsymbol{F}$

$$\boxed{\boldsymbol{v}_f = \frac{1}{q}\frac{\boldsymbol{F}\times\boldsymbol{B}}{B^2}}$$

- If $\boldsymbol{F}$ is the force of gravity $m\boldsymbol{g}$,

$$\boxed{\boldsymbol{v}_g = \frac{m}{q}\frac{\boldsymbol{g}\times\boldsymbol{B}}{B^2}}$$



- What is the difference between $\boldsymbol{v}_E$ and $\boldsymbol{v}_g$?

Time-varying $\boldsymbol{E}$ field: polarization drift

- Assume that $\boldsymbol{E} = E_0 e^{i\omega t}$, then $\dot{\boldsymbol{E}}_x = i\omega \boldsymbol{E}_x$

$$\frac{d^2 v_x}{dt^2} = -\omega_c^2 \left(v_x \mp \frac{i\omega}{\omega_c} \frac{\tilde{E}_x}{B} \right)$$
$$\frac{d^2 v_y}{dt^2} = -\omega_c^2 \left(v_y - \frac{\tilde{E}_x}{B} \right)$$

$$\boxed{\tilde{v}_E = \frac{\tilde{E}_x}{B}}$$

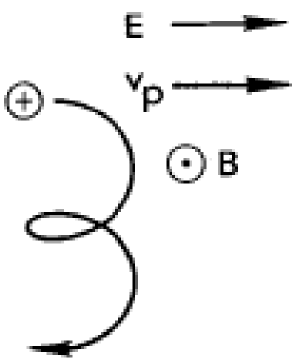
$$\boxed{\tilde{v}_p = \pm \frac{i\omega}{\omega_c} \frac{\tilde{E}_x}{B}}$$

- Particle velocity for slowly-varying E field ($\omega \ll \omega_c$)

$$v_x = v_\perp e^{i\omega_c t} + \tilde{v}_p$$
$$v_y = \pm i v_\perp e^{i\omega_c t} + \tilde{v}_E$$

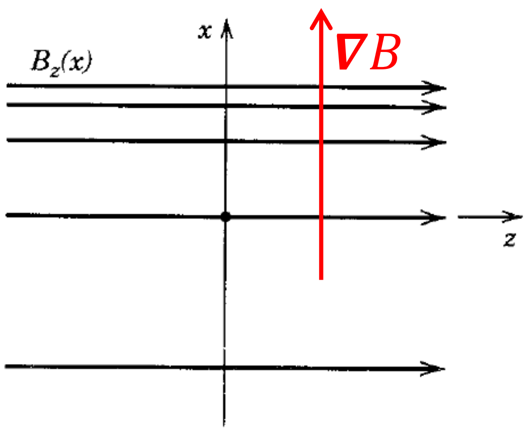
- Polarization drift

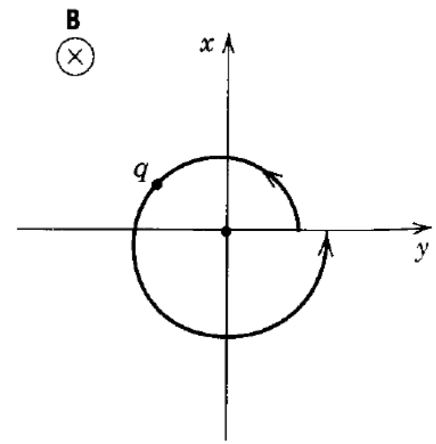
$$\boxed{\boldsymbol{v}_p = \pm \frac{1}{\omega_c B} \frac{dE}{dt}}$$



$\nabla B \perp \boldsymbol{B}$: Grad-B drift

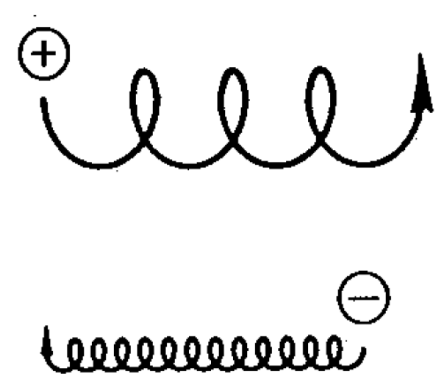
- The gradient in $|\boldsymbol{B}|$ causes the Larmor radius to be larger at the bottom of the orbit than at the top, and this should lead to a drift, in opposite directions for ions and electrons, perpendicular to both $\boldsymbol{B}$ and ∇B.





- Guiding center motion

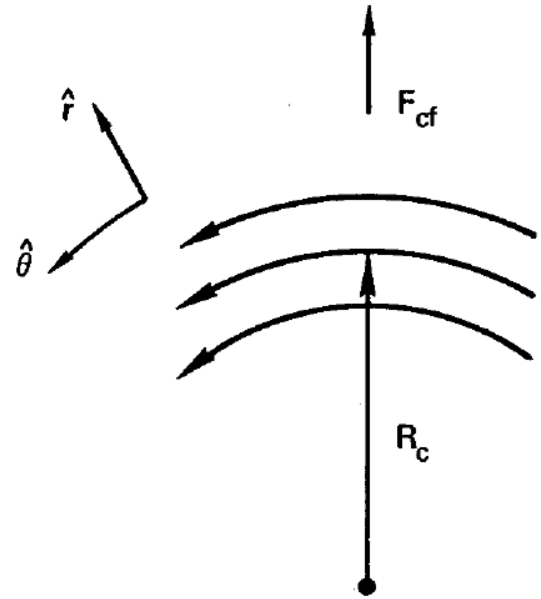
$$\boxed{\boldsymbol{v}_{\nabla B} = \pm \frac{1}{2} v_\perp r_c \frac{\boldsymbol{B} \times \nabla B}{B^2}}$$



Curved B: Curvature drift

- The average centrifugal force

$$\boldsymbol{F}_{cf} = \frac{mv_{\parallel}^{2}}{R_c}\hat{\boldsymbol{r}} = mv_{\parallel}^{2}\frac{\boldsymbol{R}_c}{R_c^{2}}$$



- Curvature drift

$$\boxed{\boldsymbol{v}_R = \frac{1}{q}\frac{\boldsymbol{F}_{cf}\times\boldsymbol{B}}{B^{2}} = \frac{mv_{\parallel}^{2}}{qB^{2}}\frac{\boldsymbol{R}_c\times\boldsymbol{B}}{R_c^{2}}}$$

$$|B| \propto \frac{1}{R_c}$$

- Total drift in a curved vacuum field (curvature + grad-B)

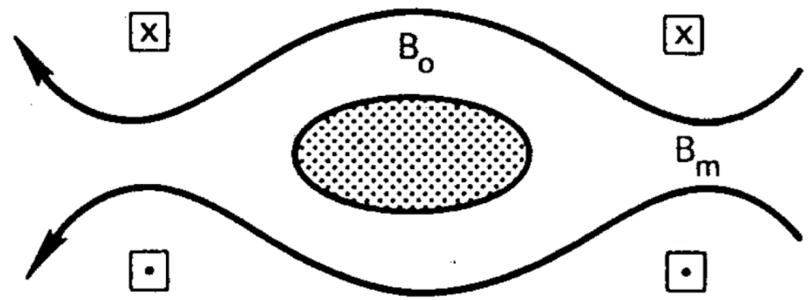
$$\boxed{\boldsymbol{v}_R + \boldsymbol{v}_{\nabla B} = \frac{m}{q}\frac{\boldsymbol{R}_c\times\boldsymbol{B}}{R_c^{2}B^{2}}\left(v_{\parallel}^{2} + \frac{1}{2}v_{\perp}^{2}\right)}$$

$$\frac{\boldsymbol{\nabla}|B|}{|B|} = -\frac{\boldsymbol{R}_c}{R_c^{2}}$$

$\boldsymbol{\nabla} B \parallel \boldsymbol{B}$: Magnetic mirror

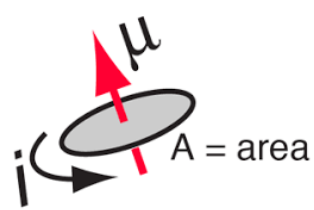
- Adiabatic invariant: Magnetic moment

$$\boxed{\mu = IA = \frac{q}{2\pi/\omega_c} \cdot \pi r_c^2 = \frac{\frac{1}{2} m {v_\perp}^2}{B}}$$



- As the particle moves into regions of stronger or weaker B, its Larmor radius changes, but μ remains invariant.

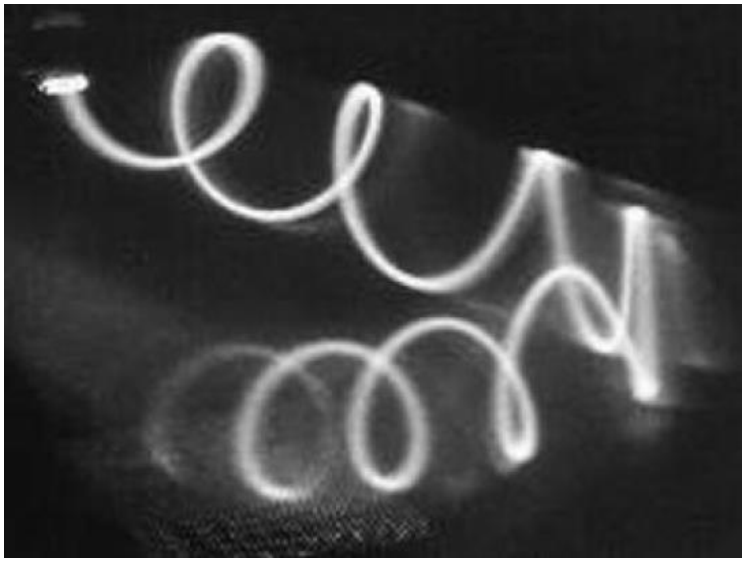
- Magnetic mirror



$$\frac{\frac{1}{2} m {v_{\perp 0}}^2}{B_0} = \frac{\frac{1}{2} m {v_{\perp m}}^2}{B_m}$$

$${v_{\perp m}}^2 + \cancel{v_{\parallel m}}^2 = {v_{\perp 0}}^2 + {v_{\parallel 0}}^2 \equiv {v_0}^2$$

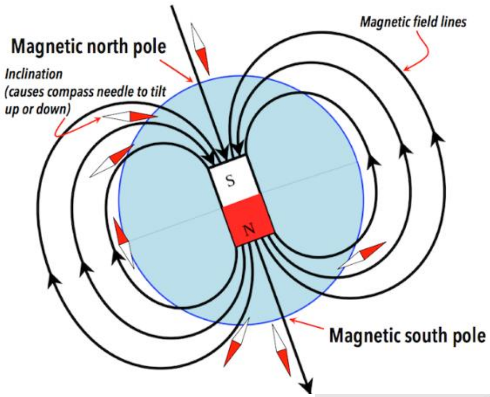
$$\frac{B_0}{B_m} = \frac{{v_{\perp 0}}^2}{{v_{\perp m}}^2} = \frac{{v_{\perp 0}}^2}{{v_0}^2} = \sin^2\theta$$

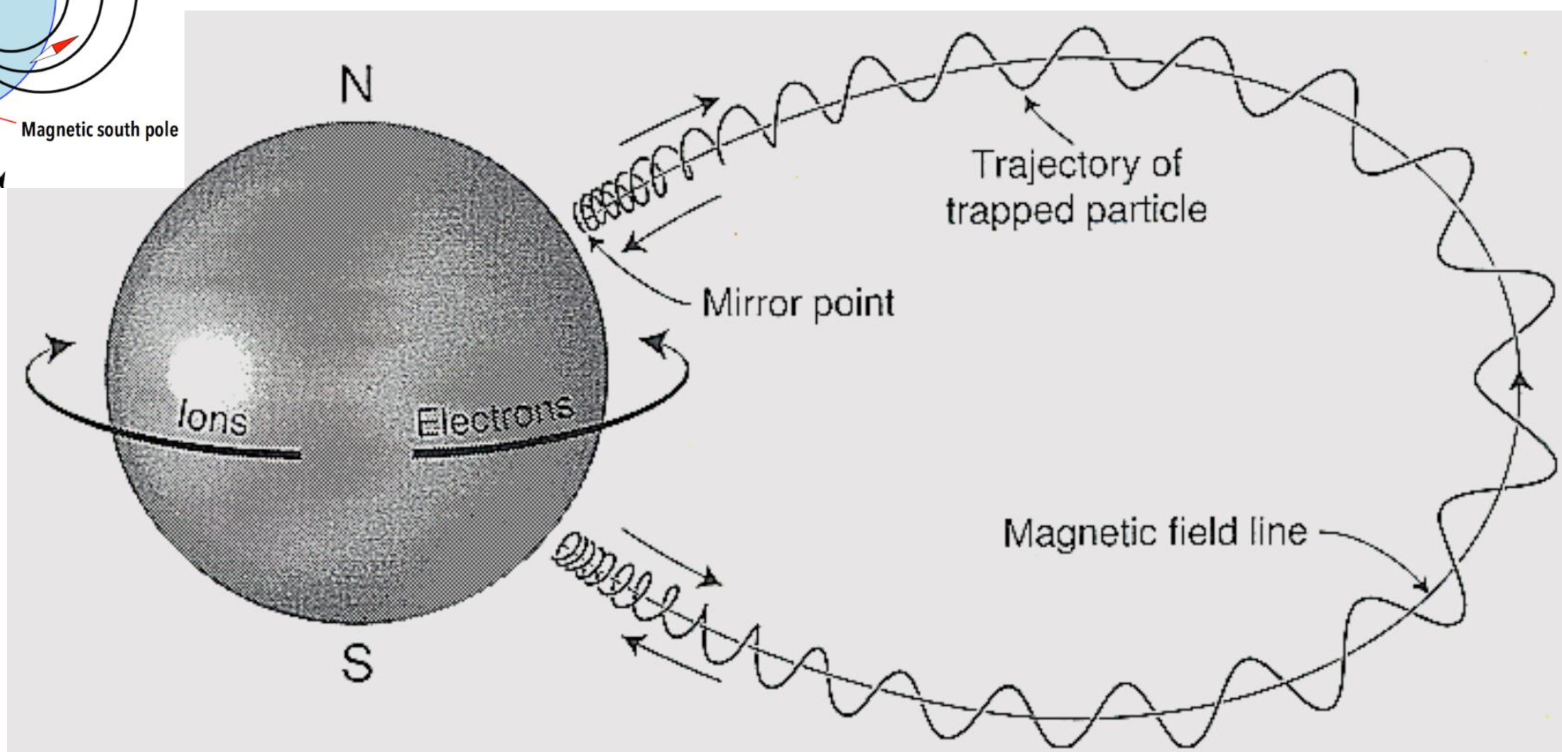


Motions of a charged particle in a dipole magnetic field

- Trajectories of particles confined in a dipole field

  → Particles experience gyro-, bounce- and drift- motions





Time-varying B field

- Since the Lorentz force is always perpendicular to v, a magnetic field itself cannot impart energy to a charged particle.

- However, time-varying magnetic field can accelerate the particles.

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

- The magnetic moment is invariant in slowly varying magnetic fields.

$$\mu = IA = \frac{\frac{1}{2} m v_{\perp}{}^{2}}{B}$$

- Generally, μ is invariant in spatially and temporally varying B fields.

  → How slowly-varying?

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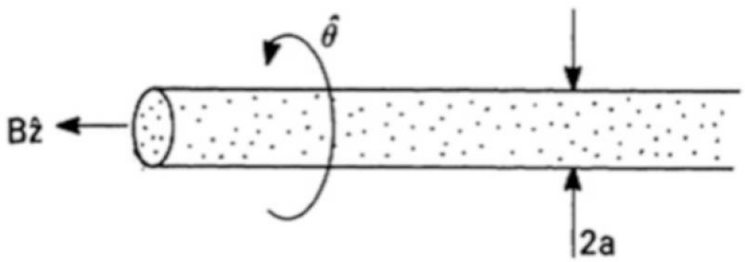
Homework

- F. Chen, Introduction to Plasma Physics and Controlled Fusion, Springer (2016), chapter 2 Problems: 2.1, 2.7, 2.21(a)

2.1. Compute r_L for the following cases if $v_{||}$ is negligible:

(a) A 10-keV electron in the earth's magnetic field of 5×10^{-5} T.
(b) A solar wind proton with streaming velocity 300 km/s, $B = 5 \times 10^{-9}$ T.
(c) A 1-keV He^+ ion in the solar atmosphere near a sunspot, where $B = 5 \times 10^{-2}$ T.
(d) A 3.5-MeV He^{++} ash particle in an 8-T DT fusion reactor.

2.7. An unneutralized electron beam has density $n_e = 10^{14}$ m^{-3} and radius $a = 1$ cm and flows along a 2-T magnetic field. If **B** is in the $+z$ direction and **E** is the electrostatic field due to the beam's charge, calculate the magnitude and direction of the **E** × **B** drift at $r = a$ (See Fig. P2.7).



2.21. An infinite straight wire carries a constant current I in the $+z$ direction. At $t = 0$, an electron of small gyroradius is at $z = 0$ and $r = r_0$ with $v_{\perp 0} = v_{||0}$. ($\perp$ and $||$ refer to the direction relative to the magnetic field.)

(a) Calculate the magnitude and direction of the resulting guiding center drift velocity.