Fundamentals of Electric and Magnetic Fields

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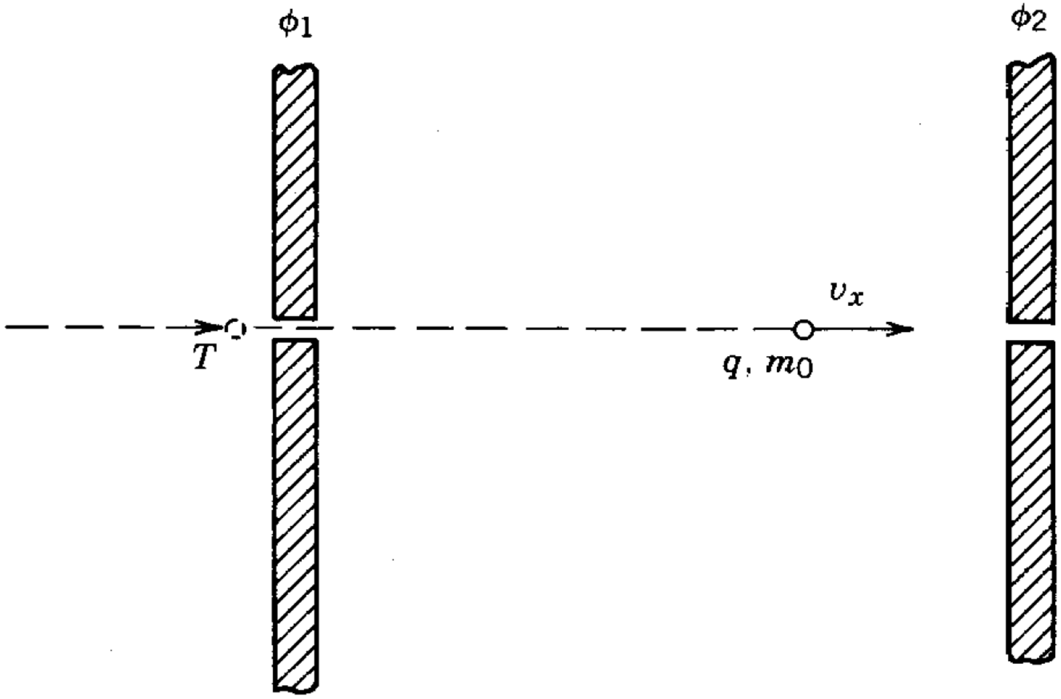
Maxwell's equations

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}}$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)*
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

*For a stationary surface S.

Particle acceleration by static electric field



$$T = (\gamma - 1)m_0c^2 = -q\Delta\phi = -q(\phi_2 - \phi_1)$$

$$\gamma = 1 - \frac{q\Delta\phi}{m_0c^2}$$

Inductive voltage and displacement current

$$\Psi = \iint \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, dS$$

$$V = -\frac{d\Psi}{dt}$$

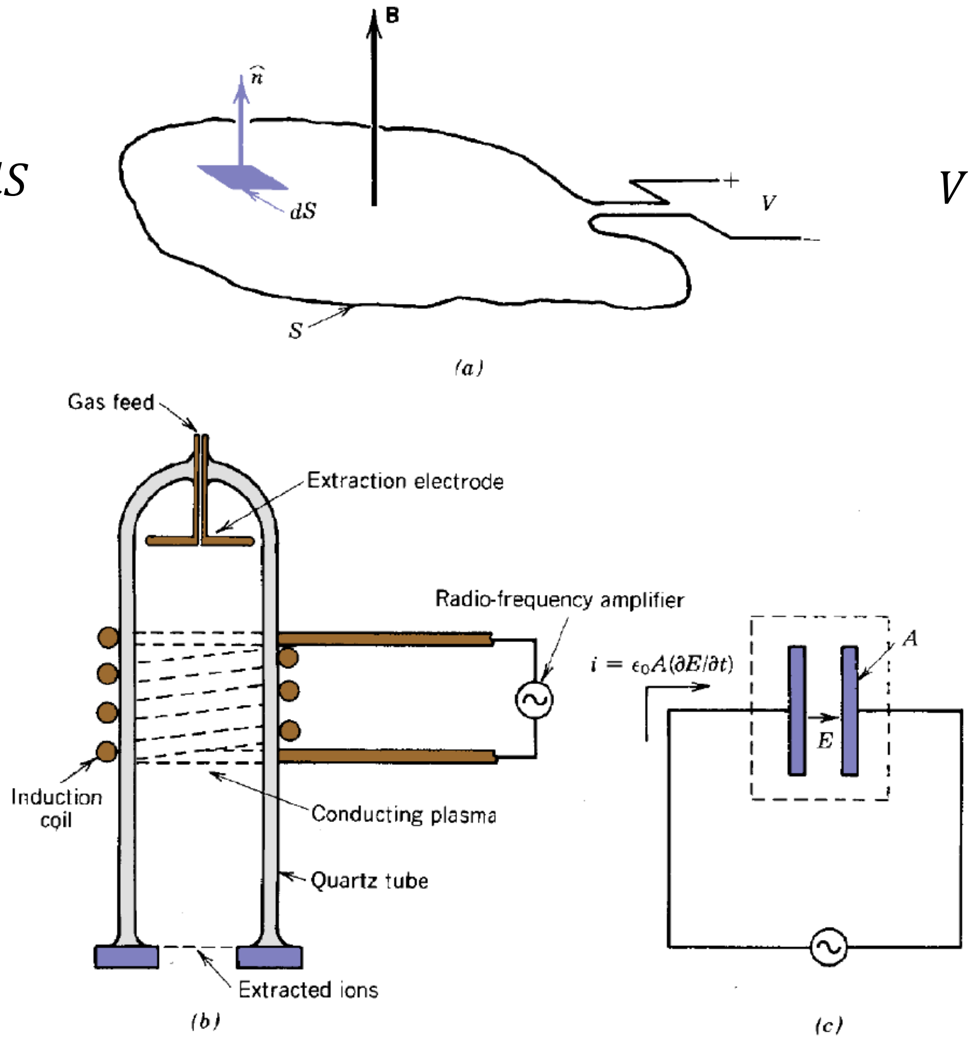


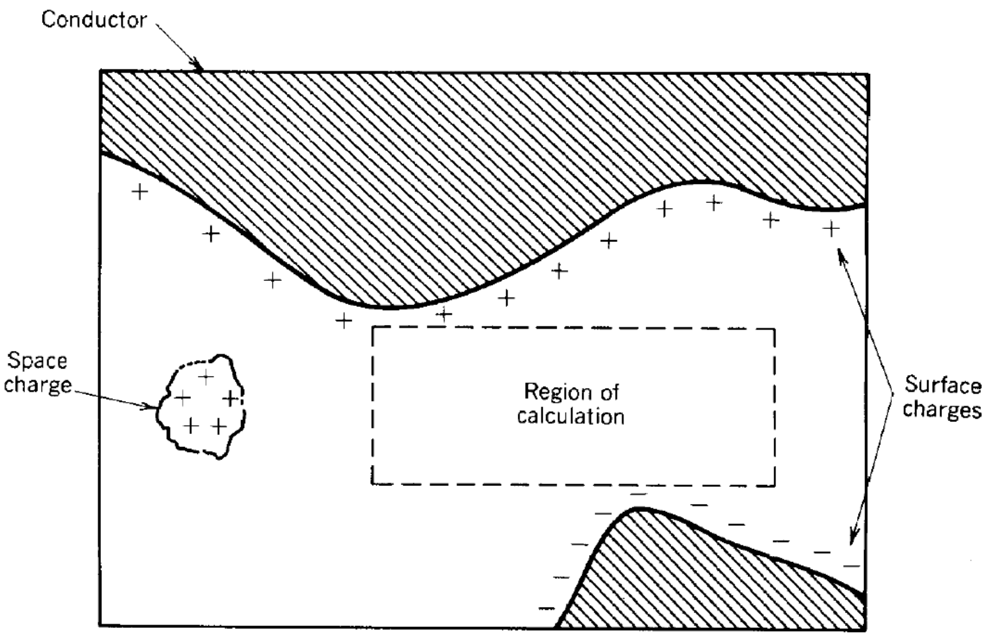
Figure 3.7 Inductive voltage and displacement currents. (*a*) Faraday's law. (*b*) Inductively coupled plasma source. (*c*) Alternating-current circuit with a parallel-plate capacitor.

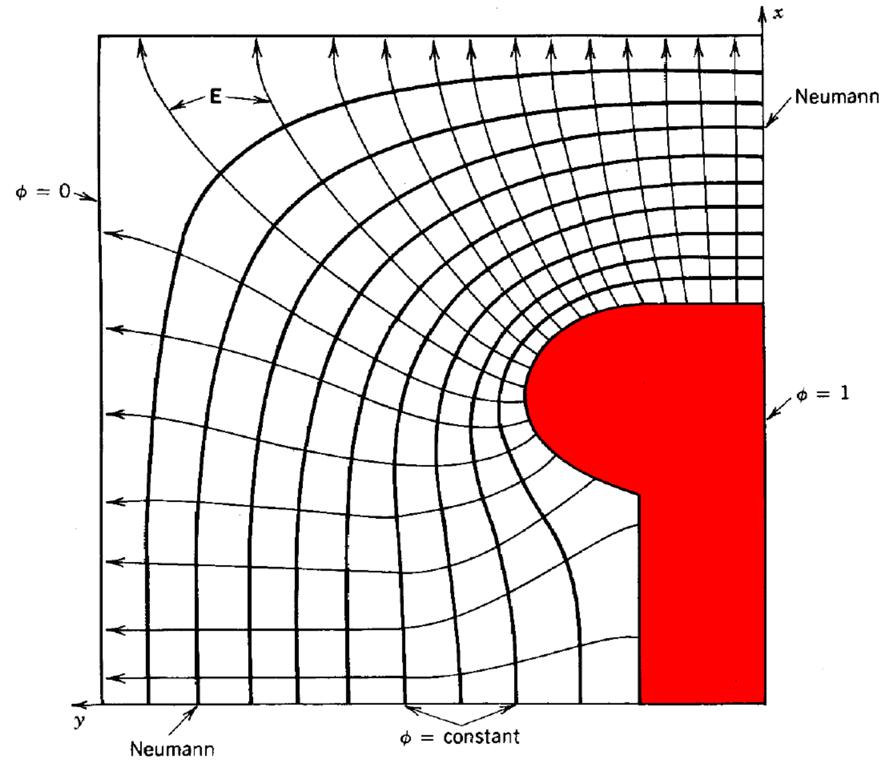
Static field equations with no sources

- When there are no charges or currents present, the Maxwell equations have the following form

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \times \boldsymbol{E} = 0$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = 0$$





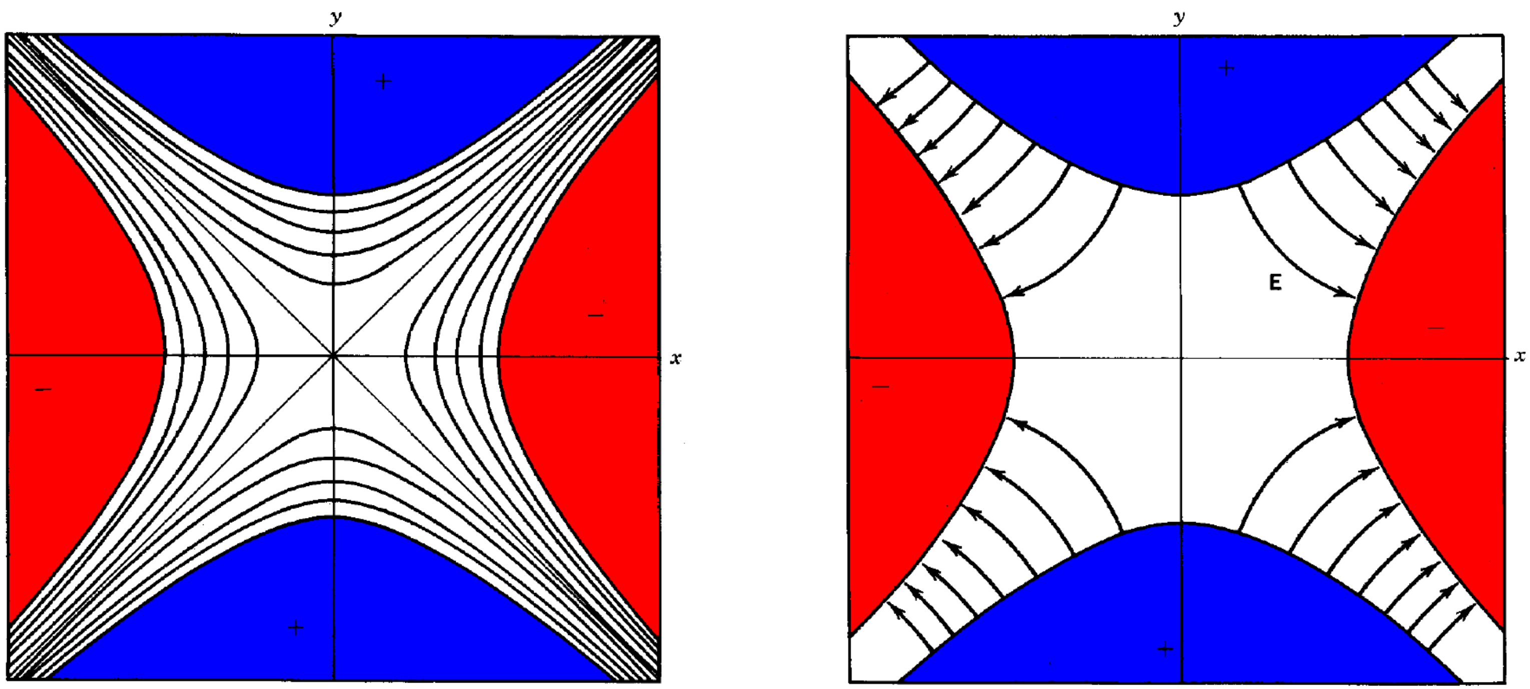
Electrostatic quadrupole field

- The desired two-dimensional electric field distribution:

$$E_x = -\frac{\partial \varphi}{\partial x} = +kx = E_0 \frac{x}{a} \qquad E_y = -\frac{\partial \varphi}{\partial y} = -ky = -E_0 \frac{y}{a}$$

- We obtain the following hyperbolic potential function with $\varphi(0,0) = 0$

$$\varphi(x,y) = \frac{E_0}{2a}(y^2 - x^2) \qquad \frac{\varphi(x,y)}{E_0 a/2} = \left(\frac{y}{a}\right)^2 - \left(\frac{x}{a}\right)^2$$



Static electric fields with space charge

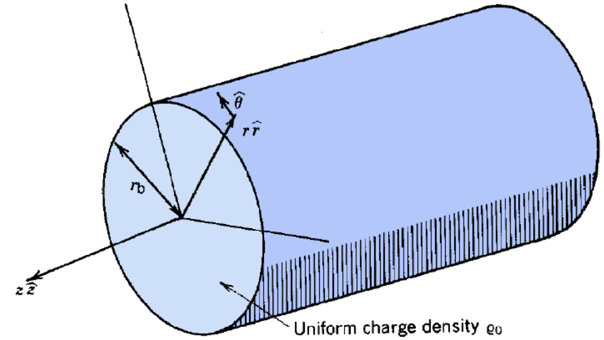
- Space charge is charge density present in the region in which an electric field is to be calculated. In accelerator applications, space charge is identified with the charge of the beam; it must be included in calculations of fields internal to the beam.

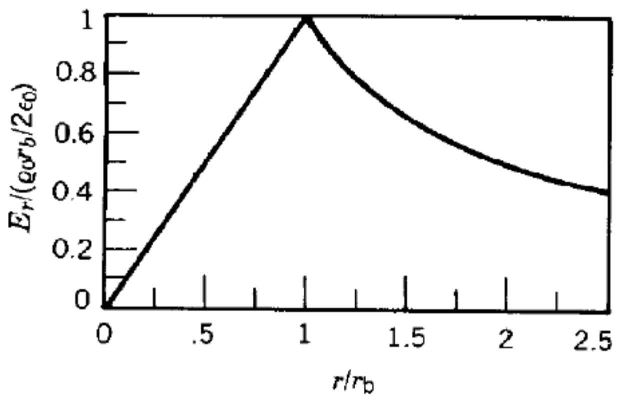
$$\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho = \underbrace{\rho_1}_{\text{applied}} + \underbrace{\rho_2}_{\text{dielectric}} + \underbrace{\rho_3}_{\text{space (free)}}$$

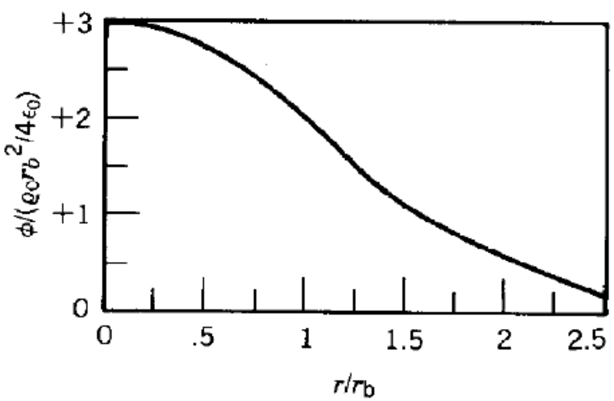
- For a cylindrical beam with uniform charge density:

$$E_r = (\rho_0/(2\epsilon_0))r, \qquad (0 \le r \le r_b)$$

$$E_r = (\rho_0/(2\epsilon_0))(r_b^2/r), \qquad (r_b \le r)$$







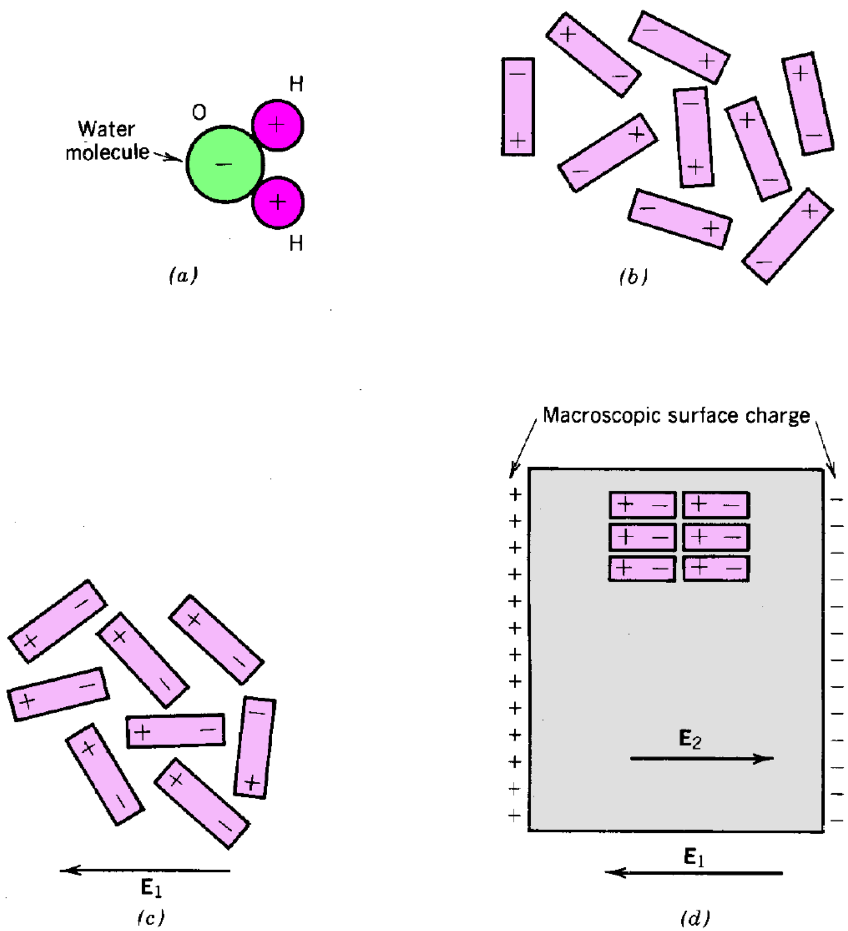
Dielectrics

- If a dielectric is inserted into a vacuum field region, the following equations hold:

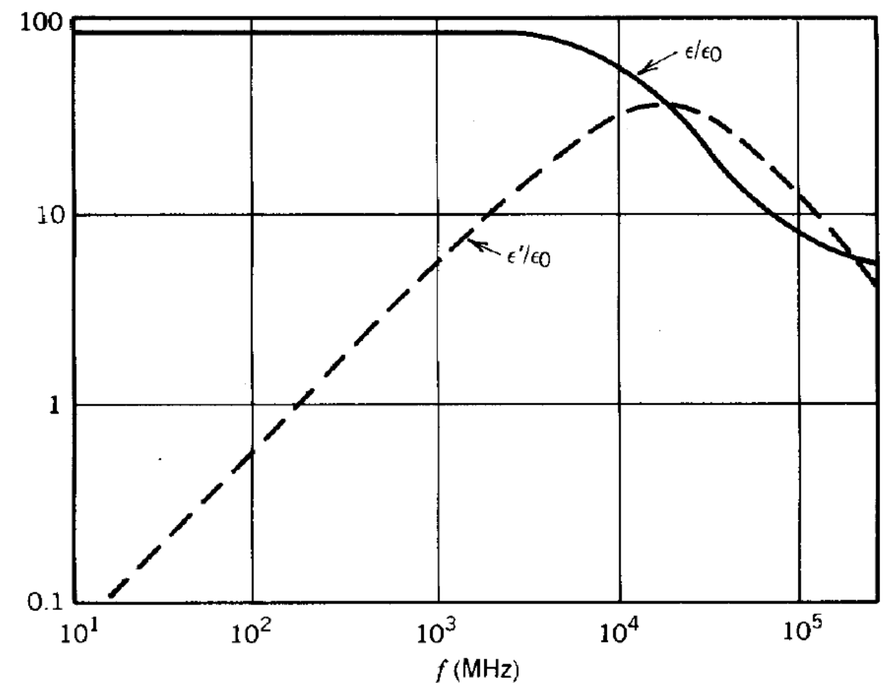
$$\nabla \cdot \boldsymbol{D} = 0 \qquad \nabla \cdot \boldsymbol{E} \neq 0$$

$$\nabla \times \boldsymbol{H} = (\sigma + j\omega\epsilon)\boldsymbol{E} = j\omega\left(\epsilon - j\frac{\sigma}{\omega}\right)\boldsymbol{E} = j\omega\epsilon_c\boldsymbol{E}$$

ϵ'



Relative permittivity of water



Plasma as a dielectric

- It is a common practice to introduce the concept of a plasma dielectric constant to describe phenomena such as the refraction of optical radiation.

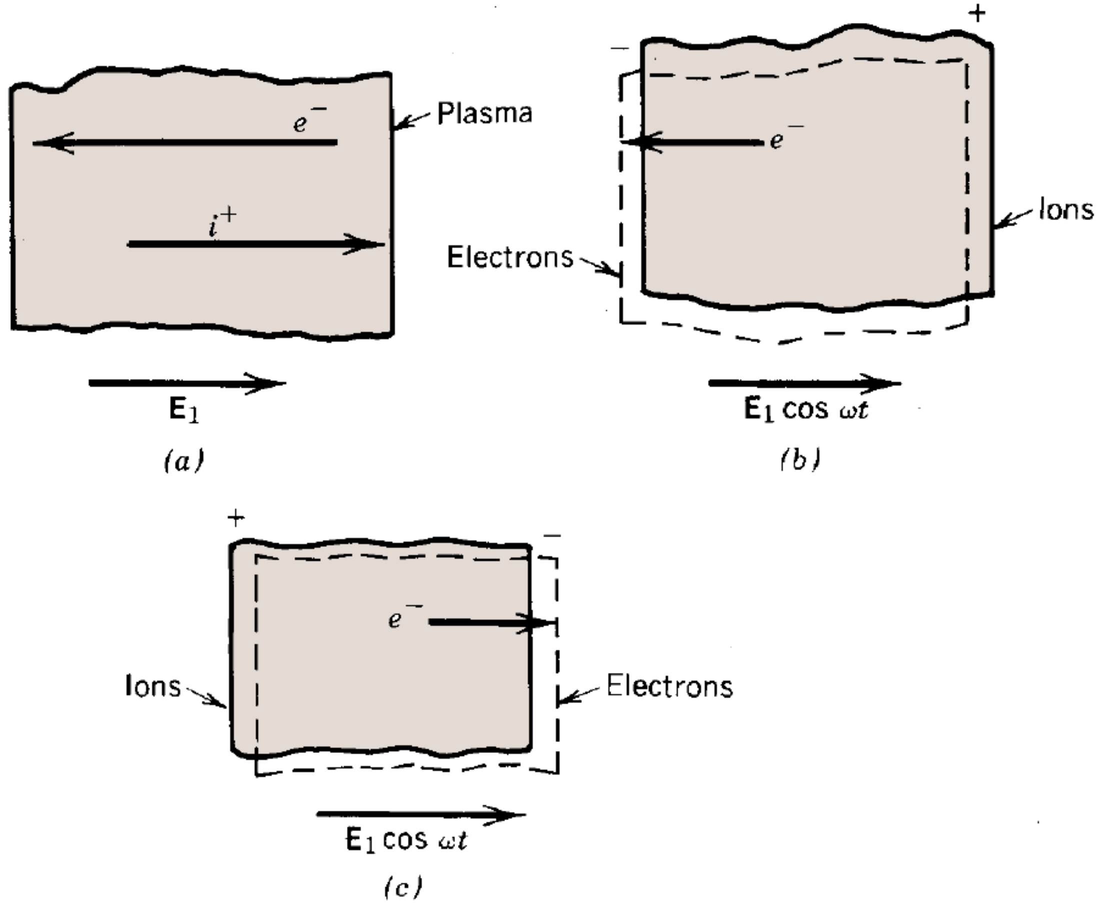


Figure 5.4 Response of particles in a plasma slab to an oscillating applied electric field. (*a*) Direct-current field (zero frequency). (*b*) Low-frequency ac field. (*c*) High-frequency ac field.

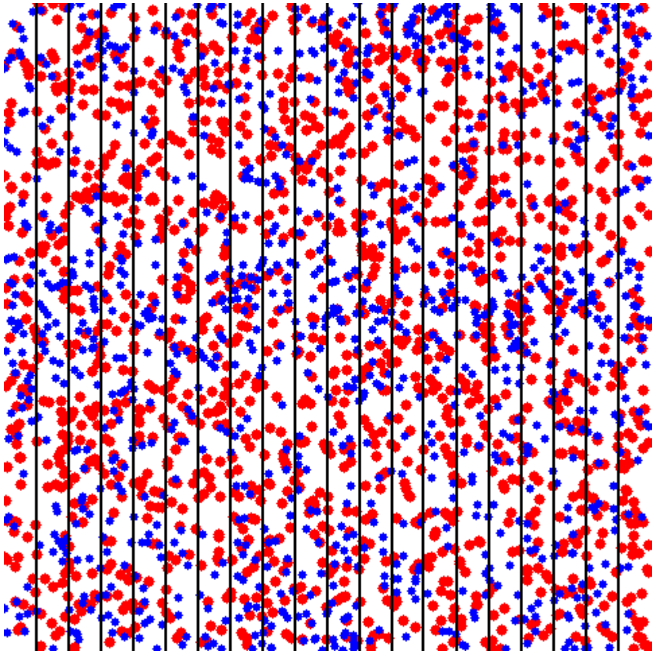
Plasma oscillation

- Electrons overshoot by inertia and oscillate around their equilibrium position with a characteristic frequency known as plasma frequency.

- Equation of motion (cold plasma)

$$m\frac{d^2\Delta x}{dt^2} = -eE_x = -e\frac{n_0 e\Delta x}{\epsilon_0}$$

$$\frac{d^2\Delta x}{dt^2} + \frac{n_0 e^2}{m\epsilon_0}\Delta x = 0 \quad \leftarrow \text{Harmonic oscillator}$$



- Electron plasma frequency

$$\boxed{\omega_{pe} = \left(\frac{n_0 e^2}{m\epsilon_0}\right)^{1/2}}$$

- If the assumption of infinite mass ions is not made, then the ions also move slightly and we obtain the natural frequency

$$\omega_p = \left(\omega_{pe}^2 + \omega_{pi}^2\right)^{1/2}$$

where, $\omega_{pi} = (n_0 e^2/M\epsilon_0)^{1/2}$ (ion plasma frequency)

Plasma frequency

- Plasma oscillation frequency for electrons and ions

$$f_{pe} = \frac{\omega_{pe}}{2\pi} = 8980\sqrt{n_0}\ [\mathrm{Hz}] \quad (n_0 \text{ in cm}^{-3})$$

$$f_{pi} = \frac{\omega_{pi}}{2\pi} = 210\sqrt{n_0/M_A}\ [\mathrm{Hz}] \quad (n_0 \text{ in cm}^{-3}, M_A \text{ in amu})$$

- Typical values for a processing plasma (Ar)

$$f_{pe} = \frac{\omega_{pe}}{2\pi} = 8980\sqrt{10^{10}}\ [\mathrm{Hz}] = 9 \times 10^{8}\ [\mathrm{Hz}]$$

$$f_{pi} = \frac{\omega_{pi}}{2\pi} = 210\sqrt{10^{10}/40}\ [\mathrm{Hz}] = 3.3 \times 10^{6}\ [\mathrm{Hz}]$$

$$\lambda_{\mathrm{De}} = \frac{v_{\mathrm{th}}}{\omega_{\mathrm{pe}}}$$

- Collective behavior

$$\boxed{\omega_{pe}\tau > 1}$$

Plasma frequency > collision frequency

Plasma response in time-varying electric field

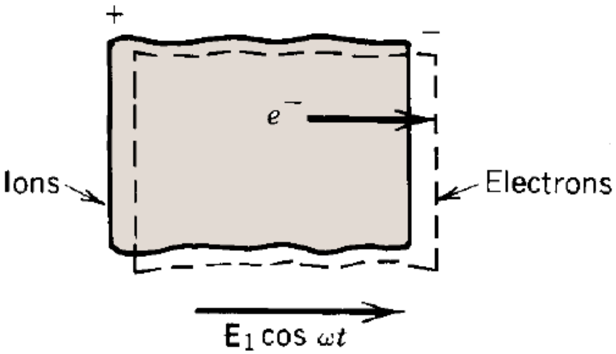
- Consider a uniform plasma in the presence of a background gas that is driven by a small amplitude time-varying electric field:

$$E_x(t) = \tilde{E}_x \cos \omega t = \mathrm{Re}\, \tilde{E}_x e^{j\omega t}$$

$$u_x(t) = \mathrm{Re}\, \tilde{u}_x e^{j\omega t}$$

- The electron force equation

$$m \frac{du_x}{dt} = -eE_x - m\nu_m u_x$$



- The complex velocity amplitude

$$\tilde{u}_x = -\frac{e}{m} \frac{1}{j\omega + \nu_m} \tilde{E}_x$$

- The total current amplitude (displacement current + conduction current)

$$\tilde{J}_{Tx} = j\omega\epsilon_0 \tilde{E}_x - en_0 \tilde{u}_x = j\omega\epsilon_0 \left[1 - \frac{\omega_{pe}^2}{\omega(\omega - j\nu_m)} \right] \tilde{E}_x$$

Dielectric constant and conductivity

- Plasma dielectric constant

$$\epsilon_p = \epsilon_0 \kappa_p = \epsilon_0 \left[1 - \frac{\omega_{pe}^2}{\omega(\omega - j\nu_m)} \right]$$

$$\nabla \times \widetilde{\boldsymbol{H}} = j\omega\epsilon_p \widetilde{\boldsymbol{E}}$$

- Plasma conductivity

$$\tilde{J}_{Tx} = (\sigma_p + j\omega\epsilon_0)\tilde{E}_x$$

$$\nabla \times \widetilde{\boldsymbol{H}} = (\sigma_p + j\omega\epsilon_0)\widetilde{\boldsymbol{E}}$$

$$\sigma_p = \frac{\epsilon_0 \omega_{pe}^2}{j\omega + \nu_m}$$

- Low frequency ($\omega \ll \nu_m$): dc plasma conductivity

$$\sigma_{dc} = \frac{\epsilon_0 \omega_{pe}^2}{\nu_m} = \frac{n_0 e^2}{m\nu_m}$$

- High frequency ($\omega \gg \nu_m$): collisionless plasma dielectric constant

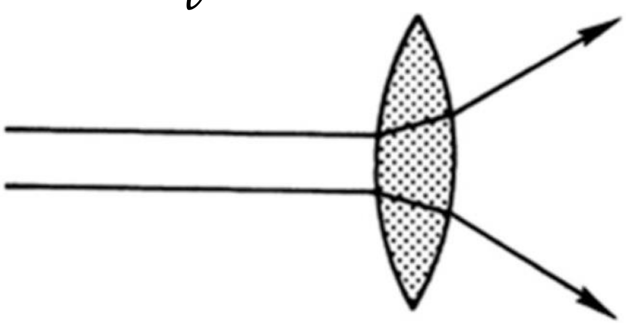
$$\epsilon_p = \epsilon_0 \kappa_p = \epsilon_0 \left[1 - \frac{\omega_{pe}^2}{\omega^2} \right]$$

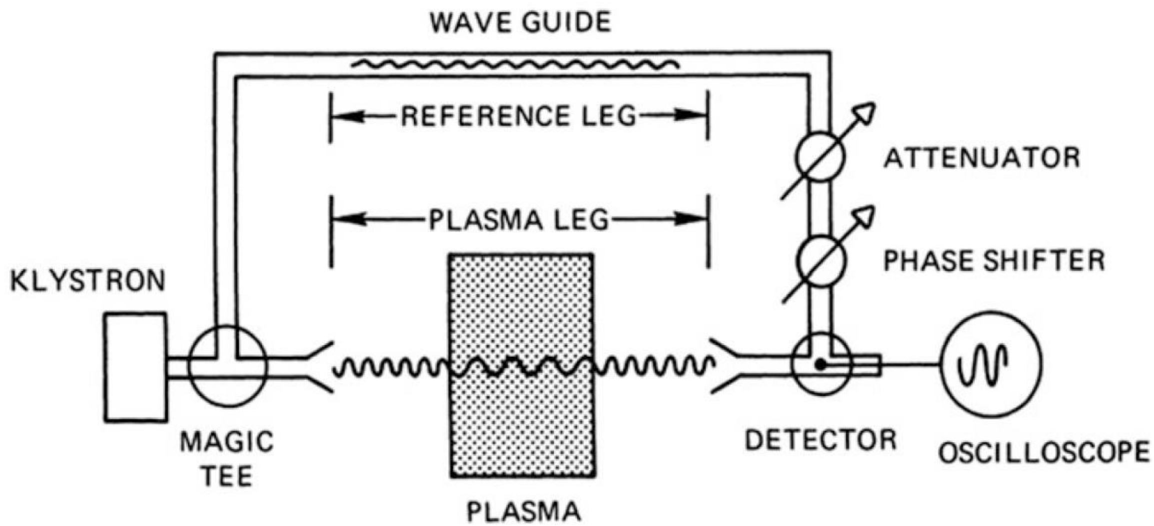
Behaviors in typical low-pressure rf discharges

- At very high driving frequencies ($\omega > \omega_{pe}$): ϵ_p is positive but less than ϵ_0

  → The plasma acts as a dielectric with a relative dielectric constant less than 1.

$$n = \frac{c}{v} \sim \sqrt{\epsilon}$$

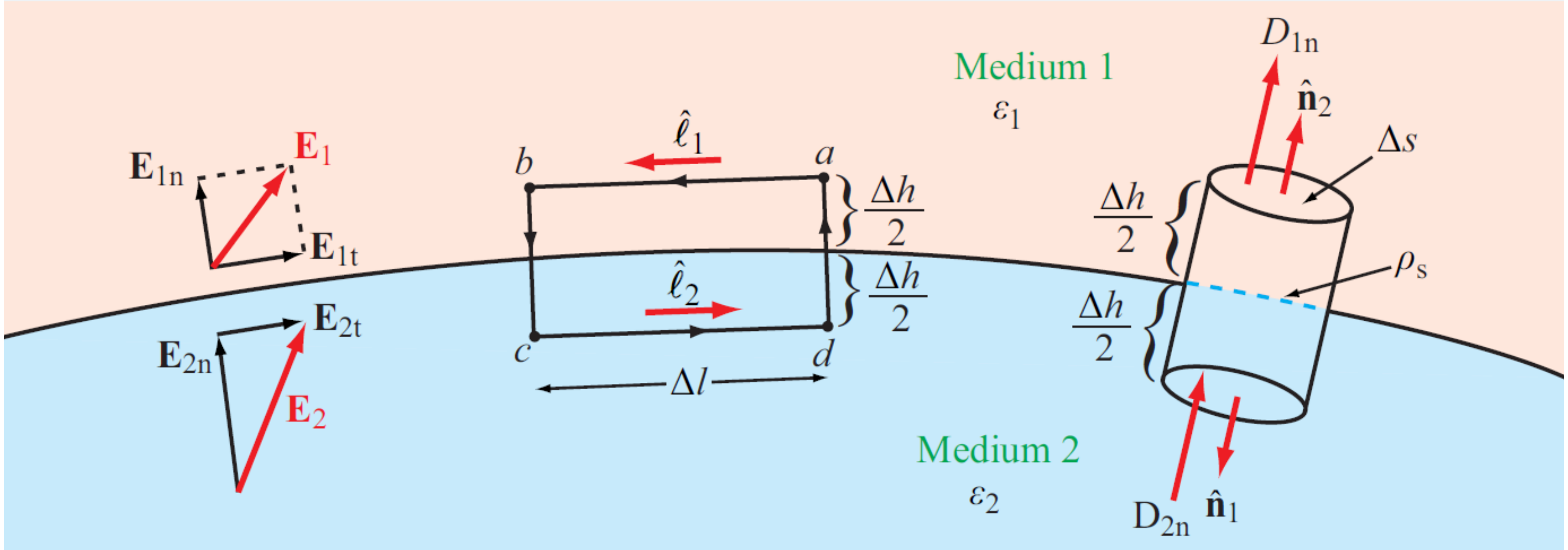




- Most discharges are driven at lower frequencies ($\omega < \omega_{pe}$): ϵ_p is negative

  → The plasma behaves like an inductor in this frequency regime.

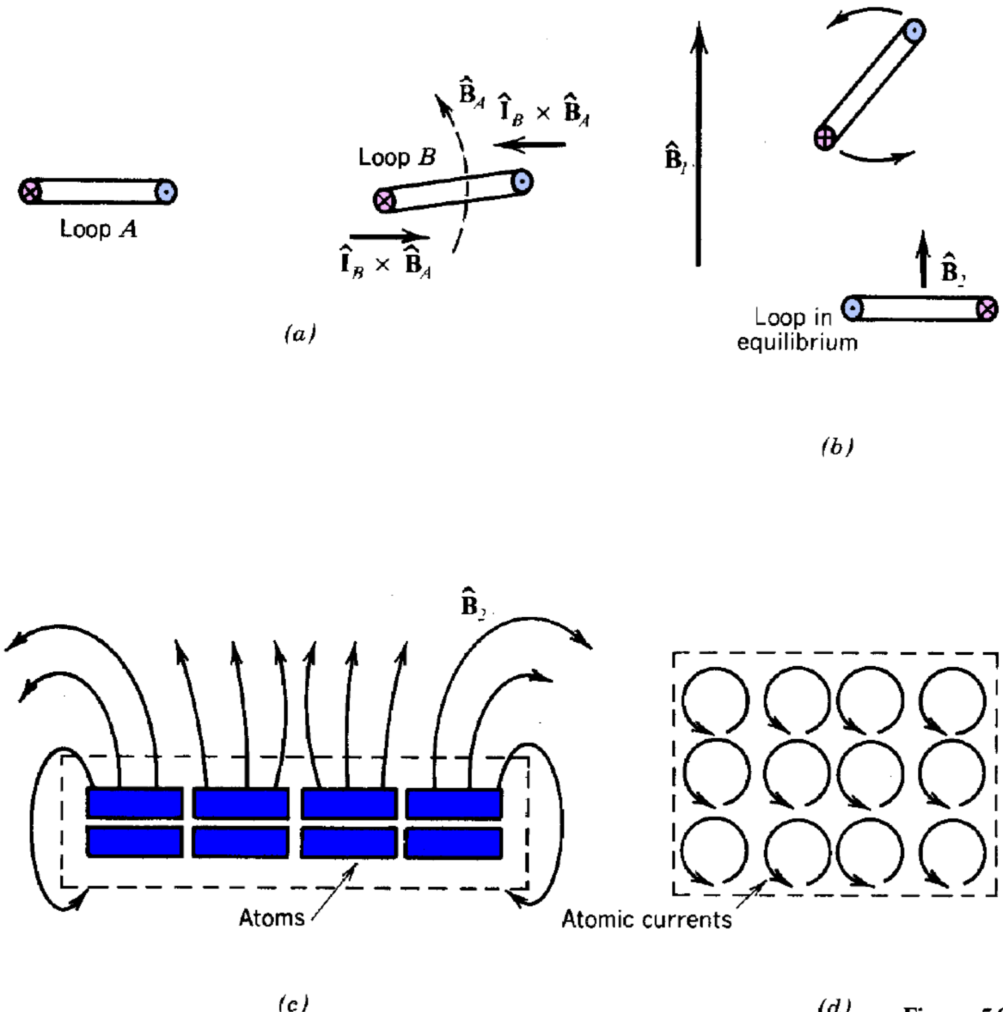
Boundary conditions at dielectric surfaces

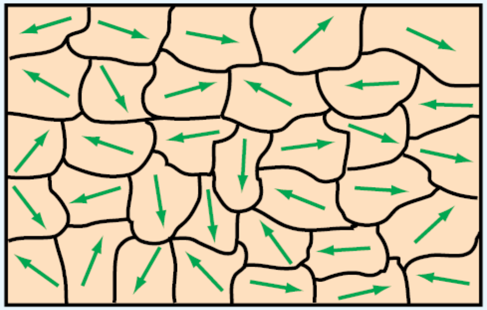


Field Component	Any Two Media	Medium 1 Dielectric ϵ_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\epsilon_1 = \mathbf{D}_{2t}/\epsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\epsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

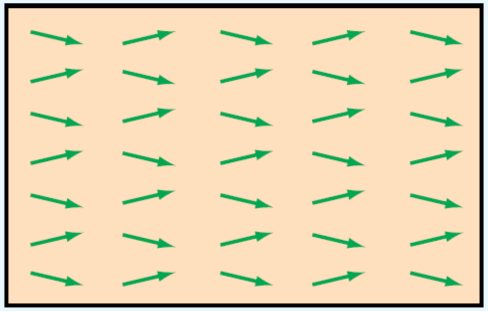
Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Behavior of magnetic materials





(a) Unmagnetized domains

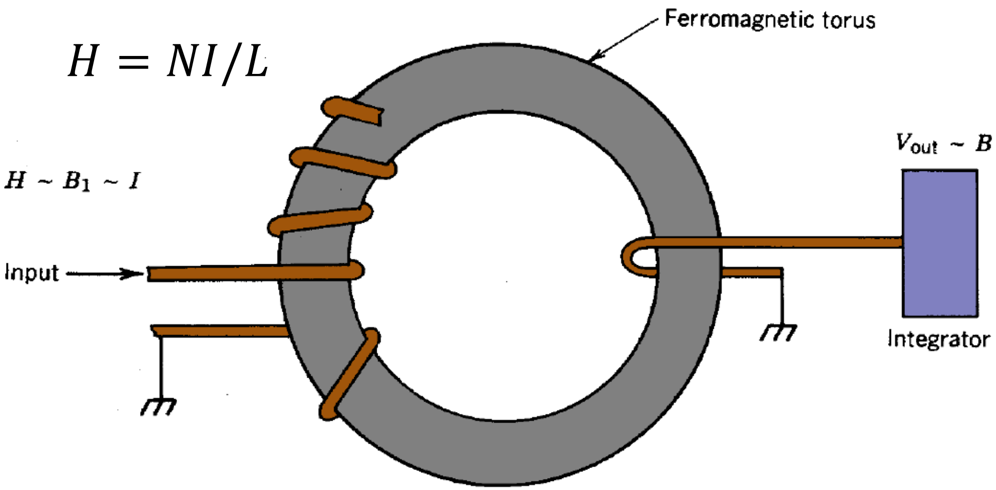


(b) Magnetized domains

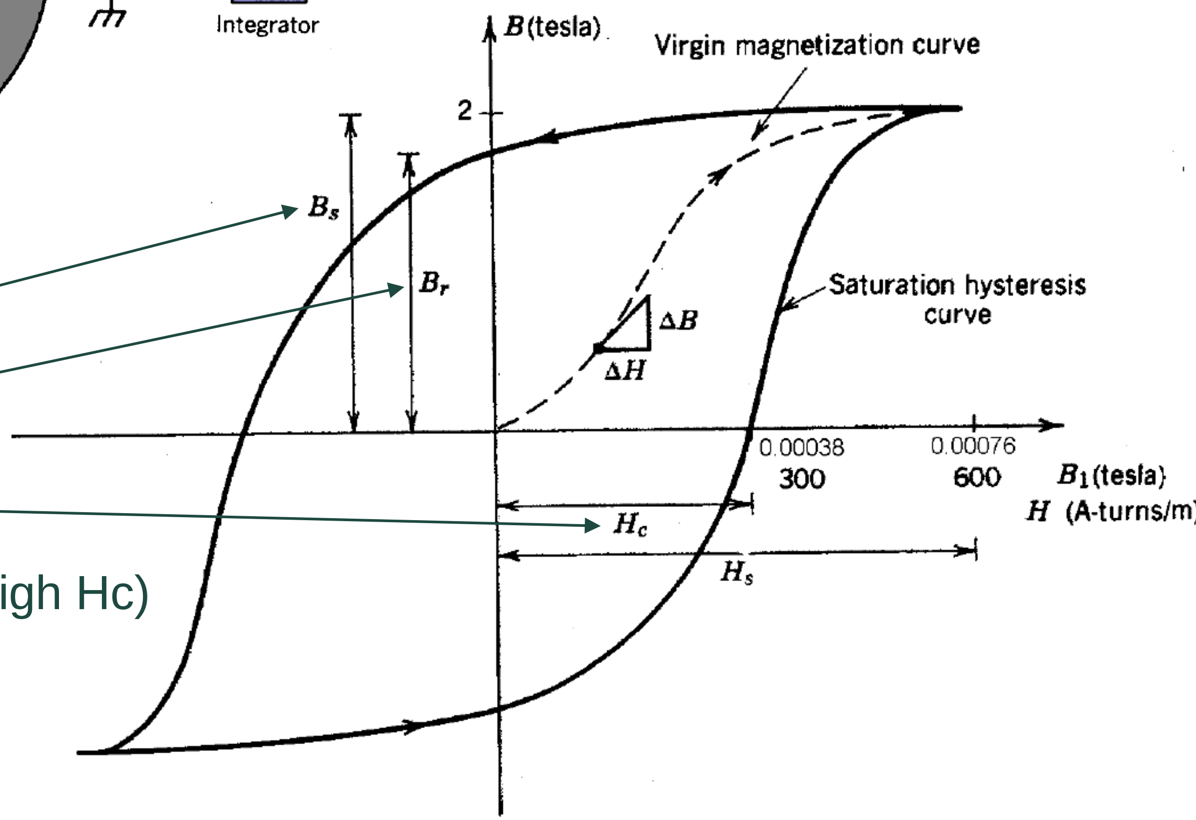
Figure 5.9 Behavior of magnetic materials. (*a*) Interactions between atoms with magnetic moments; force exerted on one current loop (*B*) by another (*A*). (*b*) Response of atoms in a paramagnetic material to an applied magnetic field. (*c*) Macroscopic magnetic fields produced by alignment of atomic currents in a material. (*d*) Macroscopic surface current in a material resulting from alignment of atoms with magnetic moments.

Static hysteresis curve for ferromagnetic materials

$$H = NI/L$$



$$BA = \int V_{out}(t)dt$$



Saturation induction

Remanence flux

Coercive force

Soft (small Hc) vs hard (high Hc)

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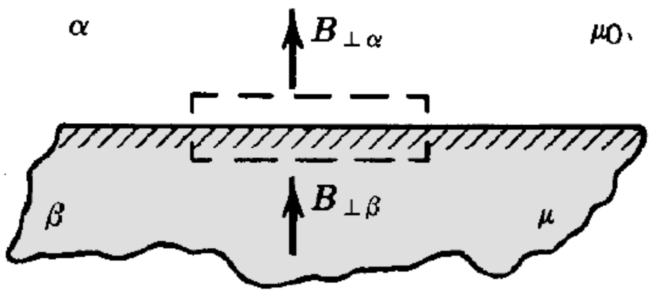
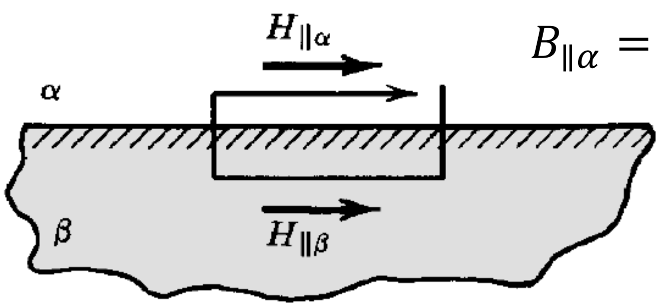
Magnetic poles

- Boundary conditions at a boundary between vacuum and a ferromagnetic material

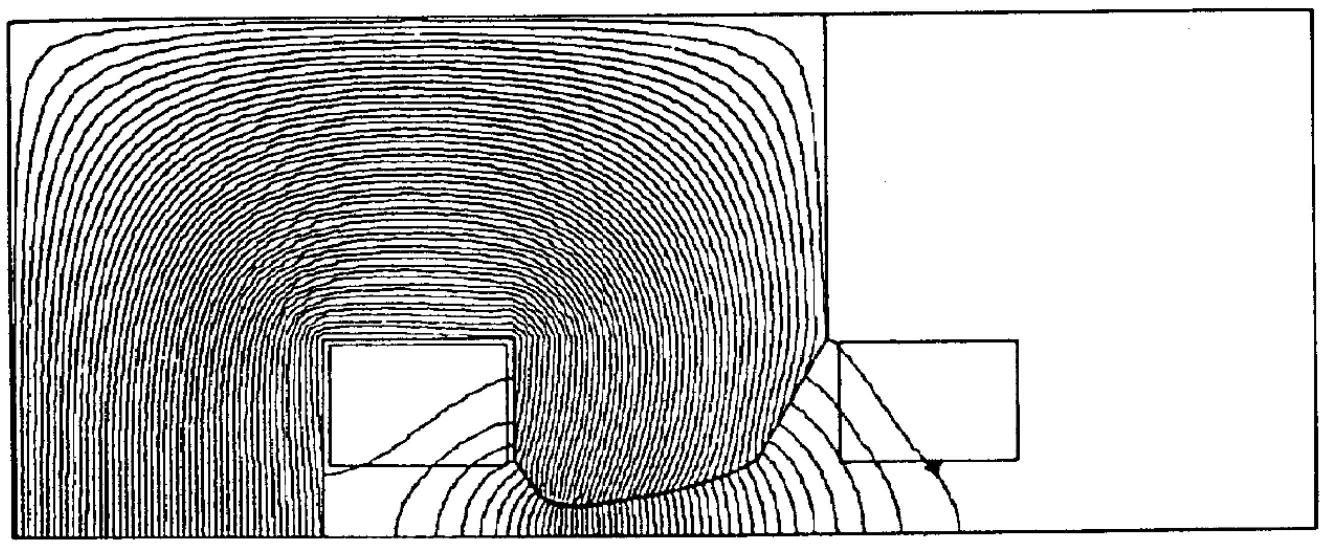


(a)



(b)

$$B_{\|\alpha} = \frac{\mu_0}{\mu} B_{\|\beta}$$



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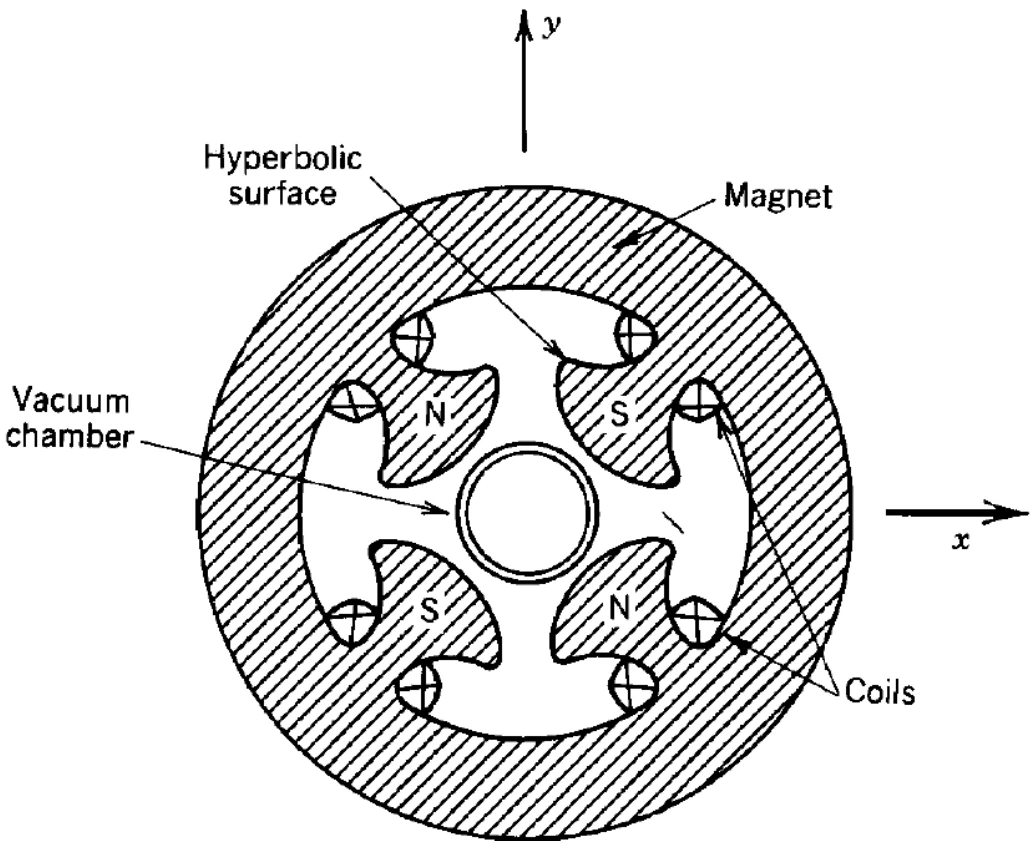
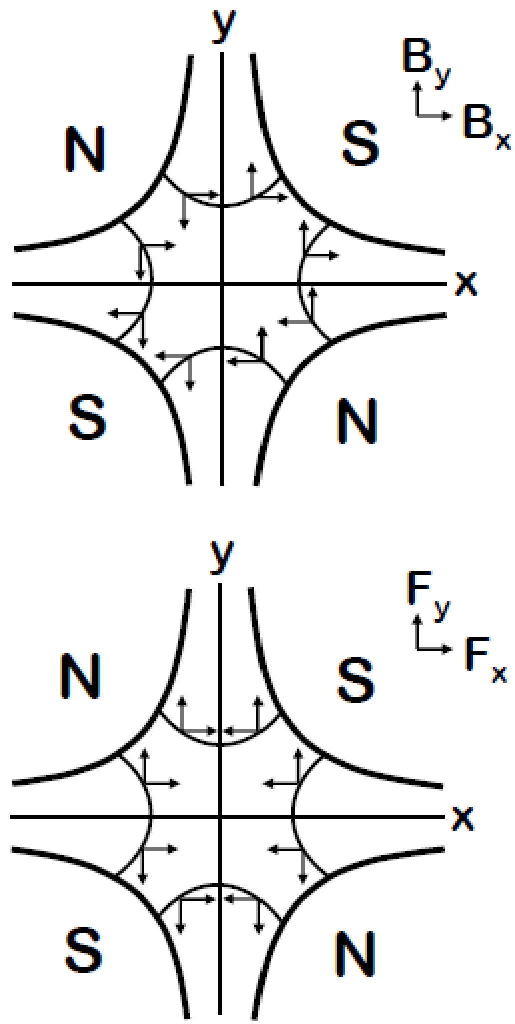
Magnetic quadrupole lens



$$B_x = B_0 \frac{y}{a} \qquad B_y = B_0 \frac{x}{a}$$



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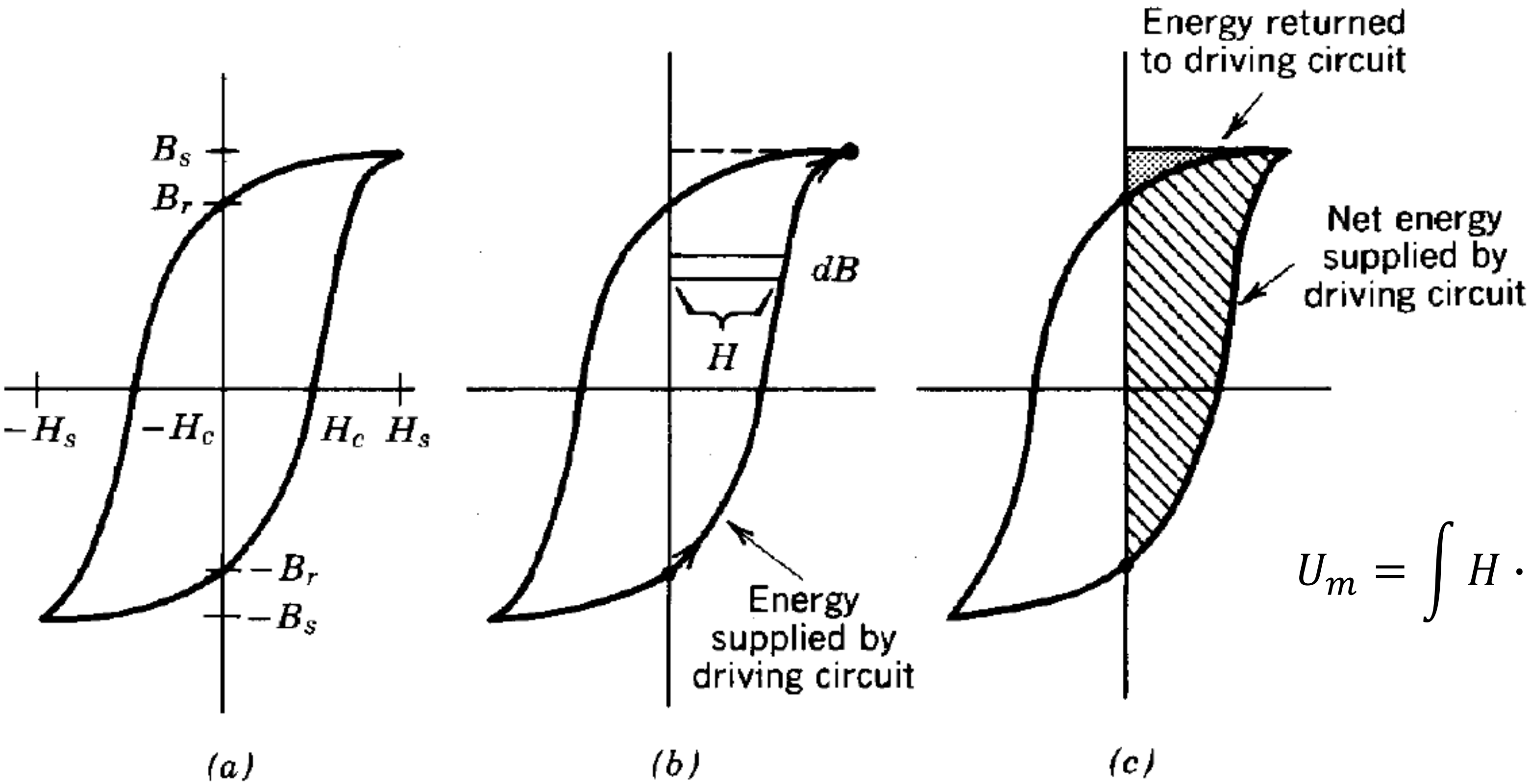
Energy required to magnetize a ferromagnetic material



$$U_m = \int H \cdot dB$$

Figure 5.17 Energy required to magnetize and demagnetize a ferromagnetic material. (*a*) Saturation hysteresis curve. (*b*) Quantities to calculate energy changes moving along saturation hysteresis curve. (*c*) Energy supplied by circuit (cross-hatched area) and energy returned to circuit (shaded portion).

Advantage of including ferromagnetic material

- The ampere turn product is related to the magnetic field in the circuit through:

$$\int \left(\frac{B}{\mu}\right) \cdot dl = NI$$

- The constant circuit flux is given by

$$\Psi = B_g A_g = B_c A_c$$

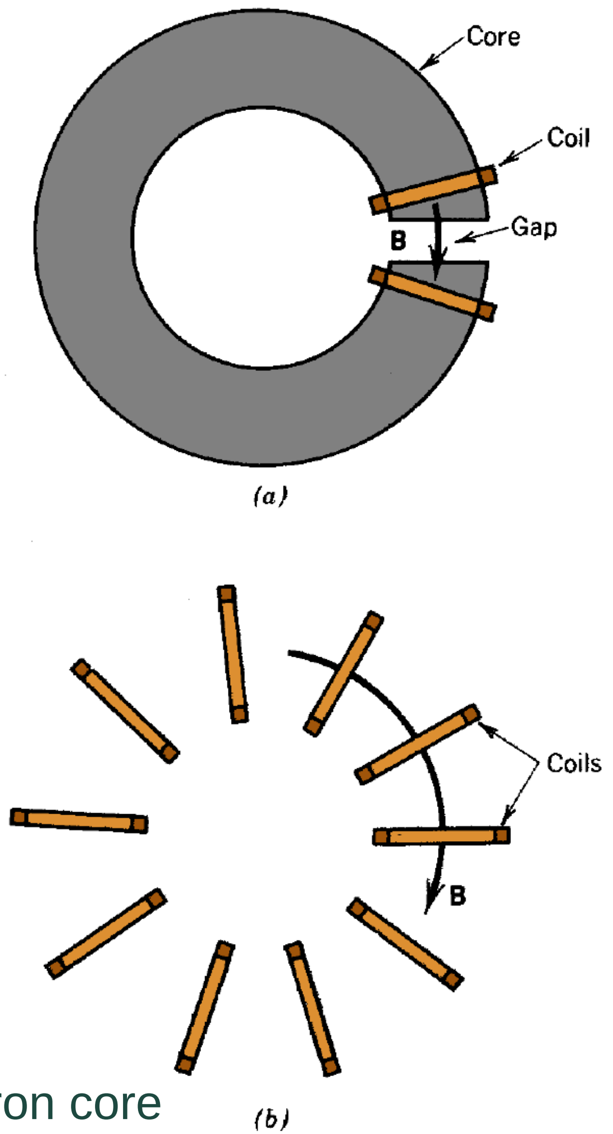
- For the air core circuit:

$$B_g \left(\frac{g}{\mu_0}\right) + B_c \left(\frac{l}{\mu_0}\right) = \Psi \left(\frac{g}{A_g \mu_0} + \frac{l}{A_c \mu_0}\right) = NI$$

- For the ferromagnetic core circuit:

$$B_g \left(\frac{g}{\mu_0}\right) + B_c \left(\frac{l}{\mu}\right) = \Psi \left(\frac{g}{A_g \mu_0} + \frac{l}{A_c \mu}\right) = NI$$

Reluctance of the air gap Reluctance of the iron core

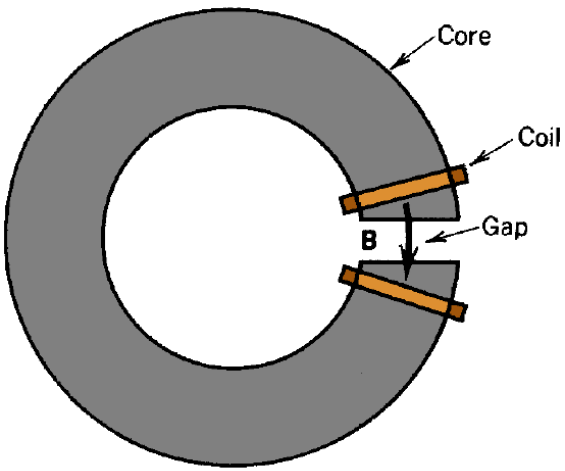


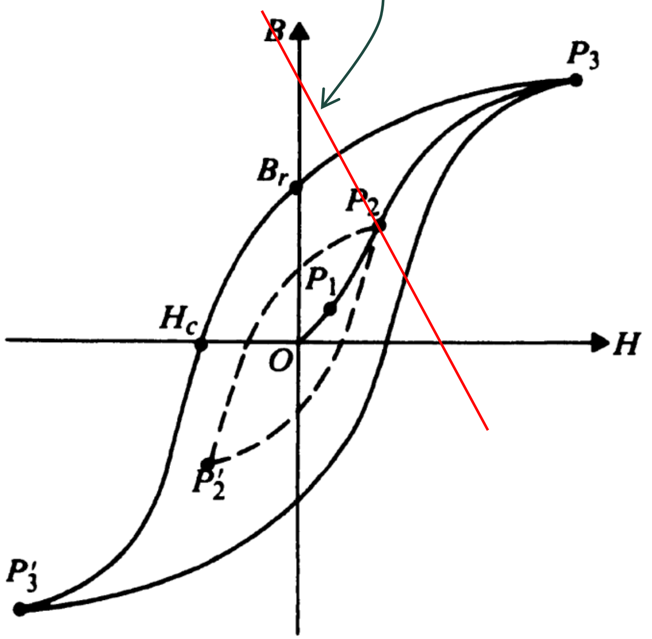
Magnetic circuits: operating point

- For the ferromagnetic core circuit for $A_g = A_c$:

$$B_g\left(\frac{g}{\mu_0}\right) + B_c\left(\frac{l}{\mu}\right) = NI \qquad B_g = B_c \qquad B_c = \mu H_c$$

$$B_c\left(\frac{g}{\mu_0}\right) + H_c l = NI \quad \Rightarrow \quad B_c = -\left(\frac{\mu_0 l}{g}\right)H_c + \frac{\mu_0 NI}{g}$$





Magnetic circuits: analogy with electric circuits

- The magnetic circuit has many analogies with electric circuits in which electrons circulate.
- The excitation windings provide the motive force (voltage), the vacuum gap is the load (resistance), and the ferromagnetic material completes the circuit (conducting wire).

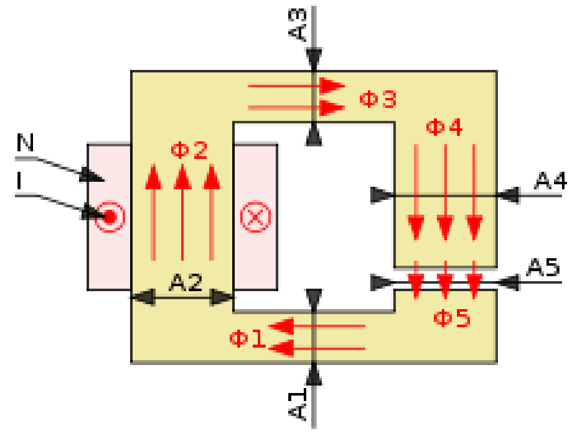
Analogy between 'magnetic circuits' and electrical circuits

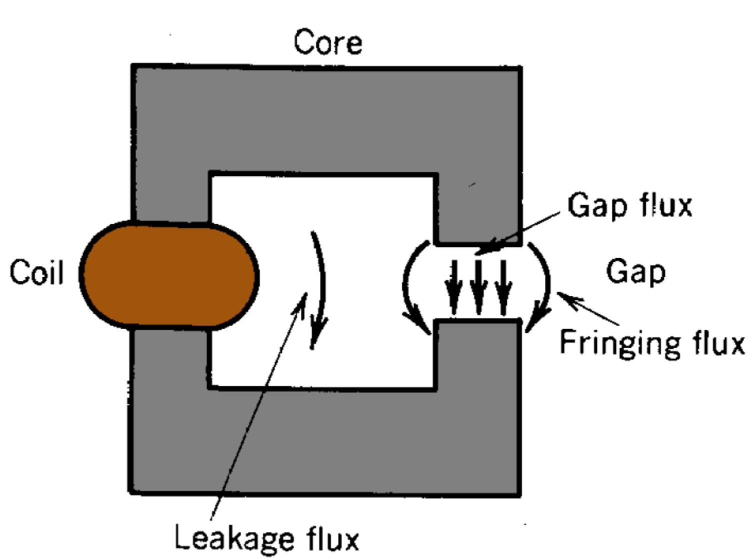
Magnetic			Electric		
Name	Symbol	Units	Name	Symbol	Units
Magnetomotive force (MMF)	$\mathcal{F} = \int \mathbf{H} \cdot \mathrm{d}\mathbf{l}$	ampere-turn	Electromotive force (EMF)	$\mathcal{E} = \int \mathbf{E} \cdot \mathrm{d}\mathbf{l}$	volt
Magnetic field	H	ampere/meter	Electric field	E	volt/meter = newton/coulomb
Magnetic flux	Φ	weber	Electric current	I	ampere
Hopkinson's law or Rowland's law	$\mathcal{F} = \Phi \mathcal{R}_m$	ampere-turn	Ohm's law	$\mathcal{E} = IR$	
Reluctance	$\mathcal{R}_m$	1/henry	Electrical resistance	R	ohm
Permeance	$\mathcal{P} = \frac{1}{\mathcal{R}_m}$	henry	Electric conductance	$G = 1/R$	1/ohm = mho = siemens
Relation between B and H	$\mathbf{B} = \mu \mathbf{H}$		Microscopic Ohm's law	$\mathbf{J} = \sigma \mathbf{E}$	
Magnetic flux density B	B	tesla	Current density	J	ampere/square meter
Permeability	μ	henry/meter	Electrical conductivity	σ	siemens/meter

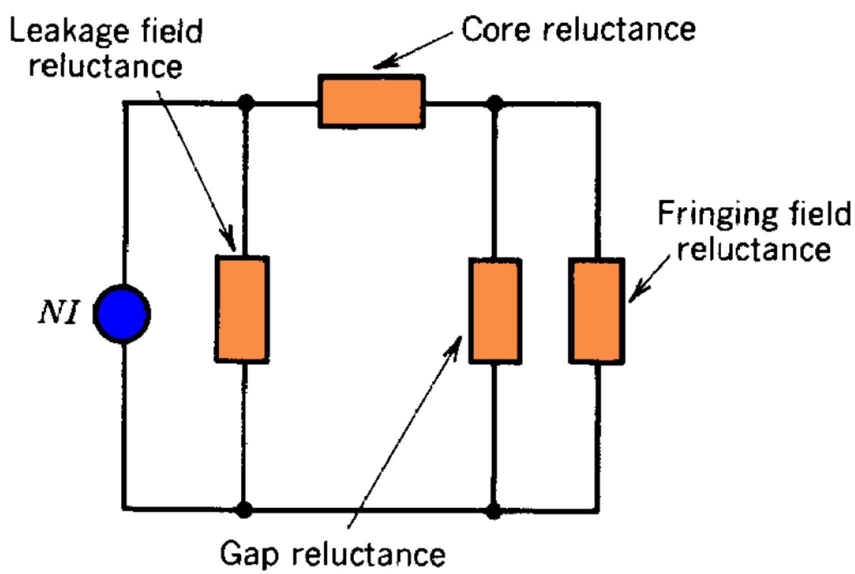
Magnetic circuits

- Magnetic circuits obey other laws that are similar to electrical circuit laws. For example, the total reluctance in series is:

$$\mathcal{R}_T = \mathcal{R}_1 + \mathcal{R}_2 + \cdots$$



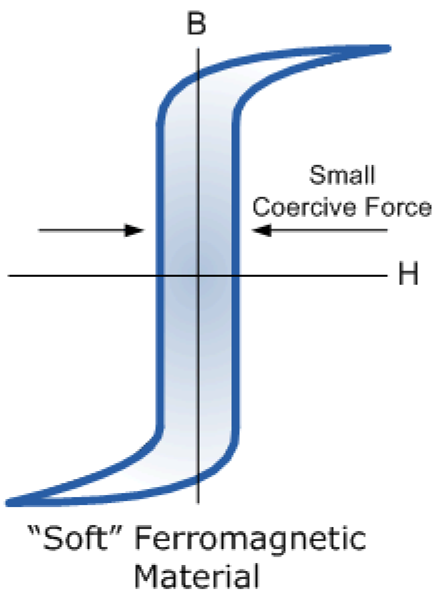


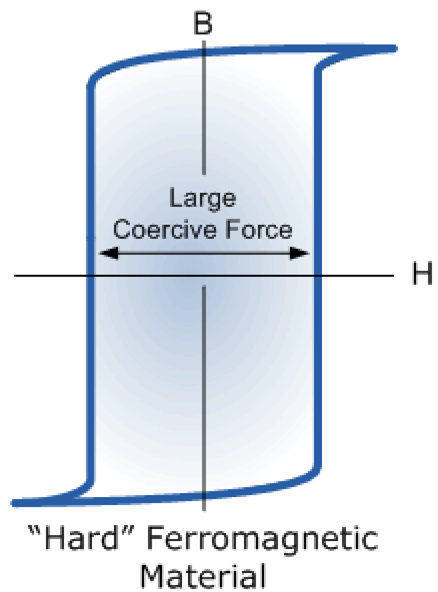


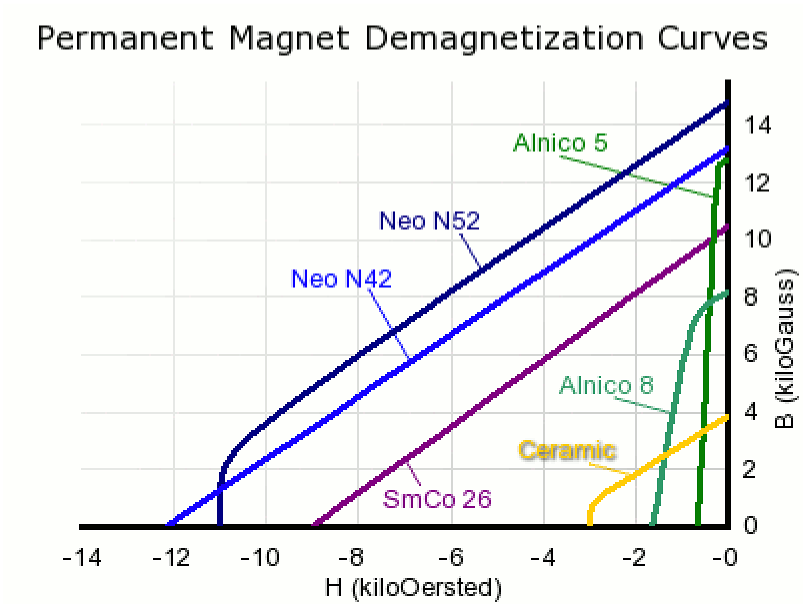
Permanent magnet circuits

- Permanent magnet circuits have the advantage that a dc magnetic field can be maintained with no power input.
- There are two drawbacks of permanent magnet circuits: (1) it is difficult to vary the field magnitude in the gap and (2) bulky magnets are needed to supply high fields over large areas.

$$B(\mathrm{G}) = \mu_r H(\mathrm{Oe}) \qquad 1\ \mathrm{Oe} = \frac{1000}{4\pi}\ \mathrm{A/m}$$







Operating point of a permanent magnet circuit

- Neglecting the reluctance of the iron core, the operating point of the permanent magnet is determined by the gap properties through

$$H_m L_m = H_g L_g = B_g L_g / \mu_0$$

$$\Phi = B_g A_g = B_m A_m$$

- Magnetic field energy in the gap is proportional to the energy product ($H_m B_m$) and the magnet volume:

$$W_g = \frac{B_g^2}{2\mu_0} A_g L_g = \frac{H_m B_m}{2} A_m L_m$$

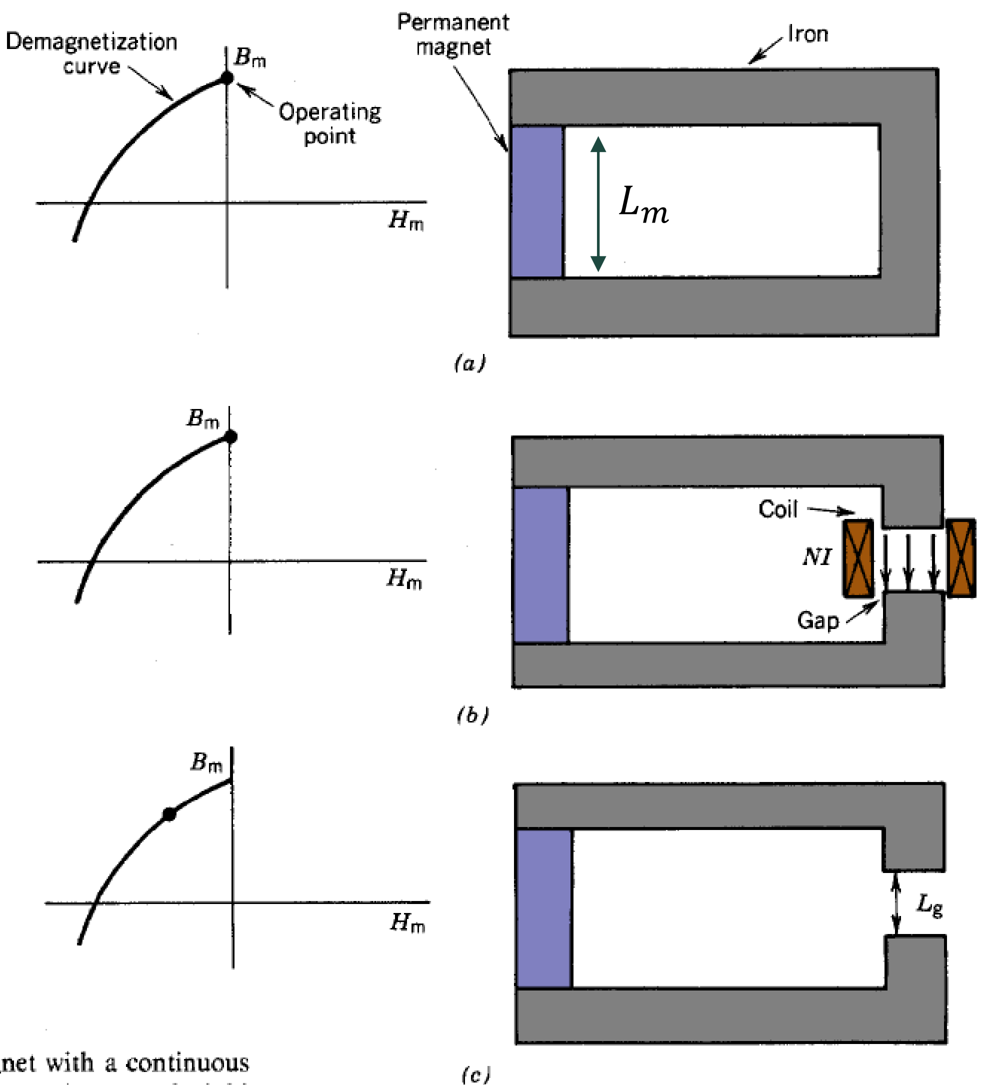


Figure 5.21 Operating point of a permanent magnet. (*a*) Permanent magnet with a continuous iron flux conductor, zero magnetizing force. (*b*) Addition of an air gap with a coil to supply field energy. (*c*) Deactivation of the gap coil.

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