Transfer Matrices and Periodic Focusing Systems

Spring, 2021

Kyoung-Jae Chung

Department of Nuclear Engineering

Seoul National University

Introduction

- Periodic focusing channels are used to confine high-energy beams in linear and circular accelerators. Periodic channels consist of a sequence of regions called focusing cells containing one or more charged particle optical elements. A focusing cell is the smallest unit of periodicity in the channel. The main application is in high-energy accelerators that utilize strong focusing.
- For example, the focusing channel of a linear ion accelerator consists typically of a series of magnetic quadrupole lenses with alternating north-south pole orientation. Thus, along either transverse axis, the lenses are alternately focusing and defocusing. Such a combination has a net focusing effect that is stronger than a series of solenoid lenses at the same field strength.
- A quadrupole focusing channel can therefore be constructed with a much smaller bore diameter than a solenoid channel of the same acceptance. The associated reduction in the size and power consumption of focusing magnets has been a key factor in the development of modern high-energy accelerators.
- Periodic focusing channels also have application at low beam energy. Configurations include the electrostatic accelerator column, the electrostatic Einzel lens array and periodic permanent magnet (PPM) channels used in high-power microwave tubes.

Transfer matrix

- Most beam transport devices, such as charged particle lenses and bending magnets, apply transverse forces that are linearly proportional to the distance of a particle from a preferred axis.
- To specify the orbit of the particle in the x-direction, we must give its position, x, and velocity, v_x. The convention in charged-particle optics is to represent particle orbits in terms of their angle relative to the main axis, rather than the transverse velocity. In the limit that $v_x \ll v_z$, the angle is

$$x' = \frac{dx}{dz} \approx \frac{v_x}{v_z}$$

- If the x-directed forces in the device are linear, then we can express the exit vector as a linear combination of the entrance vector components:

$$\begin{bmatrix} x_1 \\ x_1' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

Exit vector Transfer matrix Entrance vector

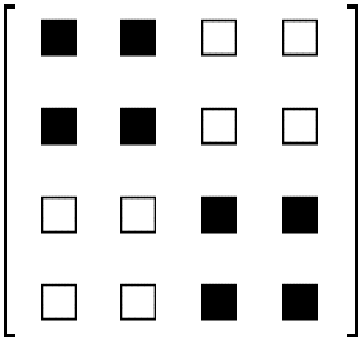
- The quantities a_{mn} depend on the distribution of forces. Without acceleration, the determinant of the transfer matrix equals unity.

Transfer matrix

- If a particle travels through linear device $\boldsymbol{A}$ and then through device $\boldsymbol{B}$, the final orbit vector is

$$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$\boldsymbol{x_2} = \boldsymbol{B}(\boldsymbol{A}\boldsymbol{x_0}) = (\boldsymbol{B}\boldsymbol{A})\boldsymbol{x_0} = \boldsymbol{C}\boldsymbol{x_0}$$



- The orbit vector transformation from any combination of one-dimensional focusing elements is a single transfer matrix, the product of the individual matrices of all the elements.
- The particle orbit vector for a two-dimensional focusing system is $\boldsymbol{x} = [x, x', y, y']$. A 4×4 matrix represents the effect of a general linear focusing element or system.
- In many practical devices, such as quadrupole lens arrays, the forces in the x and y directions are independent. Then, we can calculate motion in x and y separately using individual 2×2 matrices.

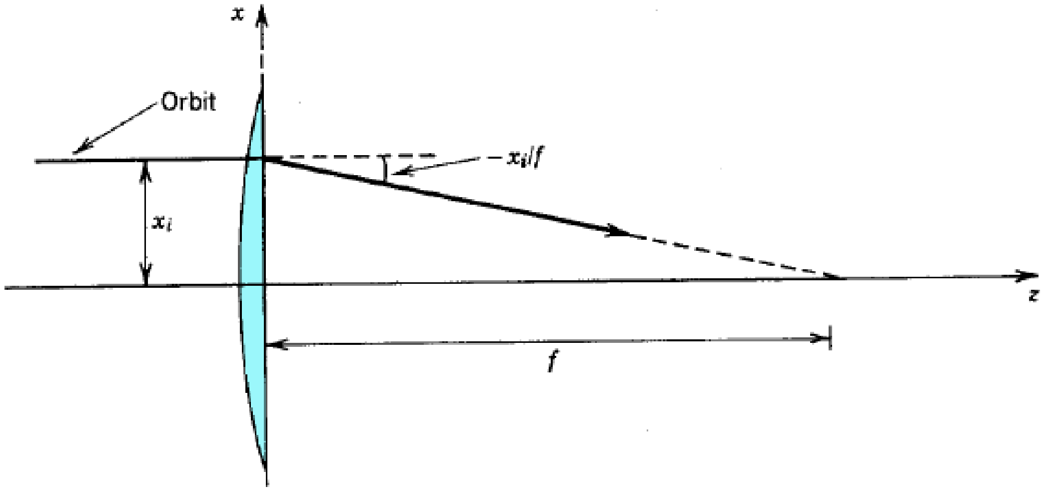
Transfer matrix for a thin lens and drift region

- The matrix for a thin lens with focal length f is

$$x_f = x_i$$

$$x_f' = x_i' - x_i/f$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

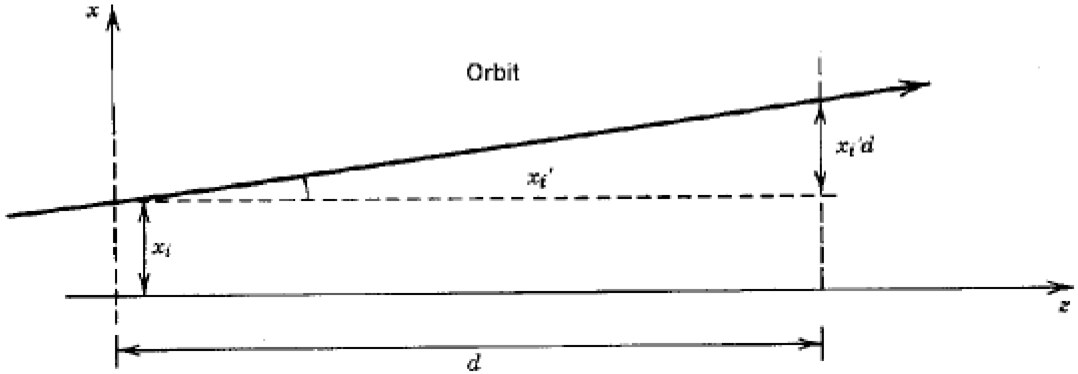


- The matrix of a field-free drift region of length d is

$$x_f = x_i + x_i' d$$

$$x_f' = x_i'$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$



Transfer matrix for a sector magnet w/ weak focusing

- The sector field below is a pie-shaped segment of the betatron field. For horizontal plane,

$$x = A\cos[\sqrt{1-n_0}z/r_g + \varphi] \qquad x' = -(\sqrt{1-n_0}/r_g)A\sin[\sqrt{1-n_0}z/r_g + \varphi]$$

$$\omega_g t = \frac{v_z t}{r_g} = \frac{z}{r_g}$$

- For input vector $(x_i, 0)$: $\varphi = 0$

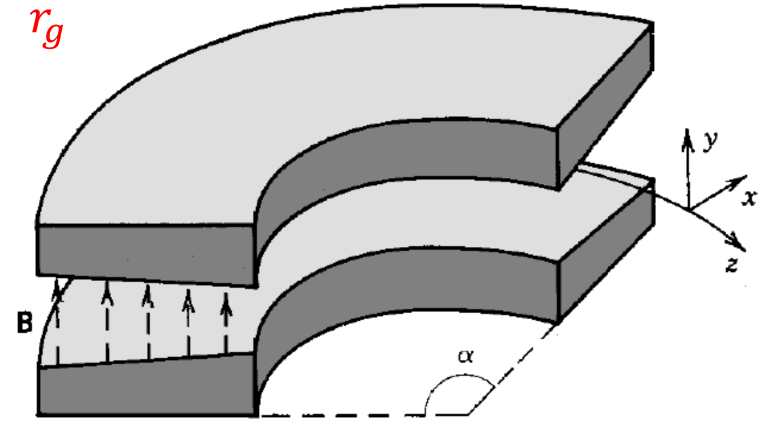
$$x_f = x_i\cos(\sqrt{1-n_0}\,d/r_g)$$

$$x_f' = -x_i\sqrt{1-n_0}/r_g\sin(\sqrt{1-n_0}\,d/r_g)$$

- For input vector $(0, x_i')$: $\varphi = \pi/2$

$$x_f = x_i' r_g \sin(\sqrt{1-n_0}\,d/r_g)/\sqrt{1-n_0}$$

$$x_f' = x_i'\cos(\sqrt{1-n_0}\,d/r_g)$$



$$\boldsymbol{A_H} = \begin{bmatrix} \cos(\sqrt{1-n_0}\,\alpha) & \dfrac{r_g\sin(\sqrt{1-n_0}\,\alpha)}{\sqrt{1-n_0}} \\ -\dfrac{\sqrt{1-n_0}\sin(\sqrt{1-n_0}\alpha)}{r_g} & \cos(\sqrt{1-n_0}\,\alpha) \end{bmatrix}$$

$$n_0 = -\frac{r_g}{B_0}\frac{\partial B_z(r,0)}{\partial r}$$

$$\frac{d}{r_g} = \alpha$$

Transfer matrix for a sector magnet w/ weak focusing

- Similarly,

$$A_V = \begin{bmatrix} \cos(\sqrt{n_0}\,\alpha) & \dfrac{r_g \sin\left(\sqrt{n_0}\,\alpha\right)}{\sqrt{n_0}} \\ -\dfrac{\sqrt{n_0}\sin\left(\sqrt{n_0}\alpha\right)}{r_g} & \cos(\sqrt{n_0}\,\alpha) \end{bmatrix}$$

- Initially parallel beams are focused in the horizontal direction. The horizontal focal length beyond the sector exit is

$$f' = \frac{r_g}{\sqrt{1-n_0}\tan(\sqrt{1-n_0}\,\alpha)}$$

- The distance to the vertical focal point is

$$f' = \frac{r_g}{\sqrt{n_0}\tan(\sqrt{n_0}\,\alpha)}$$

- If $n_0 = +0.5$, horizontal and vertical focal lengths are equal and the sector lens can produce a two-dimensional image. The dual-focusing property of the gradient field is used in charged particle spectrometers.

Transfer matrix for a sector magnet w/ weak focusing

- The transfer matrix for particles moving backward in the sector field is the inverse of the matrix for forward motion.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

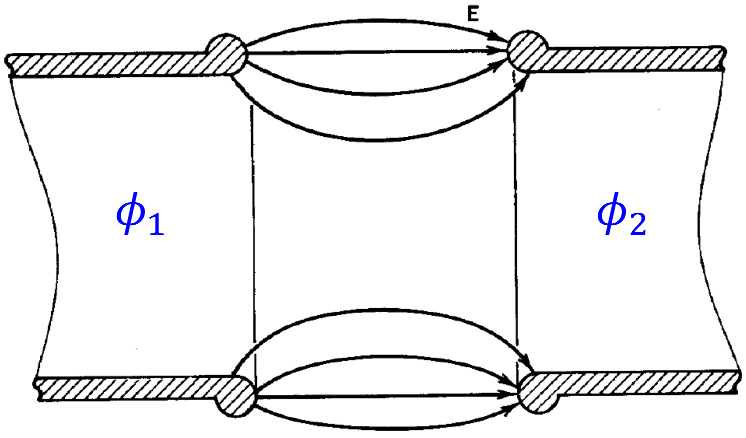
$$\boldsymbol{A}_H^{-1} = \begin{bmatrix} \cos(\sqrt{1-n_0}\,\alpha) & -\dfrac{r_g \sin\left(\sqrt{1-n_0}\,\alpha\right)}{\sqrt{1-n_0}} \\ \dfrac{\sqrt{1-n_0}\sin\left(\sqrt{1-n_0}\alpha\right)}{r_g} & \cos(\sqrt{1-n_0}\,\alpha) \end{bmatrix} \qquad \alpha \rightarrow -\alpha$$

- This result is obtained by replacing α with $-\alpha$. The negative angle corresponds to motion in the -z direction. The effect of the element is independent of the direction. The same holds true for any optical element in which the energy of the charged particle is unchanged. We can verify that in this case det A = 1.

Transfer matrix for an immersion lens (acceleration gap)

- Assume the focal length for motion of nonrelativistic particles in the accelerating direction, f_a, is known.
- The ratio of the exit velocity to the entrance velocity is

$$\xi = \sqrt{\frac{\phi_2}{\phi_1}}$$



- In the thin-lens approximation, for input vector $(x_i, 0)$, $x_f' = -x_i/f_a$, and for $(0, x_i')$, $x_f' = x_i'/\xi$. The transfer matrix of a thin electrostatic lens with acceleration is

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_a} & \dfrac{1}{\xi} \end{bmatrix} \qquad \det \boldsymbol{A} = \frac{1}{\xi} \neq 0$$

- The transfer matrix for a decelerating lens is

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -\dfrac{\xi}{f_a} & \xi \end{bmatrix} \qquad f_d = \frac{f_a}{\xi}$$

Transfer matrix for a uniform-field sector magnet

- The components a_{11} and a_{21} are related to f_1 and H_1.
- The matrix and Gaussian descriptions of linear lenses are equivalent. Lens properties are completely determined by four quantities.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad f' = -\frac{a_{11}}{a_{21}} \qquad f_1 = -\frac{1}{a_{21}}$$

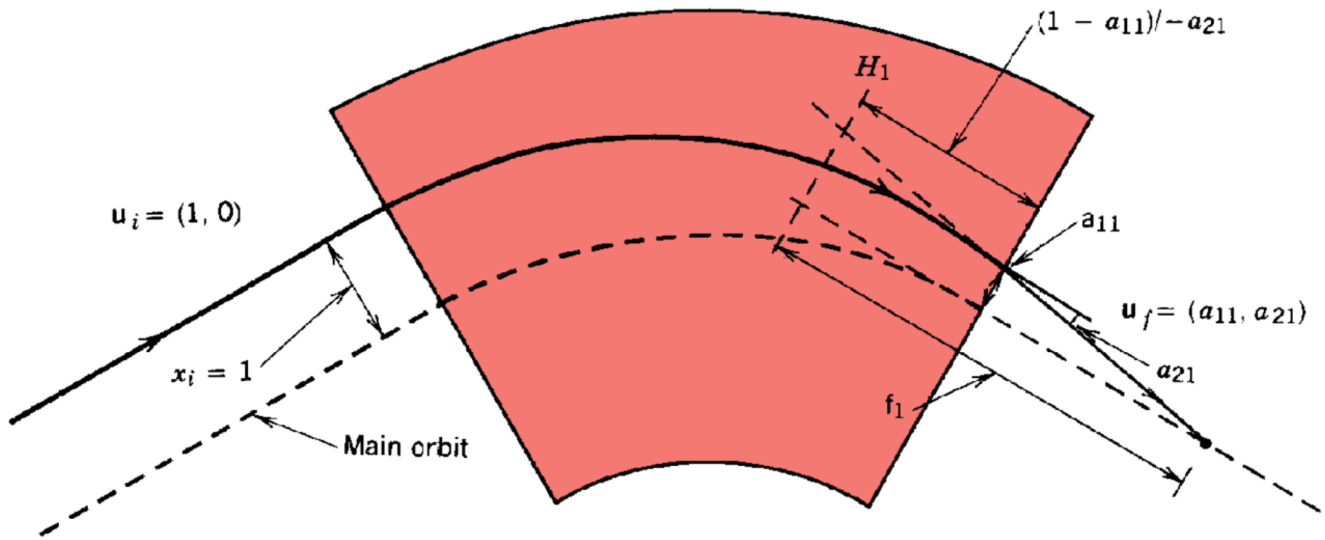


Figure 8.4 Horizontal particle motion in a uniform-field sector magnet. Relationship between the elements of the transfer matrix and the principal planes and focal lengths of Gaussian optics.

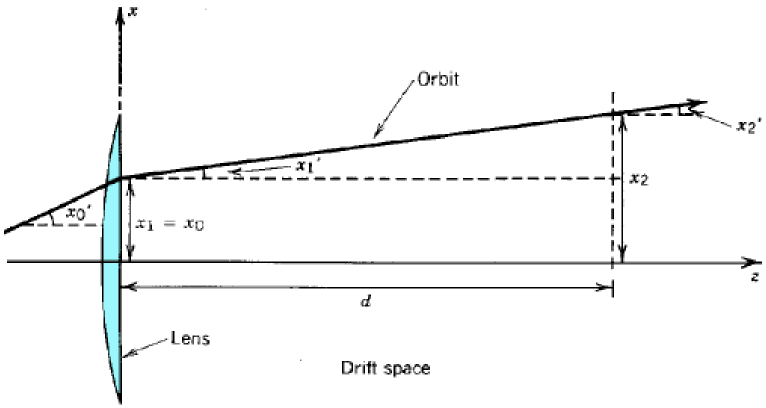
Combining optical elements

- If a particle travels through linear device $\boldsymbol{A}$ and then through device $\boldsymbol{B}$, the final orbit vector is

$$\begin{bmatrix} x_2 \\ x_2' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$\boldsymbol{x_2} = \boldsymbol{B}(\boldsymbol{A}\boldsymbol{x_0}) = (\boldsymbol{B}\boldsymbol{A})\boldsymbol{x_0} = \boldsymbol{C}\boldsymbol{x_0}$$

- For a thin lens with focal length f followed by a drift space d:



$$\boldsymbol{C} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f} & d \\ -\frac{1}{f} & 1 \end{bmatrix}$$

- For multiple lens: $\boldsymbol{C} = \boldsymbol{A_n}\boldsymbol{A_{n-1}} \cdots \boldsymbol{A_2}\boldsymbol{A_1}$

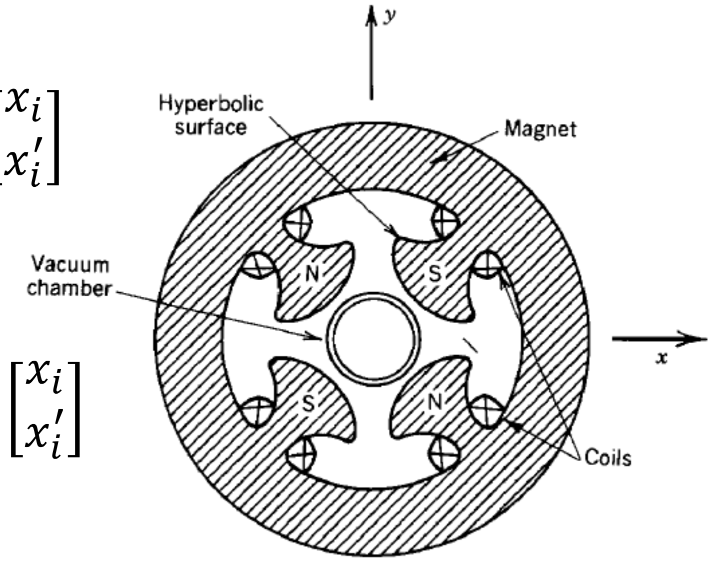
Transfer matrix of a quadrupole lens

- For focusing

$$\begin{bmatrix} x_f \\ x_f' \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{\kappa}l) & \sin(\sqrt{\kappa}l)/\sqrt{\kappa} \\ -\sqrt{\kappa}\sin(\sqrt{\kappa}l) & \cos(\sqrt{\kappa}l) \end{bmatrix} \begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \boldsymbol{A_F} \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

- For defocusing

$$\begin{bmatrix} x_f \\ x_f' \end{bmatrix} = \begin{bmatrix} \cosh(\sqrt{\kappa}l) & \sinh(\sqrt{\kappa}l)/\sqrt{\kappa} \\ \sqrt{\kappa}\sinh(\sqrt{\kappa}l) & \cosh(\sqrt{\kappa}l) \end{bmatrix} \begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \boldsymbol{A_D} \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$



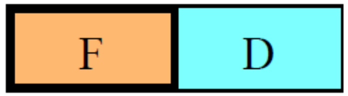
$$\kappa = \frac{qB_0}{\gamma m_0 a v_z} \qquad \Gamma = \sqrt{\kappa}l$$

- For $\Gamma \leq 1$,

$$\boldsymbol{A_F} \approx \begin{bmatrix} 1 - \frac{\Gamma^2}{2} + \frac{\Gamma^4}{24} & \left(\Gamma - \frac{\Gamma^3}{6}\right)/\sqrt{\kappa} \\ -\sqrt{\kappa}\left(\Gamma - \frac{\Gamma^3}{6}\right) & 1 - \frac{\Gamma^2}{2} + \frac{\Gamma^4}{24} \end{bmatrix} \qquad \boldsymbol{A_D} \approx \begin{bmatrix} 1 + \frac{\Gamma^2}{2} + \frac{\Gamma^4}{24} & \left(\Gamma + \frac{\Gamma^3}{6}\right)/\sqrt{\kappa} \\ \sqrt{\kappa}\left(\Gamma + \frac{\Gamma^3}{6}\right) & 1 + \frac{\Gamma^2}{2} + \frac{\Gamma^4}{24} \end{bmatrix}$$

Quadrupole doublet lens

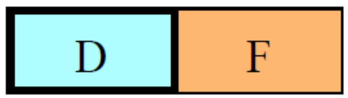
- Two-dimensional focusing can be accomplished with combinations of quadrupole lenses. → basis for high-energy particle transport system.
- For FD channel



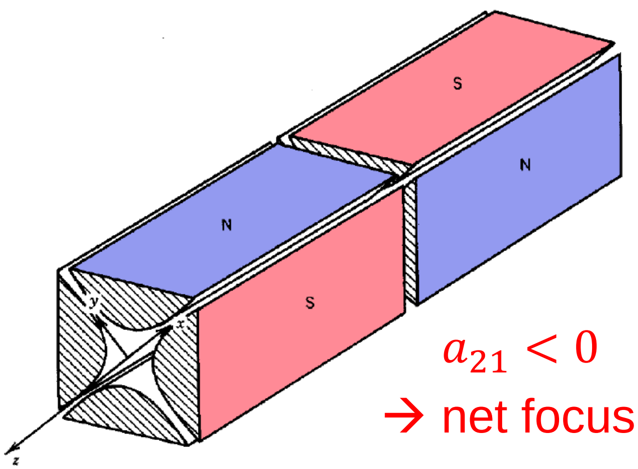
$$\Gamma = \sqrt{\kappa} l$$

$$\boldsymbol{C_{FD}} = \begin{bmatrix} \cos\Gamma\cosh\Gamma - \sin\Gamma\sinh\Gamma & (\cosh\Gamma\sin\Gamma + \cos\Gamma\sinh\Gamma)/\sqrt{\kappa} \\ \sqrt{\kappa}(\cos\Gamma\sinh\Gamma - \cosh\Gamma\sin\Gamma) & \cos\Gamma\cosh\Gamma + \sin\Gamma\sinh\Gamma \end{bmatrix}$$

- For DF channel

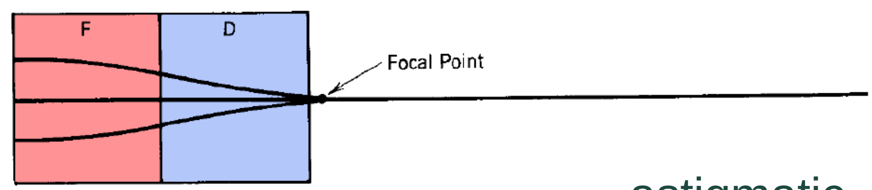


$$\boldsymbol{C_{DF}} = \begin{bmatrix} \cos\Gamma\cosh\Gamma + \sin\Gamma\sinh\Gamma & (\cosh\Gamma\sin\Gamma + \cos\Gamma\sinh\Gamma)/\sqrt{\kappa} \\ \sqrt{\kappa}(\cos\Gamma\sinh\Gamma - \cosh\Gamma\sin\Gamma) & \cos\Gamma\cosh\Gamma - \sin\Gamma\sinh\Gamma \end{bmatrix}$$

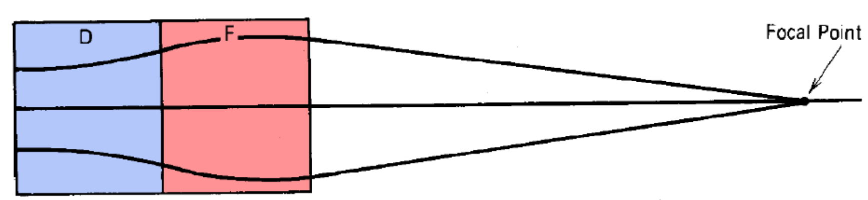


$a_{21} < 0$

→ net focusing



astigmatic



Quadrupole doublet lens

- If we take $f = \pm 1/\kappa l$, the quadrupole elements can be replaced by a drift space of length l with a thin lens at the center.

$$\boldsymbol{A_F} \approx \begin{bmatrix} 1-\frac{\kappa l^2}{2} & l-\frac{\kappa l^3}{6} \\ -\kappa l+\frac{\kappa^2 l^3}{6} & 1-\frac{\kappa l^2}{2} \end{bmatrix} \qquad \boldsymbol{A_D} \approx \begin{bmatrix} 1+\frac{\kappa l^2}{2} & l+\frac{\kappa l^3}{6} \\ \kappa l+\frac{\kappa^2 l^3}{6} & 1+\frac{\kappa l^2}{2} \end{bmatrix}$$

- [Note] The transfer matrix for a three-element optical system consisting of a drift space of length $l/2$, a thin lens with focal length $\pm f$, and another drift space is

$$\boldsymbol{A} = \begin{bmatrix} 1 & \frac{l}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \pm\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{l}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \pm \frac{l}{2f} & l \pm \frac{l^2}{4f} \\ \pm\frac{1}{f} & 1 \pm \frac{l}{2f} \end{bmatrix}$$

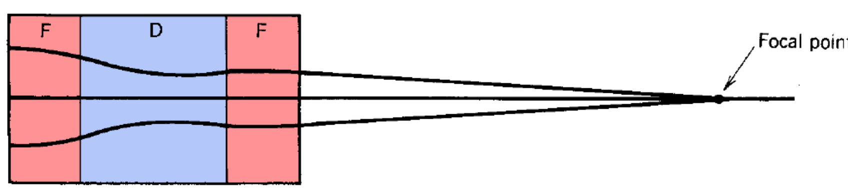
- Then, we obtain the approximate solution for doublet quadrupole lens array

$$\boldsymbol{C_{FD}} = \begin{bmatrix} 1-\kappa l^2 & 2l \\ -2\kappa^2 l^3/3 & 1+\kappa l^2 \end{bmatrix} \qquad \boldsymbol{C_{DF}} = \begin{bmatrix} 1+\kappa l^2 & 2l \\ -2\kappa^2 l^3/3 & 1-\kappa l^2 \end{bmatrix}$$

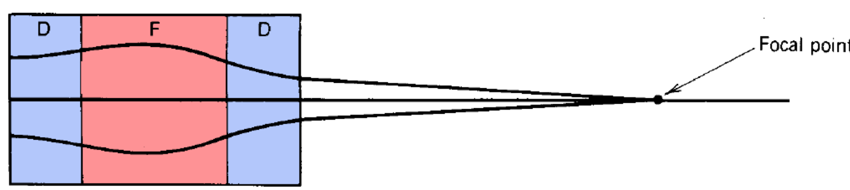
Quadrupole triplet lens

- Stigmatism can be achieved with quadrupoles in a configuration called the triplet. This consists of three quadrupole sections. The entrance and exit sections have the same length ($l/2$) and pole orientation, while the middle section is rotated 90° and has length l.
- An exact treatment shows that the exit displacements are identical in both planes for equal entrance displacements. A transfer matrix of triplet quadrupole lens array for both FDF and DFD is given by

$$\boldsymbol{C}_{\boldsymbol{FDF}} = \begin{bmatrix} 1 & 2l \\ -\frac{\kappa^2 l^3}{6} - \frac{\kappa^3 l^5}{24} & 1 \end{bmatrix} \qquad \boldsymbol{C}_{\boldsymbol{DFD}} = \begin{bmatrix} 1 & 2l \\ -\frac{\kappa^2 l^3}{6} - \frac{\kappa^3 l^5}{20} & 1 \end{bmatrix}$$



stigmatic



SEOUL NATIONAL UNIVERSITY

Focusing in a thin-lens array

- The orbit parameters at the exit of the *n*th focusing cell:

$$x_{n+1} = x_n + d\,x'_n$$

$$x'_{n+1} = x'_n - \frac{x_{n+1}}{f} = \frac{x_{n+2} - x_{n+1}}{d}$$

$$\begin{bmatrix} x_{n+1} \\ x'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & d \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{bmatrix} \begin{bmatrix} x_n \\ x'_n \end{bmatrix}$$

- We obtain the difference equation:

$$x_{n+2} - 2\left(1 - \frac{d}{2f}\right)x_{n+1} + x_n = 0$$

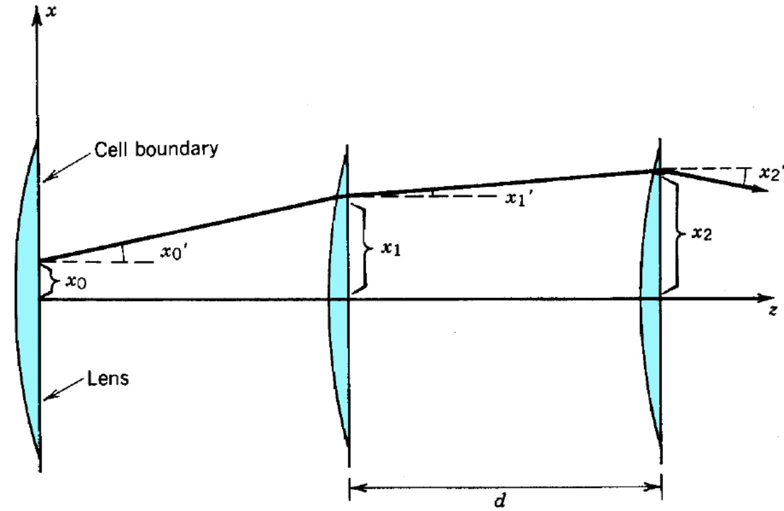
$$\Rightarrow \quad x_n = x_0 \exp(jn\mu) = x_0 \cos(n\mu + \varphi)$$

- The solution is harmonic when:

$$\left|1 - \frac{d}{2f}\right| \le 1$$

- The phase advance:

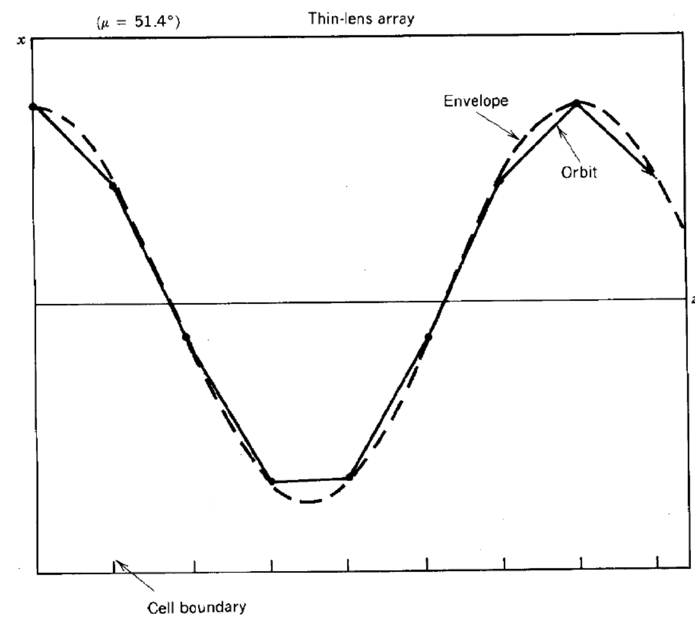
$$\mu = \pm\cos^{-1}\left(1 - \frac{d}{2f}\right)$$



Focusing in a thin-lens array

- There are parameters for which all particle orbits are unstable. Orbits are no longer harmonic but have an exponentially growing displacement when

$$\left|1-\frac{d}{2f}\right|>1$$

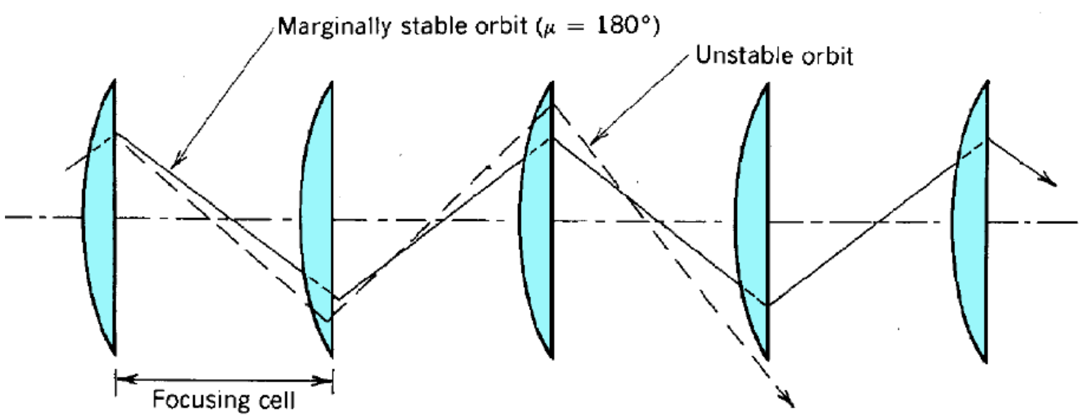


- Stability condition for transport in a thin-lens array:

$$f\geq\frac{d}{4}$$

- In general: $\boldsymbol{C}^n=\begin{bmatrix}1 & d\\ -\frac{1}{f} & 1-\frac{d}{f}\end{bmatrix}^n$

$$\left|\frac{\mathrm{Tr}\,\boldsymbol{C}}{2}\right|\leq 1$$



General treatment using matrix

- Transfer matrices have eigenvalues and eigenvectors.

$$\boldsymbol{C}\boldsymbol{v_i} = \lambda_i \boldsymbol{v_i}$$

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

- Any orbit vector at the entrance to a periodic focusing system can be written in the form

$$\boldsymbol{u_0} = a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2}$$

- The orbit vector after n focusing cells of a periodic system is given by

$$\boldsymbol{u_n} = a_1\lambda_1^n\boldsymbol{v_1} + a_2\lambda_2^n\boldsymbol{v_2}$$

- Eigenvalues

$$\lambda^2 - (\mathrm{Tr}\boldsymbol{C})\lambda + \det\boldsymbol{C} = \lambda^2 - (\mathrm{Tr}\boldsymbol{C})\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 = \frac{\mathrm{Tr}\boldsymbol{C}}{2} \pm \sqrt{\left(\frac{\mathrm{Tr}\boldsymbol{C}}{2}\right)^2 - 1}$$

- Stability condition

$$\left|\frac{\mathrm{Tr}\ \boldsymbol{C}}{2}\right| \leq 1 \qquad \lambda_1, \lambda_2 = \frac{\mathrm{Tr}\boldsymbol{C}}{2} \pm j\sqrt{1 - \left(\frac{\mathrm{Tr}\boldsymbol{C}}{2}\right)^2} = \cos\mu \pm j\sin\mu = \exp(\pm j\mu)$$

Quadrupole focusing channels

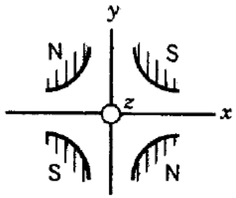
- For FD channel

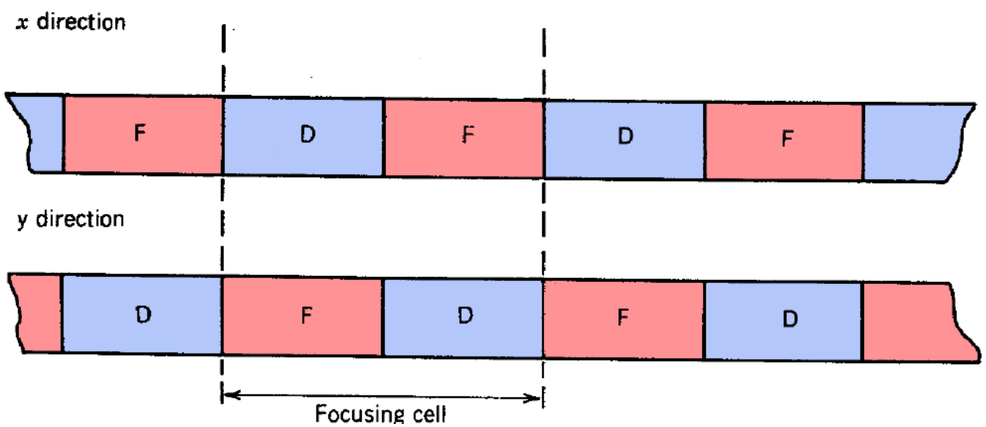
$$C_{FD} = \begin{bmatrix} \cos\Gamma\cosh\Gamma - \sin\Gamma\sinh\Gamma & (\cosh\Gamma\sin\Gamma + \cos\Gamma\sinh\Gamma)/\sqrt{\kappa} \\ \sqrt{\kappa}(\cos\Gamma\sinh\Gamma - \cosh\Gamma\sin\Gamma) & \cos\Gamma\cosh\Gamma + \sin\Gamma\sinh\Gamma \end{bmatrix}$$

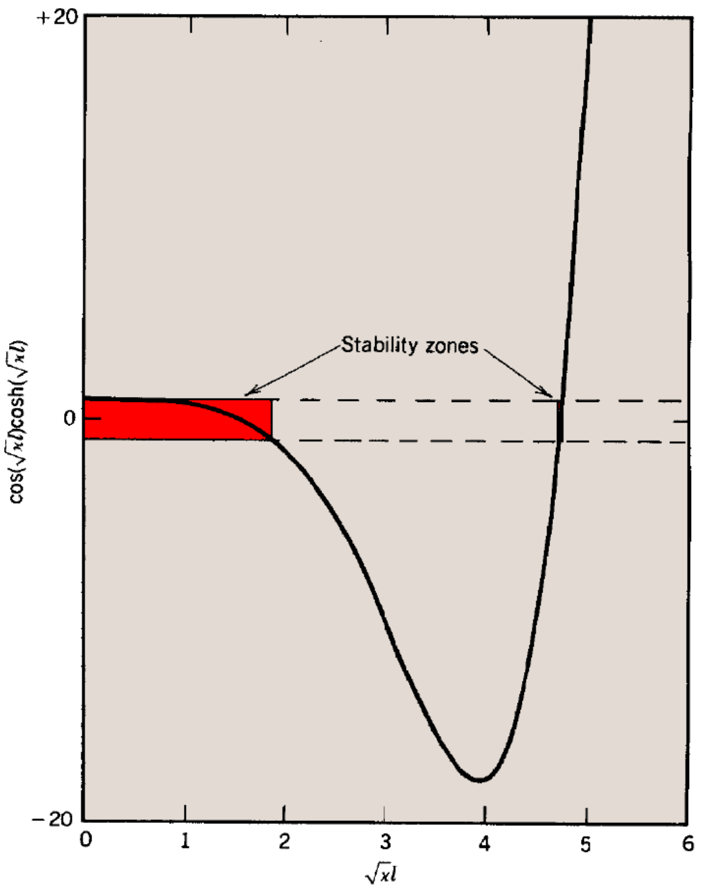
- For stable orbits

$$-1 \leq \cos\Gamma\cosh\Gamma \leq 1 \qquad \Gamma = \sqrt{\kappa}l$$

$$0 \leq \Gamma \leq 1.86$$







Actual particle orbits in an FD quadrupole array

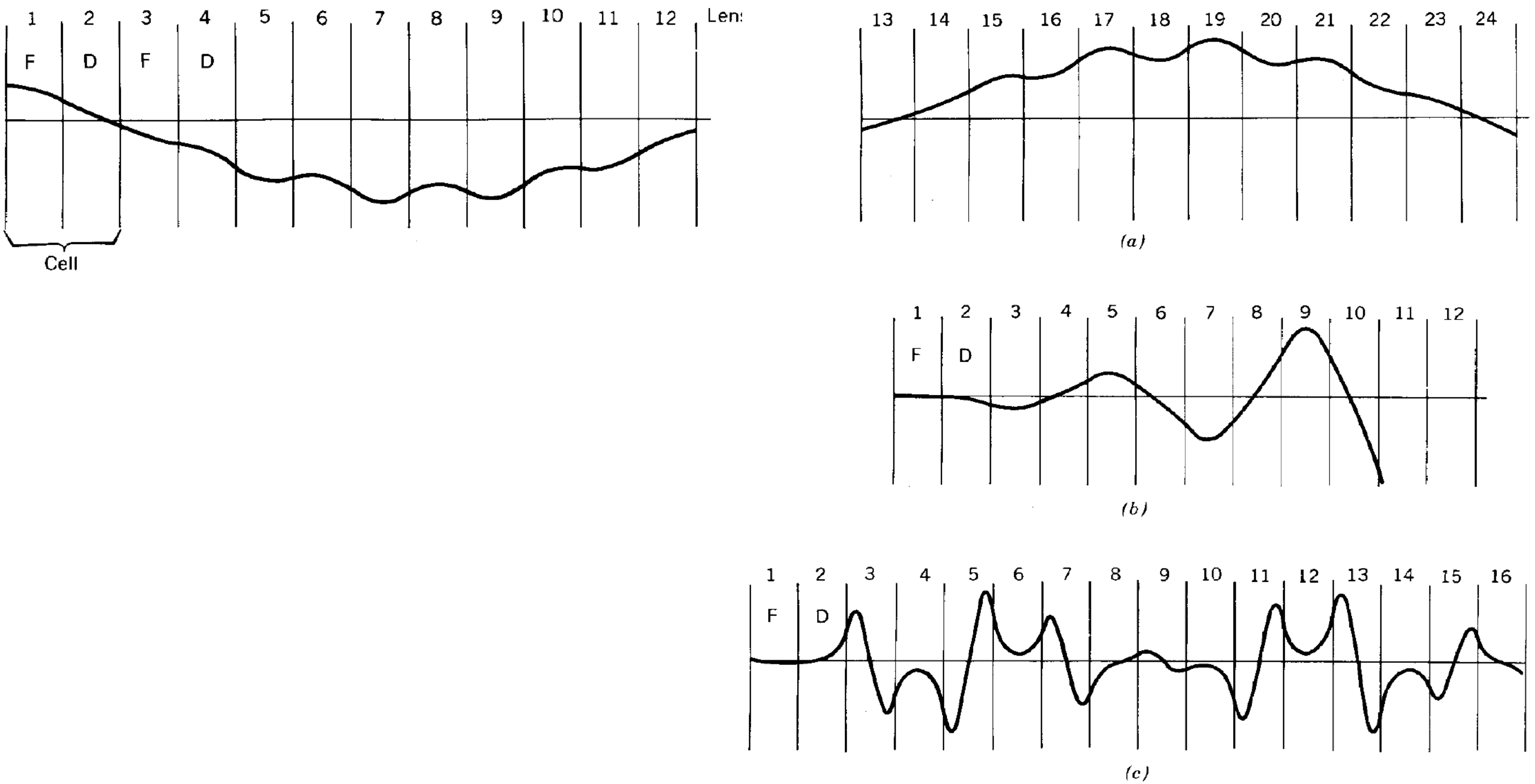


Figure 8.14 Numerical calculations of actual particle orbits in an FD quadrupole array. (*a*) $\sqrt{\kappa}\, l = 1$ (stable orbit, first passband). (*b*) $\sqrt{\kappa}\, l = 1.9$ (unstable orbit). (*c*) $\sqrt{\kappa}\, l = 4.7$ (stable orbit, second passband).

Orbital stability in quadrupole array with unequal focusing strength

- In many strong focusing systems, alternate cells may not have the same length or focusing strength, i.e. $\Gamma_F \neq \Gamma_D$.
- The transfer matrix for DF channel is

$$\boldsymbol{C_{DF}} = \begin{bmatrix} \cos\Gamma_1 & \sin\Gamma_1/\sqrt{\kappa_1} \\ -\sqrt{\kappa_1}\sin\Gamma_1 & \cos\Gamma_1 \end{bmatrix}\begin{bmatrix} \cosh\Gamma_2 & \sinh\Gamma_2/\sqrt{\kappa_2} \\ \sqrt{\kappa_1}\sinh\Gamma_2 & \cosh\Gamma_2 \end{bmatrix}$$

- The phase advance for DF operation:

$$\cos\mu_{DF} = \cos\Gamma_1\cosh\Gamma_2 + \left(\frac{\sin\Gamma_1\sinh\Gamma_2}{2}\right)\left(\frac{\Gamma_2 l_1}{\Gamma_1 l_2} - \frac{\Gamma_1 l_2}{\Gamma_2 l_1}\right)$$

- Similarly,

$$\cos\mu_{FD} = \cosh\Gamma_1\cos\Gamma_2 + \left(\frac{\sinh\Gamma_1\sin\Gamma_2}{2}\right)\left(\frac{\Gamma_1 l_2}{\Gamma_2 l_1} - \frac{\Gamma_2 l_1}{\Gamma_1 l_2}\right)$$

- Stability condition:

$$-1 \leq \cos\mu_{FD} \leq 1$$

$$-1 \leq \cos\mu_{DF} \leq 1$$

Necktie diagram

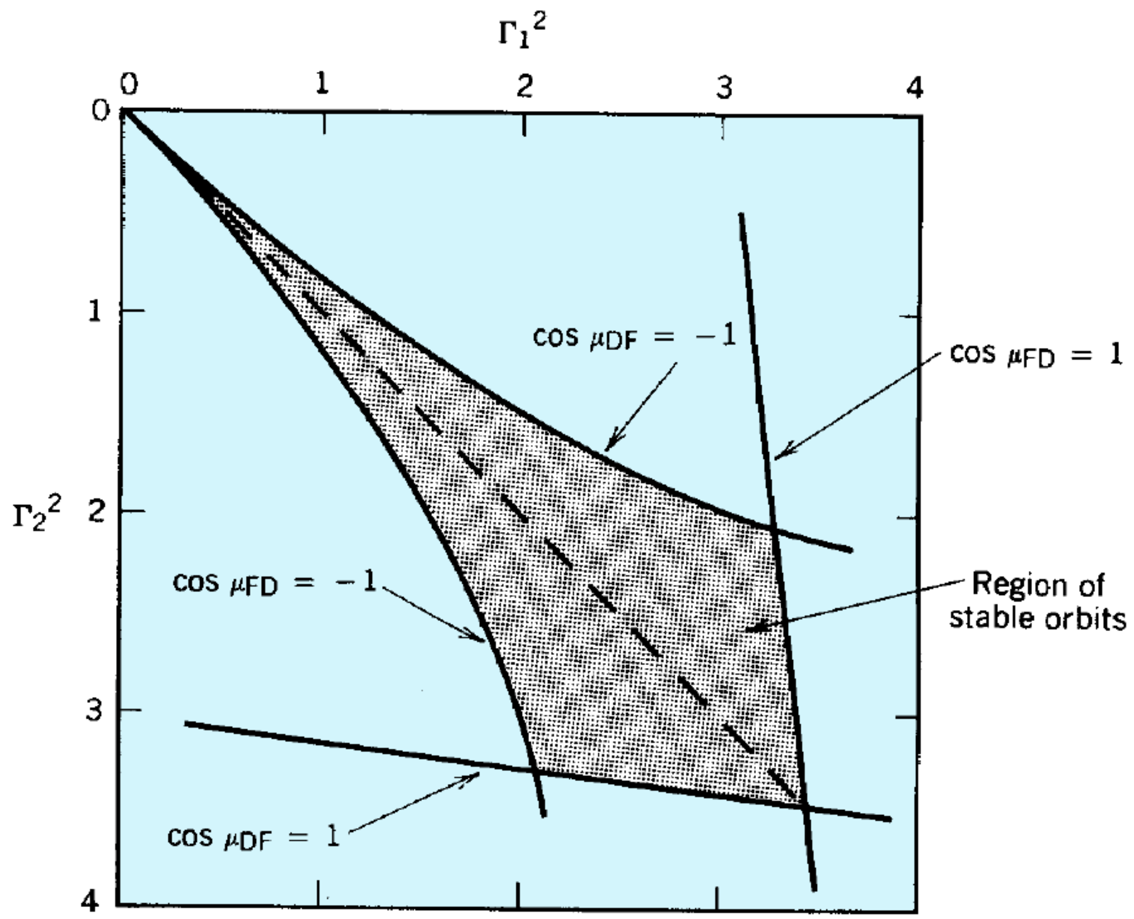


Figure 8.15 Necktie diagram. Orbital stability in an FD (DF) quadrupole array; the two lenses of a cell have unequal focusing strength. $\Gamma_1 = \sqrt{\kappa_1}\, l_1$, $\Gamma_2 = \sqrt{\kappa_2}\, l_2$. Region of parameter space with orbital stability in both x and y directions shaded. FD (lens 1 focusing, lens 2 defocusing), DF (lens 1 defocusing, lens 2 focusing).

Homework

- Derive the followings for doublet or triplet quadrupole lens arrays.

$$\boldsymbol{C_{FD}} = \begin{bmatrix} 1-\kappa l^2 & 2l \\ -2\kappa^2 l^3/3 & 1+\kappa l^2 \end{bmatrix} \qquad \boldsymbol{C_{DF}} = \begin{bmatrix} 1+\kappa l^2 & 2l \\ -2\kappa^2 l^3/3 & 1-\kappa l^2 \end{bmatrix}$$

$$\boldsymbol{C_{FDF}} = \begin{bmatrix} 1 & 2l \\ -\frac{\kappa^2 l^3}{6} & 1 \end{bmatrix} \qquad \boldsymbol{C_{DFD}} = \begin{bmatrix} 1 & 2l \\ -\frac{\kappa^2 l^3}{6} & 1 \end{bmatrix}$$