Calculation of Particle Orbits in Focusing Fields

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Transverse orbits in a continuous linear focusing force: betatron oscillation

- In many instances, particle motion transverse to a beam axis is separable along two Cartesian coordinates. This applies to motion in a magnetic gradient field and in an array of quadrupole lenses.
- Consider one-dimensional transverse paraxial particle motion along the z axis in the presence of a linear force, $F_x = -F_0(x/x_0)$.
- The equation of motion in the paraxial approximation:

$$\frac{d^2}{dt^2}(\gamma m_0 x) = -F_0 \frac{x}{x_0} \quad \xrightarrow{\frac{d}{dt} = v_z \frac{d}{dz}} \quad \frac{d^2 x}{dz^2} = -\left(\frac{F_0}{\gamma m_0 v_z^2 x_0}\right) x$$

- Particle motion is harmonic: All particle orbits have the same wavelength; they differ only in amplitude and phase.

$$x(z) = X \cos\left(\frac{2\pi z}{\lambda_z} + \varphi\right) \qquad \lambda_z = 2\pi \left(\frac{\gamma m_0 v_z^2 x_0}{F_0}\right)^{1/2}$$

- Transverse particle motions of this type in accelerators are usually referred to as betatron oscillations since they were first described during the development of the betatron [D.W. Kerst and R. Serber, Phys. Rev. 60, 53 (1941)]. The quantity λ_z is called the betatron wavelength.

Particle orbits in an array of uniform, equally spaced, thin lenses

- Consider lenses of focal length f and axial spacing d in the limit that $d \ll f$ (thin-lens approximation). We want to calculate the change in x and v_x passing through one drift space and one lens. If v_x is the transverse velocity in the drift region, then

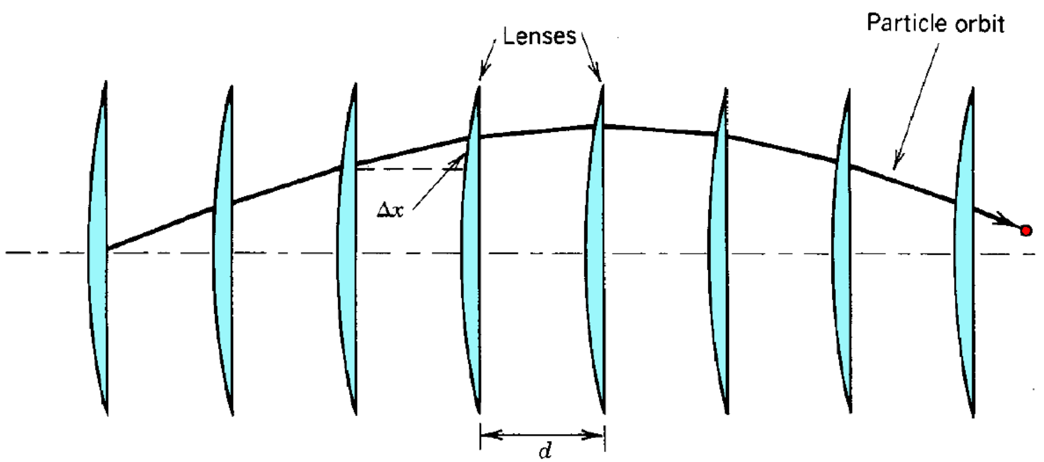
$$\Delta x = \left(\frac{v_x}{v_z}\right) d \qquad \frac{\Delta v_x}{v_z} = -\frac{x}{f} \quad \text{(Definition of the focal length)}$$

- Above equations can be converted to differential equations by associating Δz with d and letting $d \approx \Delta z \rightarrow 0$,

$$\frac{dx}{dz} = \frac{v_x}{v_z} \qquad \frac{dv_x}{dz} = -\frac{v_z}{fd}x \quad \Longrightarrow \quad \frac{d^2x}{dz^2} = -\left(\frac{1}{fd}\right)x$$

- The solution is harmonic, with $\lambda_z = (fd)^{1/2}$.
- Averaging the transverse force over many lenses gives

$$\bar{F}_x = \left(\frac{\gamma m_0 v_z^2}{fd}\right)x$$



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Acceptance

- Perfectly laminar beams cannot be achieved in practice, resulting in the particles' spread in position and velocity. The capability of a focusing system is parametrized by the range of particle positions and transverse velocities that can be transported without beam loss. This parameter is called the acceptance.
- Acceptance is the set of all particle orbit parameters (x_i, v_{xi}) at the entrance to a focusing system or optical element that allow particles to propagate through the system without loss. Acceptance is indicated as a bounded area on a phase space plot.
- Matching a beam to a focusing system consists of operating on the beam distribution so that it is enclosed in the acceptance.

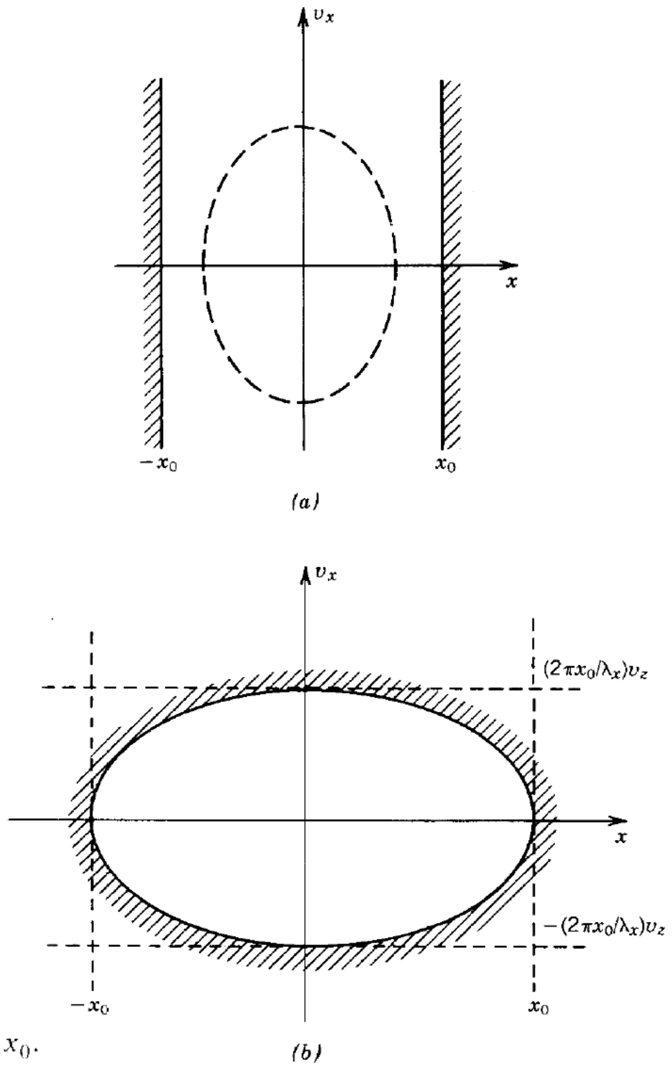


Figure 7.3 Acceptance of transport elements. (*a*) A one-dimensional aperture; slot of width $2x_0$. Dashed line indicates a beam distribution that can pass without attenuation. (*b*) One-dimensional continuous focusing system with linear forces.

Betatron: circular induction accelerator

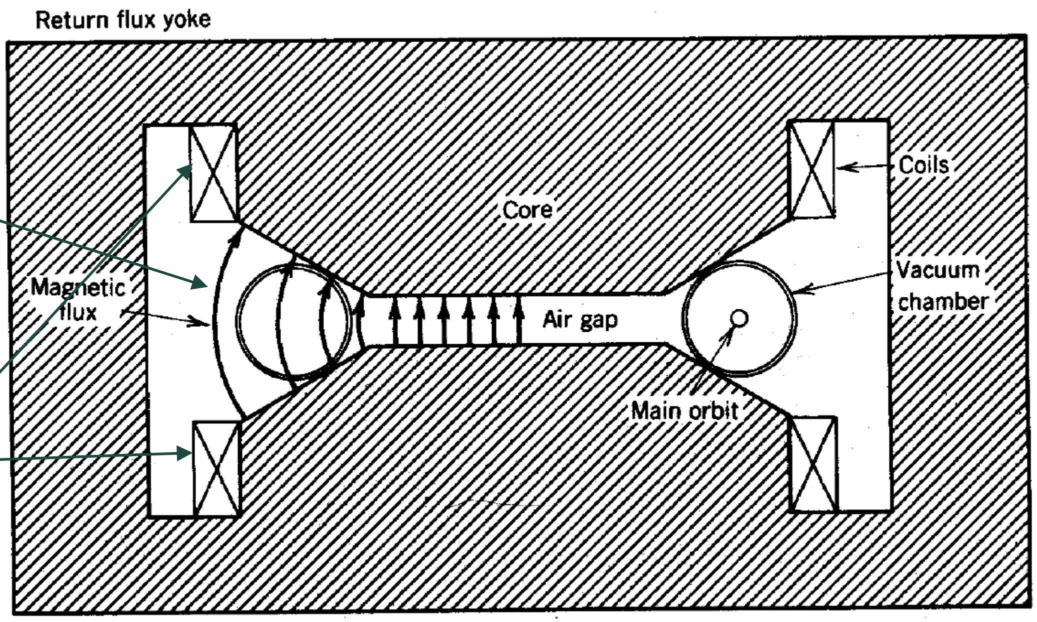
- The betatron [D.W. Kerst, Phys. Rev. 58, 841 (1940)] is a circular induction accelerator used for electron acceleration.
- Like the linear induction accelerator, the betatron is the circuit equivalent of a step-up transformer. The main difference from the linear induction accelerator is that magnetic bending and focusing fields are added to confine electrons to circular orbits around the isolation core. The beam acts as a multi-turn secondary.
- The maximum electron kinetic energy achieved by betatrons is about 300 MeV. The energy limit is determined in part by the practical size of pulsed magnets and in part by synchrotron radiation.

Basic betatron geometry

- A toroidal vacuum chamber encircles the core of a large magnet.
- The magnetic field is produced by pulsed coils; the magnetic flux inside the radius of the vacuum chamber changes with time. Increasing flux generates an azimuthal electric field which accelerates electrons in the chamber.
- In the absence of an air gap, there is little magnetic flux outside the core. An air gap is included to divert some of the magnetic flux into the vacuum chamber. By the proper choice of gap width, the vertical magnetic field can be adjusted to confine electrons to a circular orbit in the vacuum chamber.

Positive field index: focus in both the horizontal and vertical directions

Bending and acceleration



Betatron condition

- The betatron [D. W. Kerst, Phys. Rev. 58, 841 (1940)] is a circular induction accelerator used for electron acceleration.

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \Rightarrow \quad \oint \boldsymbol{E} \cdot \boldsymbol{dl} = -\frac{\partial}{\partial t} \oiint \boldsymbol{B} \cdot \boldsymbol{da}$$

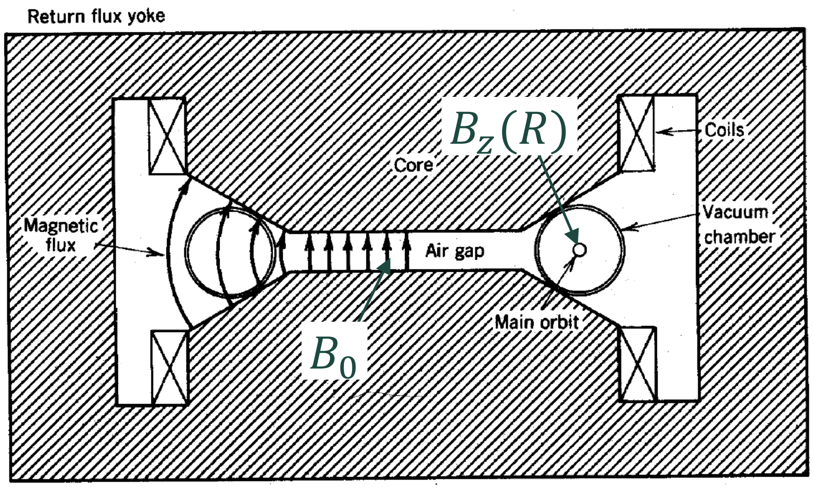
- The vertical field at R, i.e. $B_z(R)$ is related with R by

$$R = \frac{\gamma m_e v_\theta}{eB_z(R)} = \frac{p_\theta}{eB_z(R)}$$

- Particle motion on the main orbit is described by the Faraday induction law:

$$\frac{dp_\theta}{dt} = eE_0 = \frac{e}{2\pi R}\frac{d\Phi}{dt}$$

$$\Rightarrow \quad p_\theta = \frac{e[\Phi(t) - \Phi(0)]}{2\pi R} = \frac{e}{2\pi R}\Delta\Phi$$



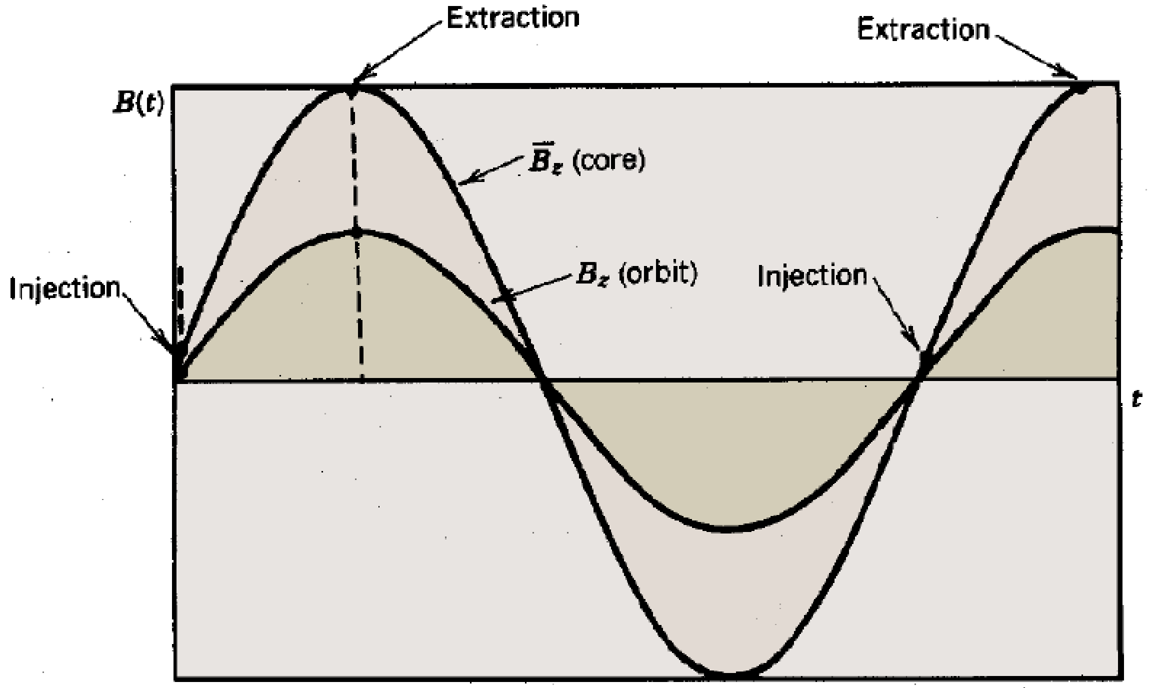
- Betatron condition:

$$B_z(R) = \frac{\Delta\Phi}{2\pi R^2} \approx \frac{B_0 \cdot \pi R^2}{2\pi R^2} = \frac{B_0}{2}$$

Betatron condition

- The betatron pole piece is designed so that vertical field at the average beam radius is equal to one-half the flux change in the core divided by the area inside the particle orbit.

$$B_z(R) = \frac{\Delta\Phi}{2\pi R^2} \approx \frac{B_0 \cdot \pi R^2}{2\pi R^2} = \frac{B_0}{2}$$



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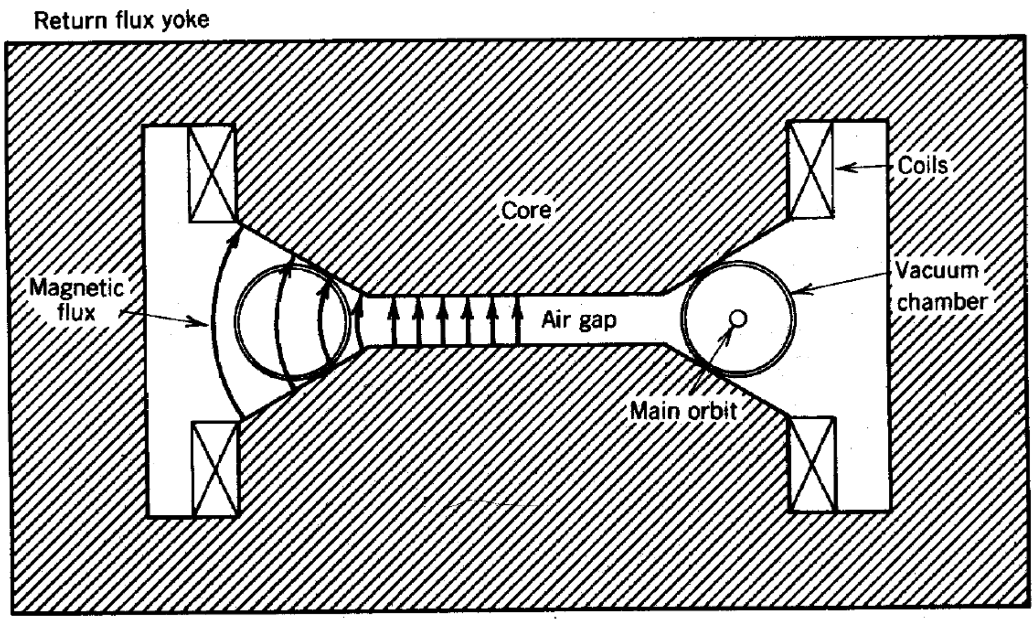
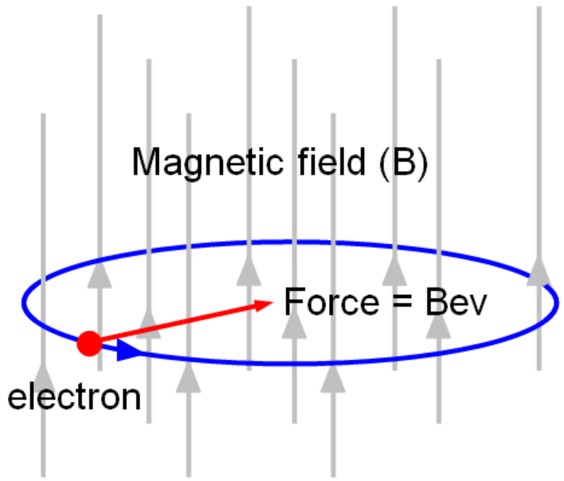
Betatron oscillations: introduction

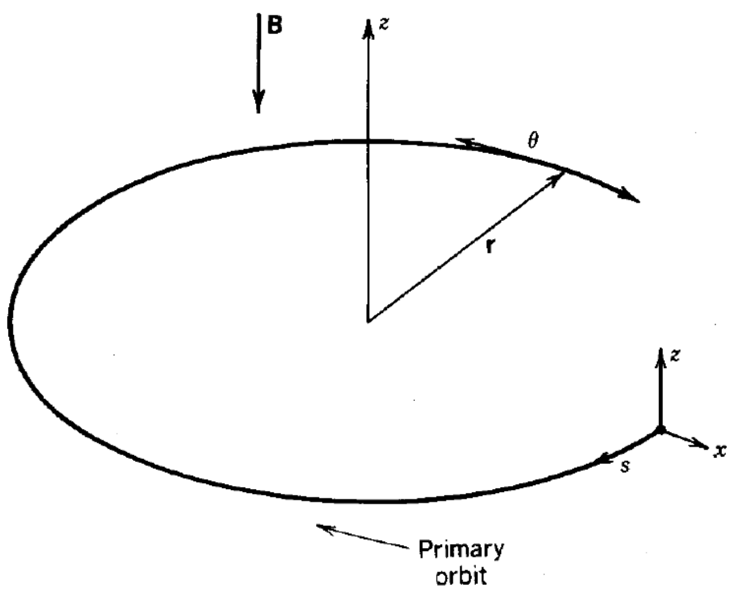
- The most familiar example of focusing in an accelerator by continuous fields occurs in the betatron. In this device, electrons are confined for relatively long times as they are inductively accelerated to high energy.
- In a betatron, particles are enclosed in a toroidal vacuum chamber centered on the main circular orbit. Beams always have spreads in angle and position of particle orbits about the main orbit. It is necessary to supplement the uniform field with additional field components so that non-ideal particle orbits oscillate about the main axis rather than drift away.

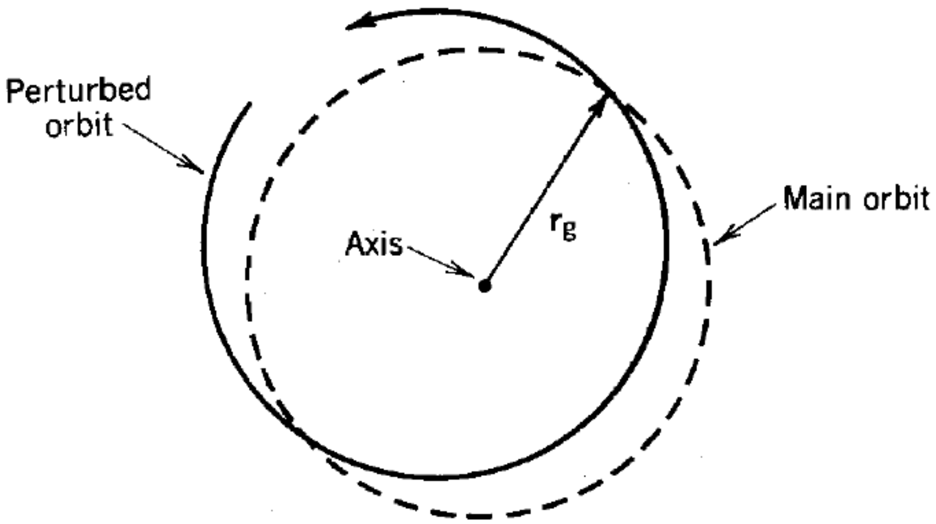




Betatron oscillations: need of radial magnetic field

- A particle distribution with a spread in v_x occupies a bounded region in x. This is not true in the z direction. A spread in v_z causes the boundary of the particle distribution to expand indefinitely.
- In order to confine a beam, we must either add additional focusing lenses around the torus to supplement the bending field or modify the bending field.
- The zero-order particle velocity is in the θ direction relative to the field axis. The magnetic field must have a radial component in order to exert a force in the z direction.



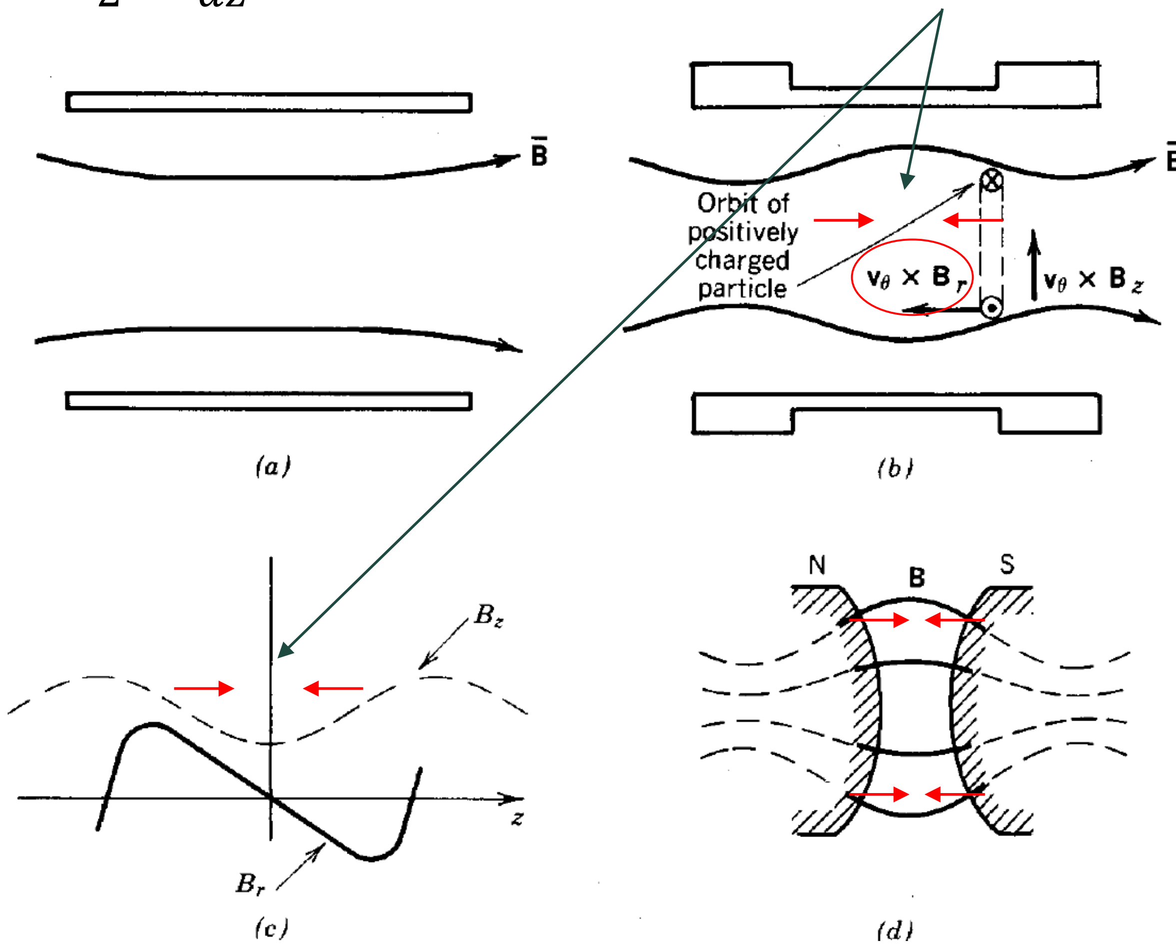


Axisymmetric magnetic field with radial gradient

- In the paraxial approximation, the bent field lines have a component B_r:

$$B_r \cong -\frac{r}{2}\frac{dB_z(0,z)}{dz}$$

Due to radial magnetic field, particles are confined about the plane of minimum B_z.



Mirror-like field can be generated by shaping magnet poles to conform to magnetic equipotential lines.

Field index

- The equations $\nabla \times \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$ imply that

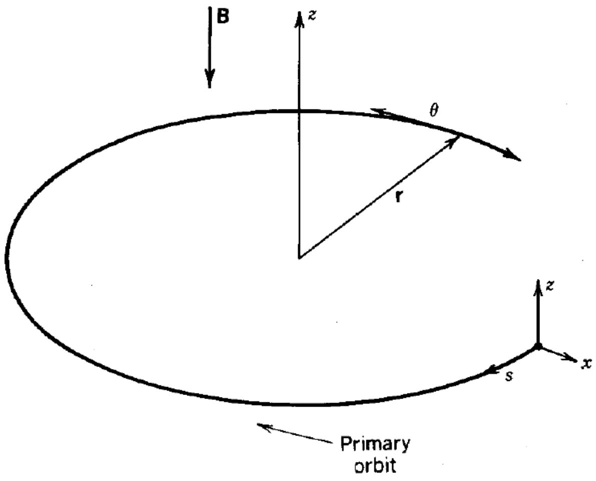
$$\frac{\partial B_z}{\partial r} = \frac{\partial B_r}{\partial z} \cong -\frac{r}{2}\frac{\partial^2 B_z(0,z)}{\partial z^2} \quad \Rightarrow \quad B_z(r,z) \cong B_0 - \frac{r^2}{4}\frac{\partial^2 B_z(0,z)}{\partial z^2}$$

- The bending field gradient is usually parametrized by the field index. The field index is a function of radius given by

$$n(r) = -\frac{r}{B_z(r,0)}\frac{\partial B_z(r,0)}{\partial r} = \frac{r^2}{2B_z(r,0)}\frac{\partial^2 B_z(0,z)}{\partial z^2} \quad > 0 \text{ for a mirror field}$$

- For ideal orbit,

$$B_0 = B_z(r_g, 0), \qquad n_0 = -\frac{r_g}{B_0}\frac{\partial B_z(r,0)}{\partial r}$$



- The magnetic field components are approximated by

$$B_r \approx B_r(r_g, 0) + \frac{\partial B_r}{\partial r}x + \frac{\partial B_r}{\partial z}z$$

$$B_z \approx B_z(r_g, 0) + \frac{\partial B_z}{\partial r}x + \frac{\partial B_z}{\partial z}z$$

$$x = r - r_g \ll r_g$$

$$z \ll r_g$$

$$v_x, v_z \ll v_\theta = v_0 \text{ (constant)}$$

Betatron oscillations

- The magnetic field expressions become

$$B_r \approx \frac{\partial B_r}{\partial z} z = \frac{\partial B_z}{\partial r} z = -\frac{n_0 B_0}{r_g} z \qquad B_z \approx B_0 + \frac{\partial B_z}{\partial r} x = B_0 - \frac{n_0 B_0}{r_g} x$$

- The relativistic equations of motion are

$$\gamma_0 m_0 \frac{d^2 r}{dt^2} = \frac{\gamma_0 m_0 v_\theta^2}{r} - q v_\theta B_z \qquad \gamma_0 m_0 \frac{d^2 z}{dt^2} = q v_\theta B_r$$

- Substituting $x = r - r_g$, we obtain

$$\frac{d^2 x}{dt^2} = \frac{v_\theta^2}{r_g + x} - \frac{q v_\theta B_0}{\gamma_0 m_0}\left(1 - \frac{n_0}{r_g} x\right) \approx \cancel{\frac{v_\theta^2}{r_g}} - \frac{v_\theta^2}{r_g^2} x - \cancel{\frac{v_\theta^2}{r_g}\Big(1} - \frac{n_0}{r_g} x\Big) = -\omega_g^2 (1 - n_0) x$$

$$\frac{d^2 z}{dt^2} = -\omega_g^2 n_0 z$$

- Finally we obtain betatron oscillations:

$$x = x_0 \cos[\sqrt{1 - n_0}\,\omega_g t + \varphi_1]$$

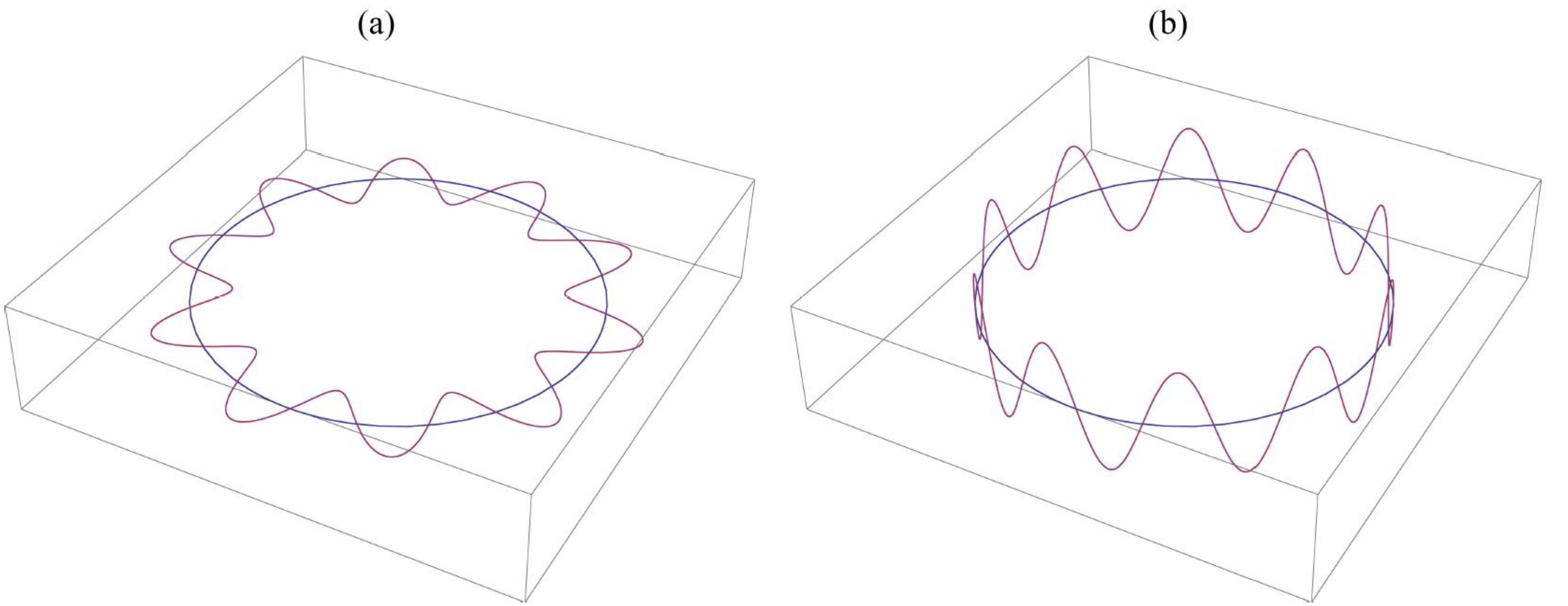
$$z = z_0 \cos[\sqrt{n_0}\,\omega_g t + \varphi_2]$$

Betatron oscillations

- In a uniform field ($n_0 = 0$), first-order radial particle motion is harmonic about the ideal orbit with frequency ω_g (displaced circle).
- When a field gradient is added, the transverse oscillation frequency is no longer equal to the rotation frequency. We can write

$$\frac{\omega_x}{\omega_g} = \sqrt{1 - n_0} = \nu_x \qquad \frac{\omega_z}{\omega_g} = \sqrt{n_0} = \nu_z$$

← Tune (normalized frequency)



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Betatron oscillations: weak focusing

- When n_0 is positive (a negative field gradient, or mirror field), ν_x is less than unity and the restoring force decreases with radius. Particles moving outward are not reflected to the primary axis as soon. Conversely, particles moving inward are focused back to the axis more strongly.
- When $n_0 > 1$, the field drops off too rapidly to restore outward moving particles to the axis. In this case, ν_x is imaginary so that x grows exponentially. This is an example of an orbital instability.
- In the z direction, particle orbits are stable only when $n_0 > 0$.

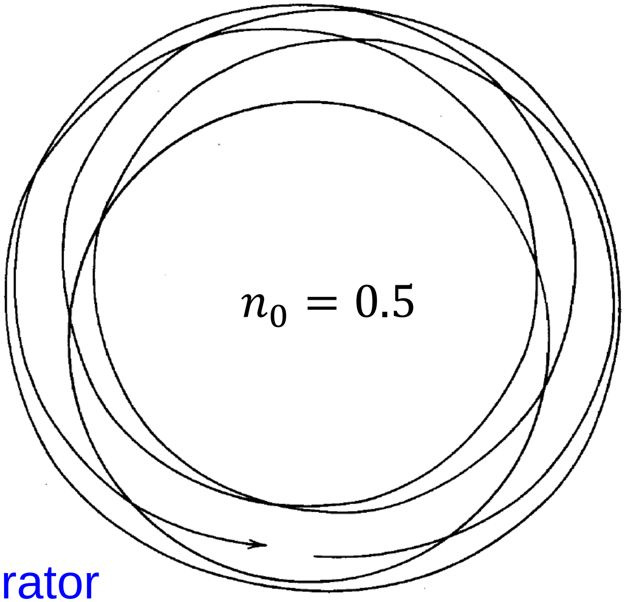
- For stability, the field index should be

$$0 < n_0 < 1$$

→ Weak focusing

- Tunes should be

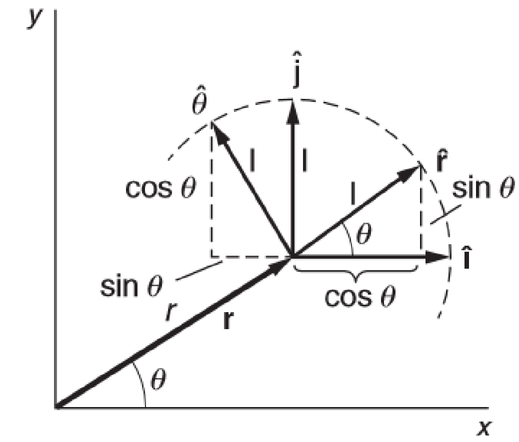
$$\frac{\omega_x}{\omega_g} = \sqrt{1 - n_0} = \nu_x < 1 \qquad \frac{\omega_z}{\omega_g} = \sqrt{n_0} = \nu_z < 1$$



→ Limitation of weak focusing in larger circular accelerator

Newton's law in cylindrical (polar) coordinates

- Let $\vec{r}$ be the position vector of a moving particle.



$$\vec{r} = (r\cos\theta, r\sin\theta) = r\hat{r}$$

- Unit vectors and their time derivative:

$$\hat{r} = (\cos\theta, \sin\theta) \qquad \hat{\theta} = (-\sin\theta, \cos\theta)$$

$$\frac{d\hat{r}}{dt} = (-\sin\theta, \cos\theta)\frac{d\theta}{dt} = \dot{\theta}\hat{\theta} \qquad \frac{d\hat{\theta}}{dt} = -(\cos\theta, \sin\theta)\frac{d\theta}{dt} = -\dot{\theta}\hat{r}$$

- Velocity in polar coordinates:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = v_r\hat{r} + v_\theta\hat{\theta}$$

- Acceleration in polar coordinates:

Linear acceleration in tangential direction

Linear acceleration in radial direction

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

Centrifugal acceleration

Coriolis acceleration

Azimuthal motion of particles in cylindrical beams

- Consider a region of azimuthally symmetric, paraxial electric and magnetic fields where particles move almost parallel to the z axis. We seek an equation that describes the azimuthal motion of charged particles in terms of on-axis quantities.
- The Lorentz force law for azimuthal motion is

$$\gamma m_0(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta = q(v_z B_r - v_r B_z)$$

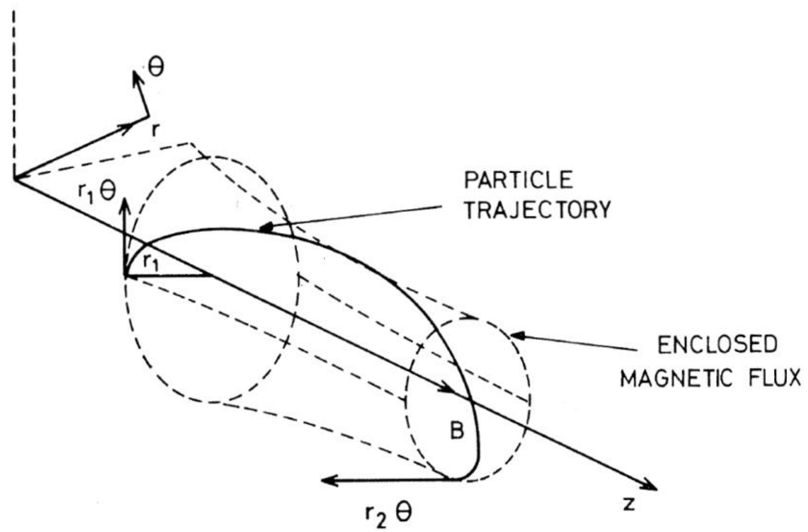
$$\gamma m_0 \frac{d}{dt}\left(r^2\dot{\theta}\right) = qr(v_z B_r - v_r B_z)$$

$$\Delta\Psi = B_z \cdot 2\pi r\Delta r + \pi r^2 \cdot \left(\frac{\partial B_z}{\partial z}\right)\Delta z$$

$$= 2\pi r(B_z\Delta r - B_r\Delta z)$$

$$d\left(r^2\dot{\theta}\right) = \frac{q}{2\pi\gamma m_0}(2\pi r dz B_r - 2\pi r dr B_z)$$

$$d\left(r^2\dot{\theta}\right) = -\frac{q}{2\pi\gamma m_0}d\Psi$$



$$\gamma m_0 r^2\dot{\theta} + \frac{q}{2\pi}\Psi = \text{const.}$$

Busch's theorem [H. Busch, Z. Phys. 81, 974 (1926)]

Azimuthal motion of particles in cylindrical beams

- Integrating Busch's theorem, we obtain

$$r^2\dot{\theta} - r_0^2\dot{\theta}_0 = -\frac{q}{2\pi\gamma m_0}(\Psi - \Psi_0)$$

- When $\dot{\theta}_0$ is zero at z_0 (e.g. at the cathode)

$$\dot{\theta} = -\frac{q}{2\pi\gamma m_0 r^2}(\Psi - \Psi_0) \qquad v_\theta = -\frac{q}{2\pi\gamma m_0 r}(\Psi - \Psi_0)$$

- In most applications of interest in linear-beam tubes, Ψ changes slowly with z, so

$$\Psi = \pi r^2 B_z \qquad \Psi_0 = \pi r_0^2 B_{z0}$$

- Then,

$$\dot{\theta} = -\frac{q}{2\pi\gamma m_0 r^2}(\pi r^2 B_z - \pi r_0^2 B_{z0}) = -\frac{q}{2\gamma m_0}\left(B_z - B_{z0}\frac{r_0^2}{r^2}\right)$$

- Note that

$$\Psi = \int_0^r 2\pi r' dr' \frac{1}{r'}\frac{\partial(r' A_\theta)}{\partial r'} = 2\pi r A_\theta \quad \Rightarrow \quad \gamma m_0 r v_\theta + q r A_\theta = \frac{q}{2\pi}\Psi_0$$

Conservation of canonical angular momentum

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Paraxial ray equation

- The paraxial ray equation is derived by combining the properties of paraxial fields, the conservation of canonical angular momentum, and the conservation of energy, under the assumption of

  1. The beam is cylindrically symmetric.
  2. Beam properties vary in space but not in time.
  3. The fields are cylindrically symmetric, with components E_r, E_z, B_r and B_z. This encompasses all axisymmetric electrostatic lenses and the solenoidal magnetic lens.
  4. The fields are static.
  5. Particle motion is paraxial.
  6. Fields are paraxial and transverse forces are linear.
  7. Particle orbits are laminar (zero emittance) and there are no self-fields (no space charge forces). Two terms are added later.

- The laminarity of orbits means that the radial projections of all particle orbits are similar. They differ only in amplitude. It is thus sufficient to treat only the boundary orbit. The axial velocity is approximately constant in any plane normal to the axis.

$$\frac{d}{dt} = v_z \frac{d}{dz} = v_z(\,)' \qquad \dot{\theta} = -\frac{q}{2\pi\gamma m_0 r^2}(\Psi - \Psi_0)$$

Paraxial ray equation

- The quantity γ may vary with axial position since zero-order longitudinal electric fields are included. The only nontrivial equation of motion is in the radial direction:

$$\frac{d}{dt}(\gamma m_0 \dot{r}) - \gamma m_0 r \dot{\theta}^2 = q(E_r + v_\theta B_z)$$

- Conservation of energy: $(\gamma - 1) m_0 c^2 = \int_{z_0}^{z} q E_z dz' = q\phi$

- Thus a change in position leads to a change in γ

$$\Delta\gamma = \frac{q E_z \Delta z}{m_0 c^2} \quad \Rightarrow \quad \frac{d\gamma}{dz} = \frac{q E_z}{m_0 c^2} \qquad E_r = -\frac{r}{2}\frac{\partial E_z}{\partial z} = -\frac{r}{2}\frac{m_0 c^2}{q}\frac{d^2\gamma}{dz^2}$$

- We obtain the following equation:

$$r'' + \frac{\gamma'}{\gamma\beta^2} r' + \left[\frac{\gamma''}{2\gamma\beta^2} + \left(\frac{q B_z}{2\gamma m_0 \beta c}\right)^2\right] r - \left(\frac{q\Psi_0}{2\pi\gamma m_0 \beta c}\right)^2 \frac{1}{r^3} = 0$$

Non-zero angular momentum

Decrease in the envelope angle by beam acceleration

Electrostatic focusing from radial components of applied electric fields

Magnetic focusing from applied solenoidal fields

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Effect of non-zero angular momentum

- When Ψ_0 is non-zero, the , the final term in the paraxial ray equation has a strong radial dependence through the $1/r^3$ factor. The term has a dominant defocusing effect when the beam is compressed to a small spot.

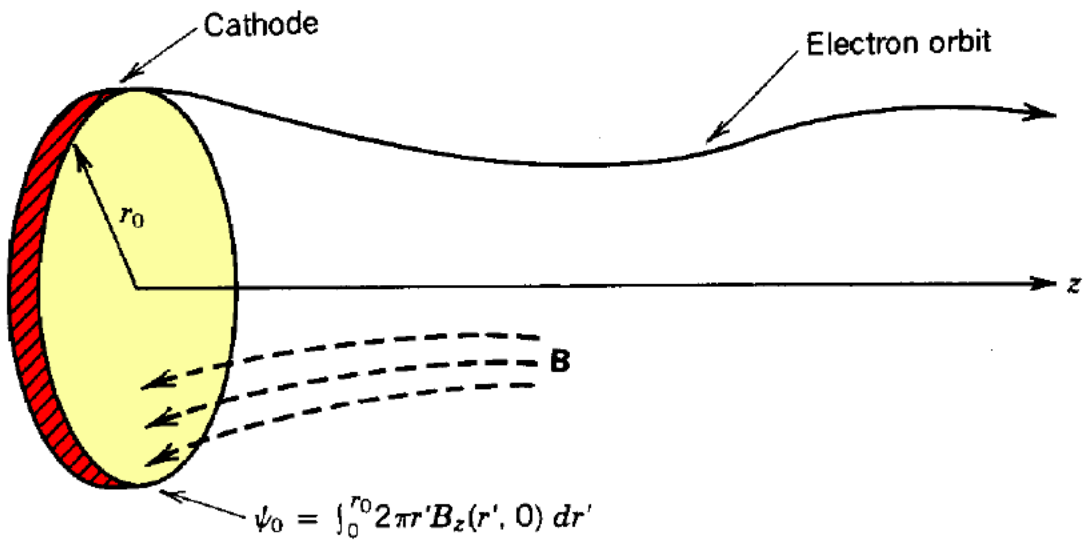


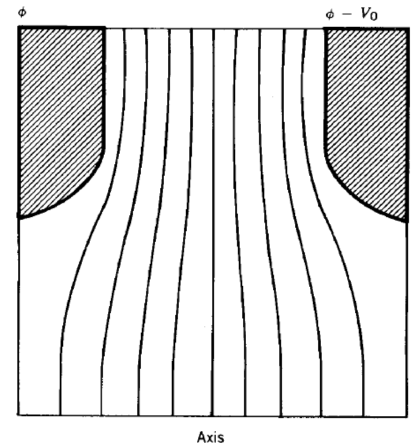
Figure 7.11 Electron emission from an immersed cathode.

- Thus, care should be taken to exclude magnetic fields from the cathode of the electron source in applications calling for fine focusing.

Paraxial ray equation: nonrelativistic approximation

- When $\Psi_0 = 0$, the paraxial ray equation is linear since the remaining terms contain only first powers of r'', r', and r.
- The nonrelativistic approximation can be used for beams of ions or low-energy electrons. Substituting $\gamma = 1 + q\phi/m_0c^2$ in the limit $q\phi \ll m_0c^2$, we obtain

$$2\phi r'' + \phi' r' + \left[\frac{\phi''}{2} + \frac{qB_z^2}{4m_0}\right] r - \frac{q\Psi_0^2}{4\pi^2 m_0}\frac{1}{r^3} = 0$$

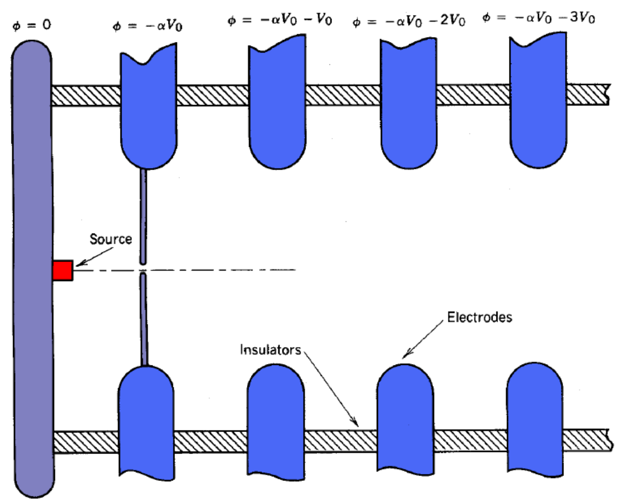


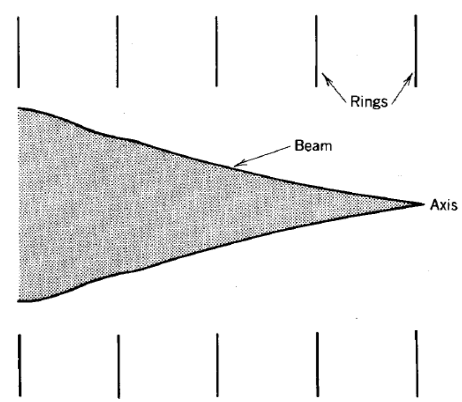
- If there are only electric fields present:

$$r'' + \frac{\phi'}{2\phi} r' + \frac{\phi''}{4\phi} r = 0$$

Decrease in the envelope angle by beam acceleration

Electrostatic focusing from radial components of applied electric fields





Homework

- Derive the following paraxial ray equation.

$$r'' + \frac{\gamma'}{\gamma\beta^2} r' + \left[\frac{\gamma''}{2\gamma\beta^2} + \left(\frac{qB_z}{2\gamma m_0 \beta c}\right)^2\right] r - \left(\frac{q\Psi_0}{2\pi\gamma m_0 \beta c}\right)^2 \frac{1}{r^3} = 0$$