Phase Space and Beam Emittance

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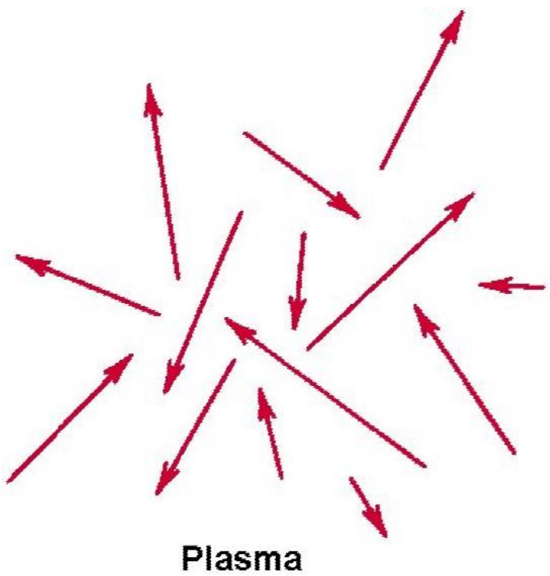
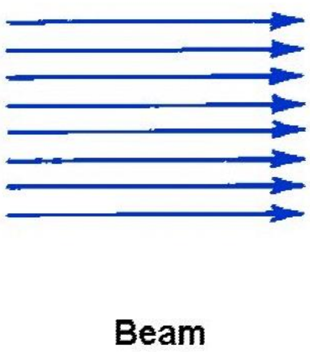
Beam

- Although single charged particles may be useful for some physics experiments, we need large numbers of energetic particles for most applications. A flux of particles is a beam when the following two conditions hold:
  - The particles travel in almost the same direction.
  - The particles have a small spread in kinetic energy.
- A beam is an ordered flow of charged particles. A disordered set of particles, such as a thermal plasma, is not a beam.
- The degree of order in a flow of particles is called coherence.





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Configuration space vs phase space

- Laminar phase flow is the foundation for theories of collective behavior.

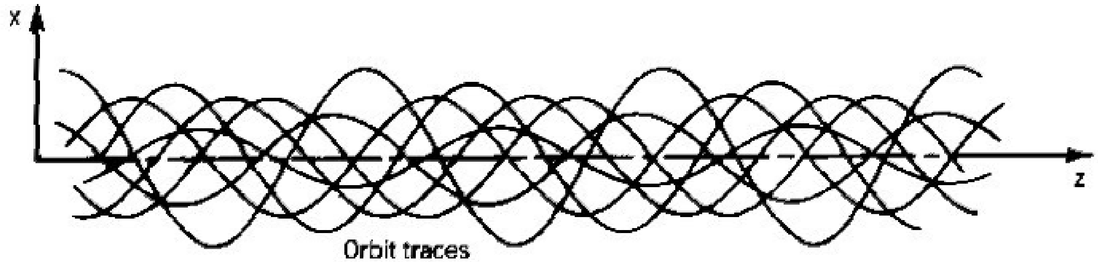


Figure 2.1. The orbits of particles in a beam portrayed in configuration space. The particles move in a non-linear transverse force with mixed amplitudes, phases, and frequencies.

$$(x(t), y(t), z(t))$$

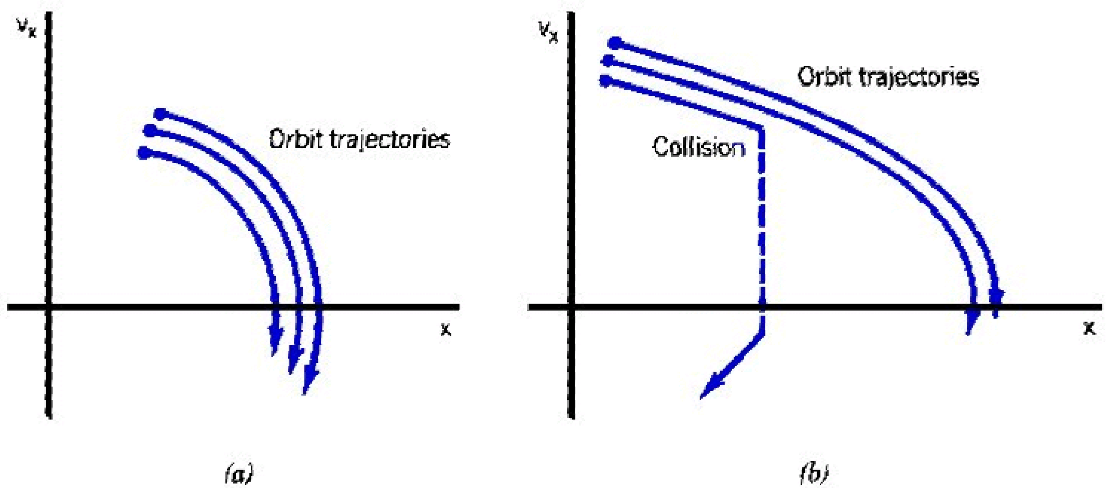
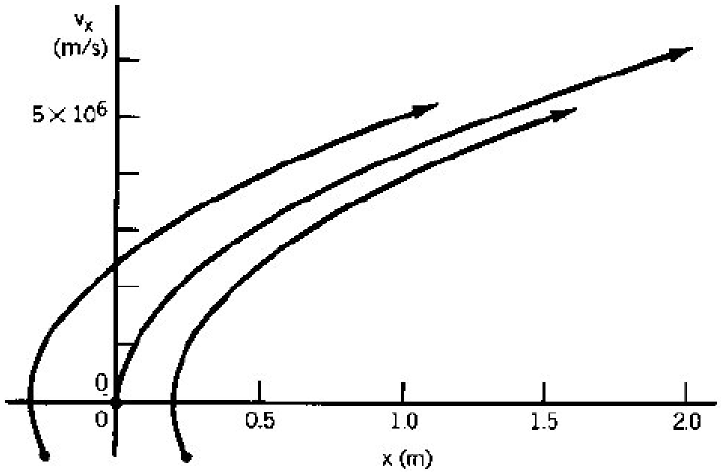


Figure 2.2. Representation of particle motions in phase-space. (*a*) Laminar phase-space trajectories of particle orbit vectors with no collisions. (*b*) Effect of a collision on the phase-space position of a particle orbit vector.

$$(x, y, z, v_x, v_y, v_z)$$

Examples of phase space description

- The trajectories of particles accelerated by a constant axial electric field E_z:



$$z(t) = z_0 + v_{z0}t + (qE_z/m_0)t^2/2$$

$$v_z(t) = v_{z0} + (qE_z/m_0)t$$

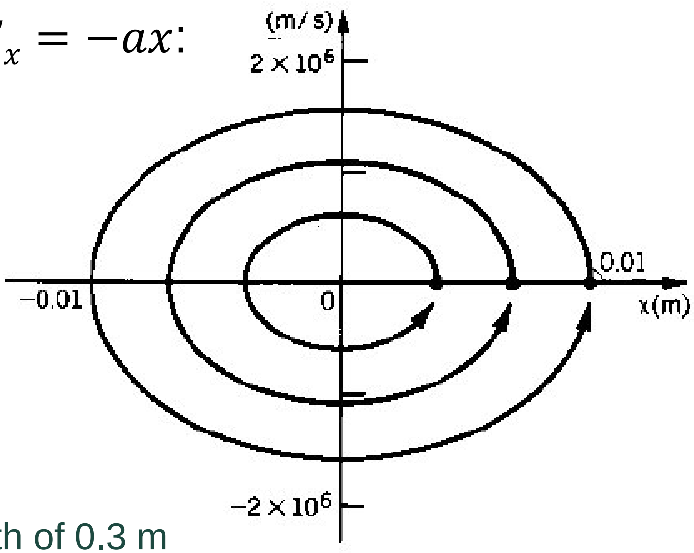
Protons in an electric field of 10^5 V/m

- Trajectories of particles in a linear focusing force $F_x = -ax$:

$$x(t) = x_0 \cos(\omega t + \phi)$$

$$v_x(t) = -x_0\omega \sin(\omega t + \phi)$$

$$\omega = \sqrt{a/m_0}$$



300 keV protons with a betatron wavelength of 0.3 m

Phase space description for relativistic particles

- The relativistic equations of motion

$$(\gamma m_0 c^2)^2 = (cp_x)^2 + (m_0 c^2)^2$$

$$\frac{dx}{dt} = \frac{p_x}{\gamma m_0} = \frac{p_x}{m_0\sqrt{1 + p_x^2/m_0^2 c^2}}$$

$$\frac{dp_x}{dt} = -ax$$

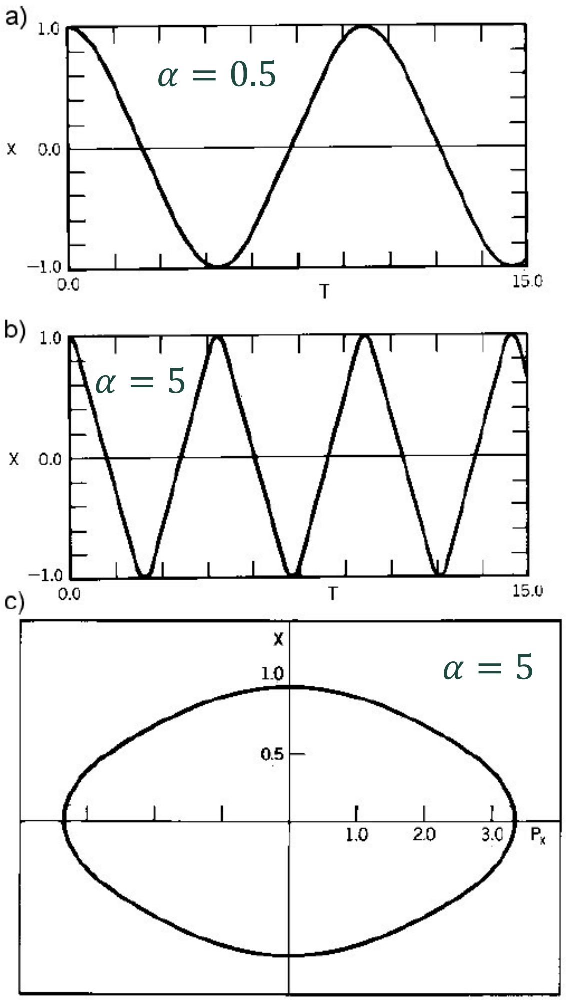
- Normalization:

$$T = t/(x_0/c) \qquad X = x/x_0 \qquad P_x = p_x/m_0 c$$

$$\frac{dX}{dT} = \frac{P_x}{\sqrt{1 + P_x^2}}$$

$$\frac{dP_x}{dT} = -\alpha X$$

where, $\alpha = a x_0^2/m_0 c^2$



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Conservation of phase space volume

- Conservation of the phase-space volume occupied by a particle distribution is a fundamental theorem of collective physics. (Liouville's theorem: the principle of incompressiblity of a phase fluid)

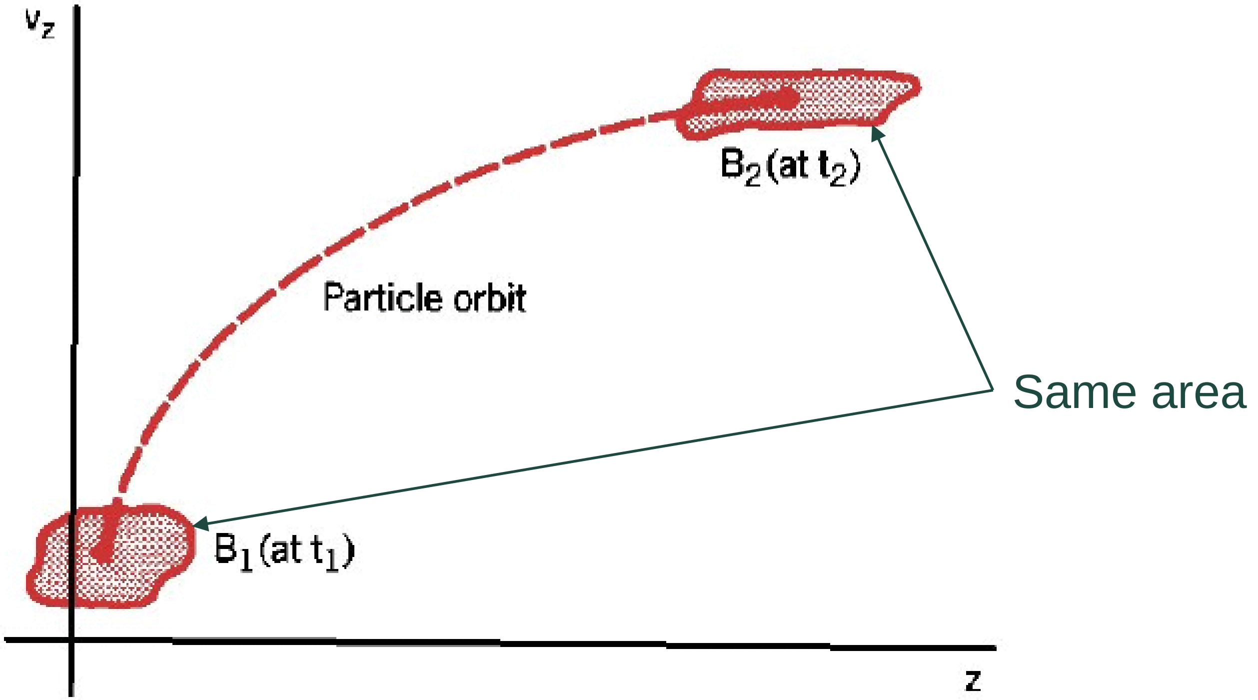


Figure 2.9. Illustration of phase-volume conservation - the boundary around the included group of particle orbit vectors changes shape with time but has the same area. The dashed line shows the orbit-vector trajectory of a particle at the center of the group.

Maxwell distribution

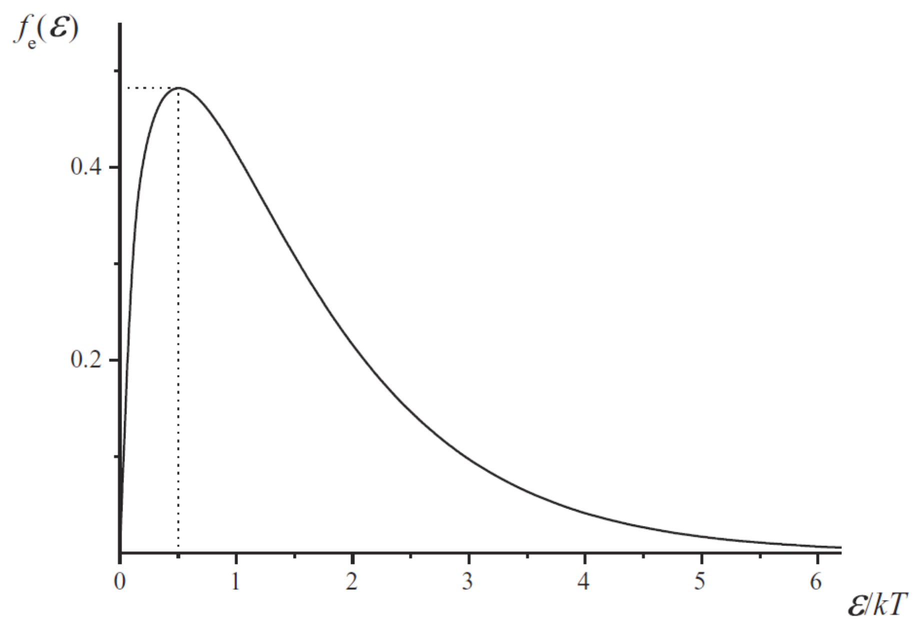
- Particles in an isotropic Maxwell distribution are in thermal equilibrium. They have a spread in kinetic energy.
- For a single species in thermal equilibrium with itself (e.g. electrons), in the absence of time variation, spatial gradients, and accelerations, the Boltzmann equation reduces to

$$\left.\frac{\partial f}{\partial t}\right|_c = 0$$

- Then, we obtain the Maxwell-Boltzmann velocity distribution

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 exp\left(-\frac{mv^2}{2kT}\right)$$

$$f(\varepsilon) = \frac{2}{\sqrt{\pi}}\left(\frac{1}{kT}\right)^{3/2} \varepsilon^{1/2}\exp\left(-\frac{\varepsilon}{kT}\right)$$



- The mean speed

$$\bar{v} = \left(\frac{8kT}{\pi m}\right)^{1/2}$$

Displaced Maxwell distribution

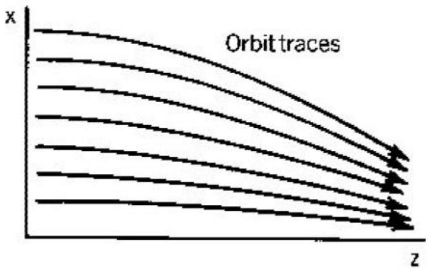
- Charged particle beams are non-isotropic and usually almost monoenergetic. In a sense, the primary goal of beam technology is to create non-Maxwellian distributions and to preserve them over time scales set by the application.
- A common assumption used in beam theory is that the particles have a Maxwell distribution when observed in the beam rest frame. The transformed distribution observed in the stationary frame of the accelerator is called a displaced Maxwell distribution.
- For example, consider a nonrelativistic ion beam extracted from a plasma source with ion temperature T_i. The beam is axially bunched passing through a radio-frequency quadrupole accelerator. The beam emerges from the accelerator with kinetic energy E_0. We can represent the exit beam distribution in the stationary frame as

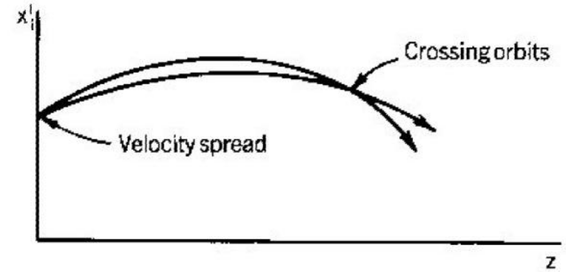
$$f(v_x, v_y, v_z) \sim exp\left(-\frac{m_i(v_x^2 + v_y^2)}{2kT_i}\right) exp\left(-\frac{m_i(v_z - v_0)^2}{2kT_i'}\right)$$

$$v_0 = \left(\frac{2E_0}{m_i}\right)^{1/2}$$

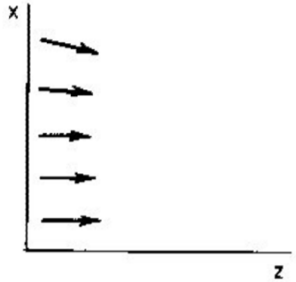
Laminar and non-laminar beams

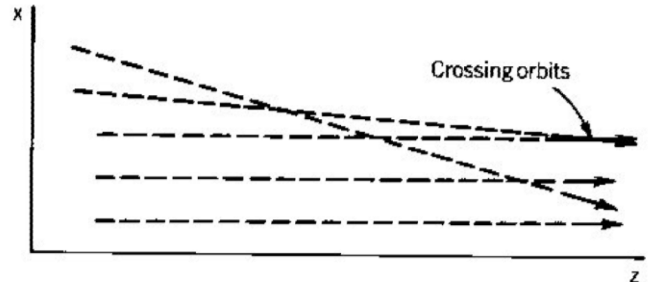
- Beams with good parallelism are easier to transport than beams with large random transverse velocity components. Ordered beams can focus to a small spot size.
- The ideal charged particle beam has laminar particle orbits. Orbits in a laminar beam flow in layers that never intersect. A laminar beam satisfies two conditions:
  - All particles at a position have identical transverse velocities. If this is not true, the orbits of two particles that start at the same position could separate and later cross each other.



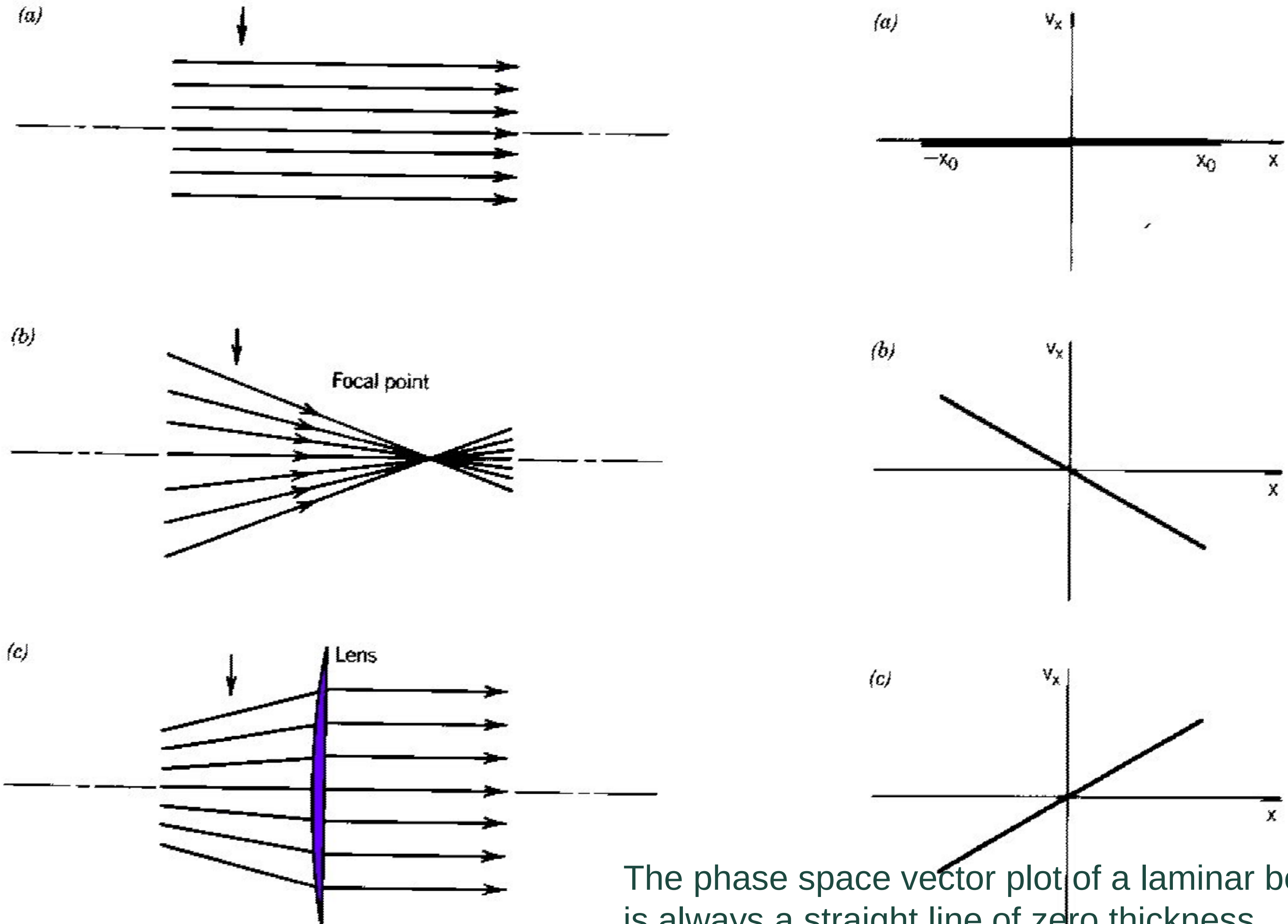


  - The magnitude of the transverse particle velocity is linearly proportional to the displacement from the axis of beam symmetry.





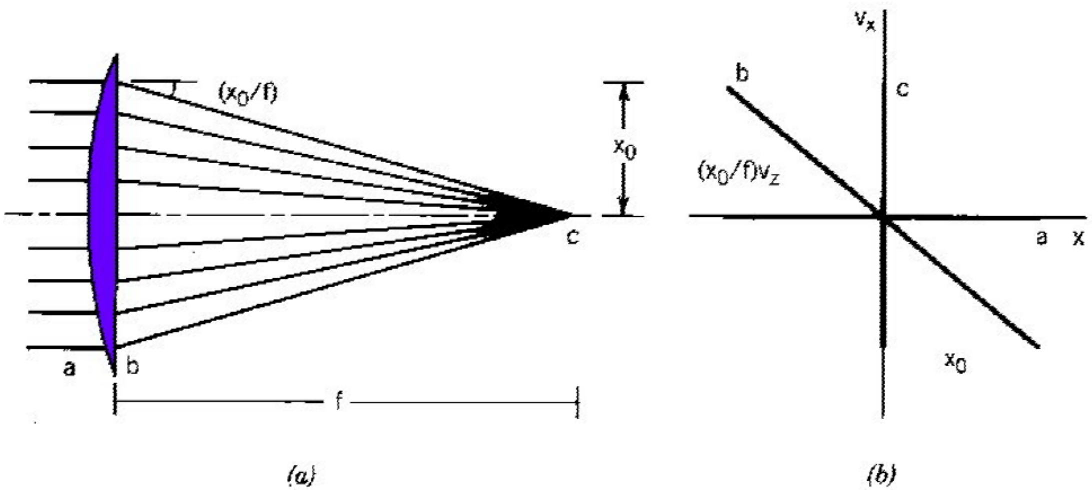
Laminar beam orbit-vector distributions viewed in transverse phase-space



The phase space vector plot of a laminar beam is always a straight line of zero thickness

Focusing a laminar beam

- The incident beam distribution is a straight line of length $2x_0$ aligned along the x axis.
- The lens displaces the distribution in the v_x direction while preserving the projected length along the x axis. The velocity displacement has a maximum value of $(x_0/f)v_z$ at the beam edge.
- During subsequent transit through the drift region of length f, the orbit vectors converge toward $x = 0$.
- The orientation of the distribution changes until it aligns with the v_x axis at the focal point. Here the distribution has dimension equal to zero in x and a halfwidth along v_x of $\pm(x_0/f)v_z$.



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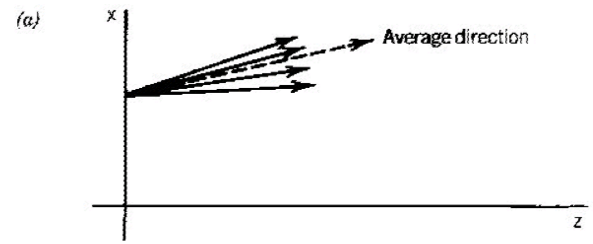
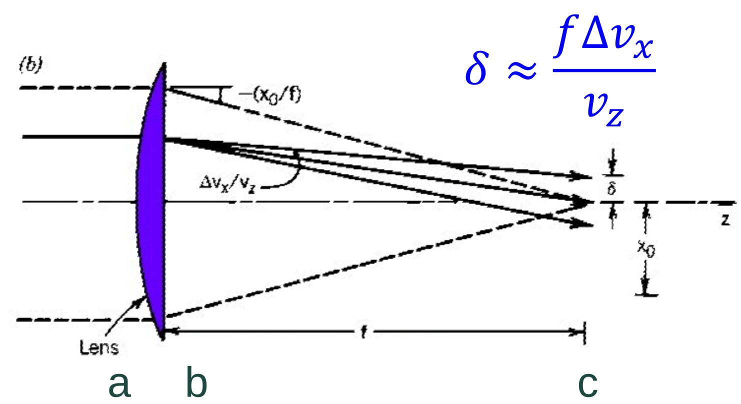
Properties of non-laminar beams

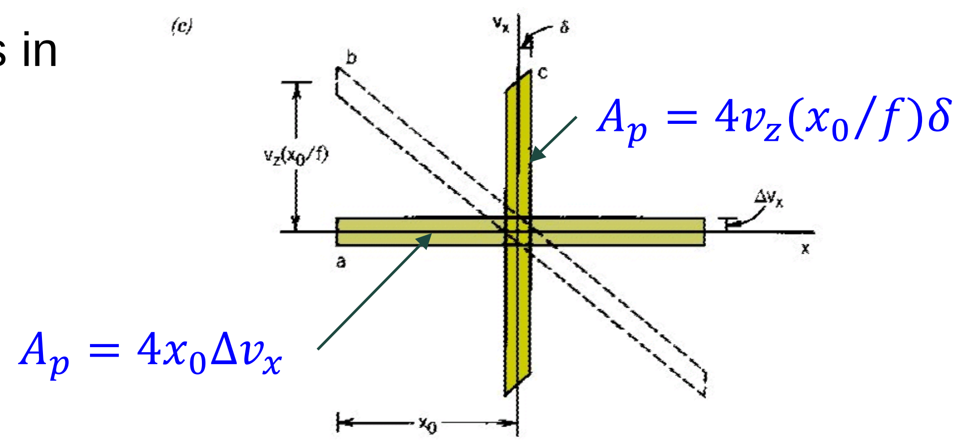
a. Configuration-space view of particle orbits — particles at the same point move in different directions.



b. Configuration-space view of the transverse focusing of a non-laminar beam. The width of the focal spot is proportional to the transverse velocity spread of the incident beam.

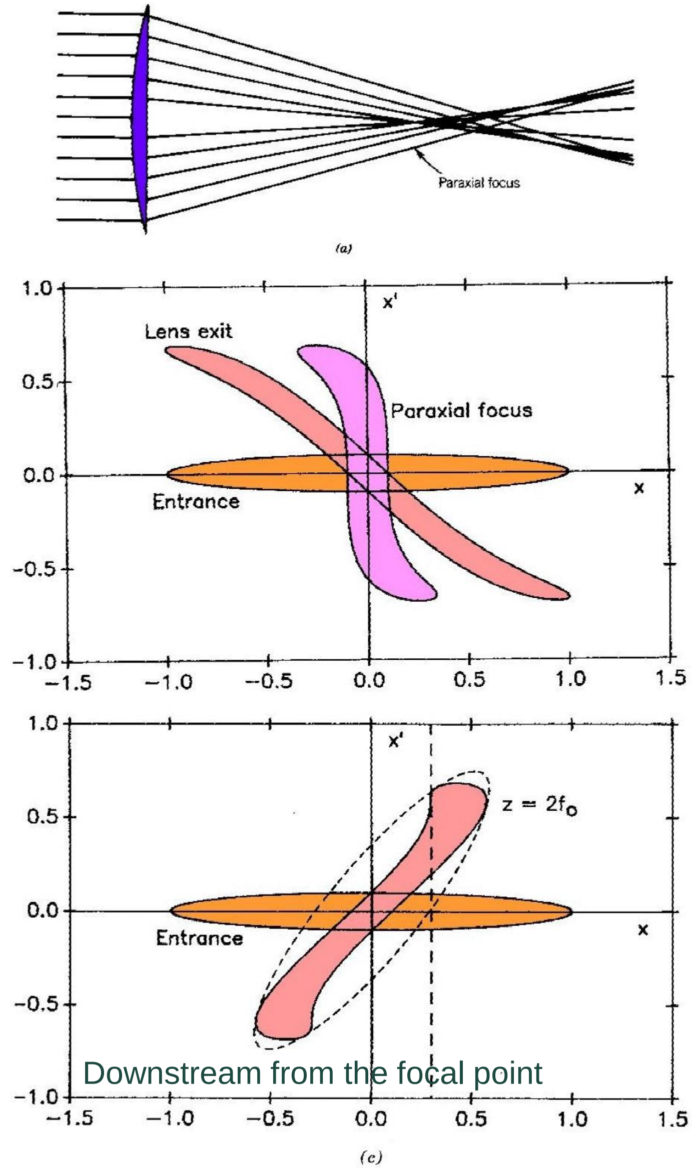


c. Snapshots of orbit vector distributions in phase-space at positions a, b and c.



Beam focusing by a non-linear lens with soft force

- Irreversible processes change distributions in such a way that they cannot be restored by macroscopic forces.
- The transverse force of the lens is soft; the lens force is weak at large displacement compared with a linear lens. As a result, the lens underfocuses particles on the periphery.
- There are positions in the beam, shown by dashed lines in Fig. (c), where the particles have two different values of transverse velocity.
- Note that the phase-space area filled by the beam is unchanged. Nonetheless, the distorted distribution surrounds regions of unoccupied phase space - the effective area of the distribution is larger. If we sought to focus all particles in the distribution of Fig. (c) to a spot with an ideal linear lens, the relevant phase space area is that inside a boundary surrounding all particles. Figure (c) shows such a boundary as a dashed line.



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Emittance

- To designate the quality of a beam for an application, we must adopt a figure of merit based on the effective volume (area) occupied by the distribution. This quantity is the emittance.
- A beam will occupy a finite volume in the six-dimensional phase space, $dxdydzdp_x dp_y dp_z$, this is defined as the beam emittance.
- If the momentum in the z-direction is much greater than in the transverse direction (x-, y-direction) (paraxial beam), the radial momentum can be replaced by the orbital angle,

$$\frac{p_x}{p_z} = \frac{dx}{dz} = x', \qquad \frac{p_y}{p_z} = \frac{dy}{dz} = y'$$

- The coordinates (x, x', y, y') are usually treated as functions of z rather than time, t. They describe the trace of a particle orbit along the axial direction, [x(z), y(z)], rather than the time-dependent position [x(t), y(t)]. Hence, the space defined by the coordinates is called trace space.
- In beam physics, it is often more convenient to work in trace space (the x-x' plane) than the x-v_x or x-p_x plane. This is partly because the inclination angle x' is much more useful for visualizing the shape of the beam than the transverse velocity or momentum.

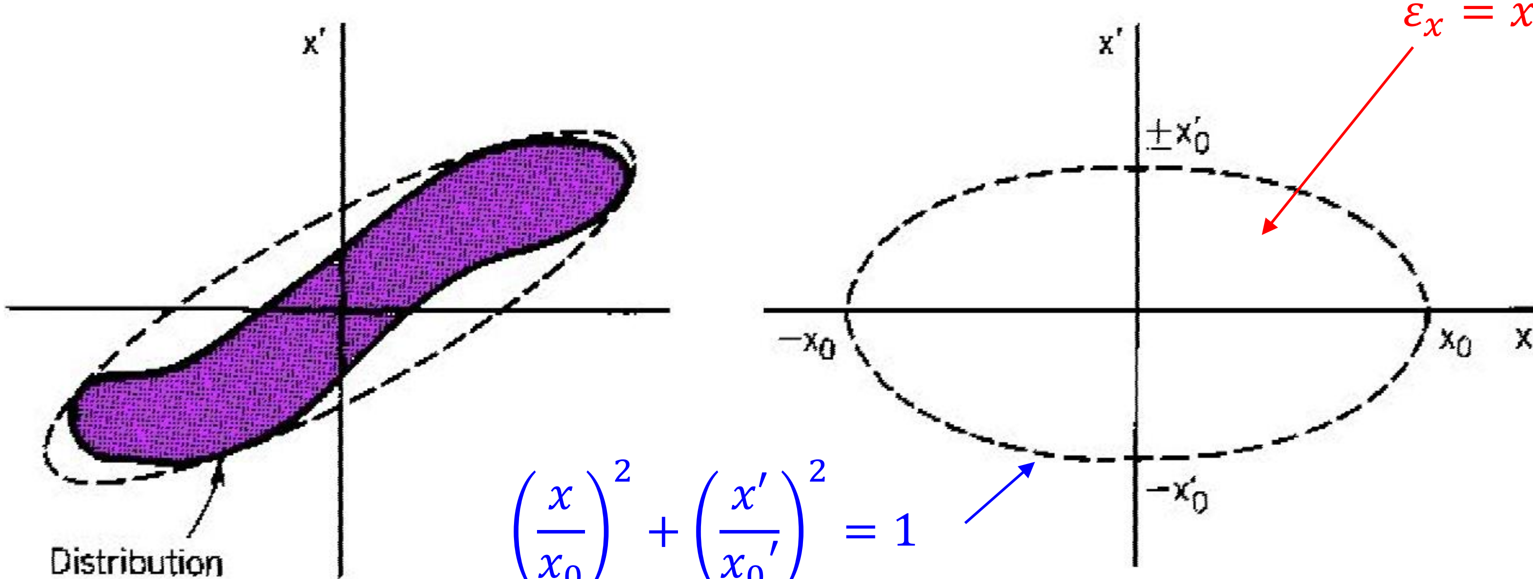
Emittance

- In many cases, the x and y motions are independent and it is convenient to consider x, x' and y, y' space projections individually. We define emittance as the area of the ellipse divided by π.

$$\varepsilon_x = \frac{1}{\pi}\iint dx dx', \qquad \varepsilon_y = \frac{1}{\pi}\iint dy dy' \qquad [\pi \text{ mm-mrad}]$$

- The smaller the phase area occupied by the beam, i.e. the smaller the beam emittance, the better the quality of the beam. Here, the term quality implies focusability or parallelism.

$$\varepsilon_x = x_0 x_0'$$



$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{x'}{x_0'}\right)^2 = 1$$

Origins of non-zero emittance in an electron beam injector

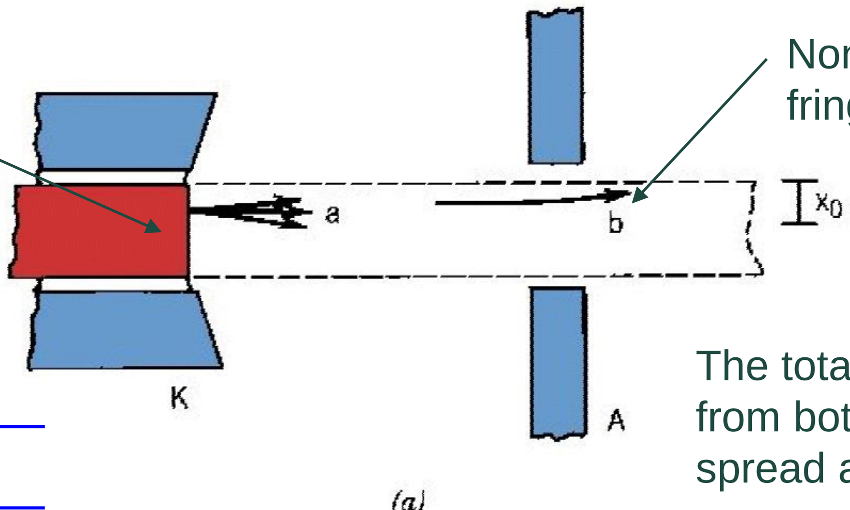
- The final emittance of a beam represents the sum of the intrinsic emittance from the source and emittance growth during acceleration

Thermionic emitter operating at T

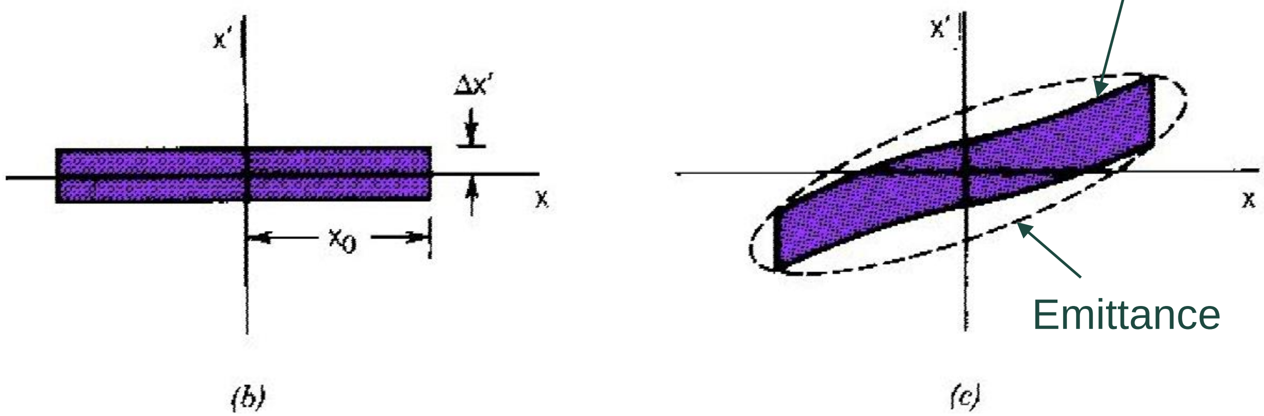
$$\overline{v_x} = \sqrt{2kT/\pi m}$$

$$\Delta x' = \frac{\overline{v_x}}{v_z} = \sqrt{\frac{kT}{\pi(m v_z^2/2)}}$$

Nonlinear deflection by fringing electric field



The total area reflects contributions from both the thermal velocity spread and distortions from optical errors



Emittance

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Elliptical distribution

- We have seen that beam distributions enclosed by elliptical boundaries play an important role in accelerators with linear focusing systems. Trace-space ellipses have geometric properties that lead to a compact beam transport theory.
- A linear transformation always transforms an elliptical distribution into another ellipse. Furthermore, with no acceleration, a linear transformation does not change the area of a distribution ellipse — the beam emittance is constant. If the optical element has transfer matrix $\boldsymbol{M}$, emittance conservation holds if $\det \boldsymbol{M} = 1$.
- The phase-space equation for the boundary of upright ellipse:

$$(x/X_0)^2 + (x'/X_0')^2 = 1$$

- Multiplying both sides by $X_0 X_0'$ gives a standard form for elliptical boundaries:

$$(X_0'/X_0)x^2 + (X_0/X_0')x'^2 = X_0 X_0' = \epsilon$$

- We shall now derive the general form for a skewed distribution ellipse by applying a linear transformation to an upright ellipse.

$$\begin{aligned} x_1 &= m_{11}x_0 + m_{12}x_0' \\ x_1' &= m_{21}x_0 + m_{22}x_0' \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_0 &= m_{22}x_1 - m_{12}x_1' \\ x_0' &= -m_{21}x_1 + m_{11}x_1' \end{aligned}$$

General mathematical form for a distribution ellipse

- By arranging the equation and dropping subscript 1, we obtain the general mathematical form for a distribution ellipse:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon}$$

$$\gamma = (X_o'/X_o)\, m_{22}^2 + (X_o/X_o')\, m_{21}^2 ,$$

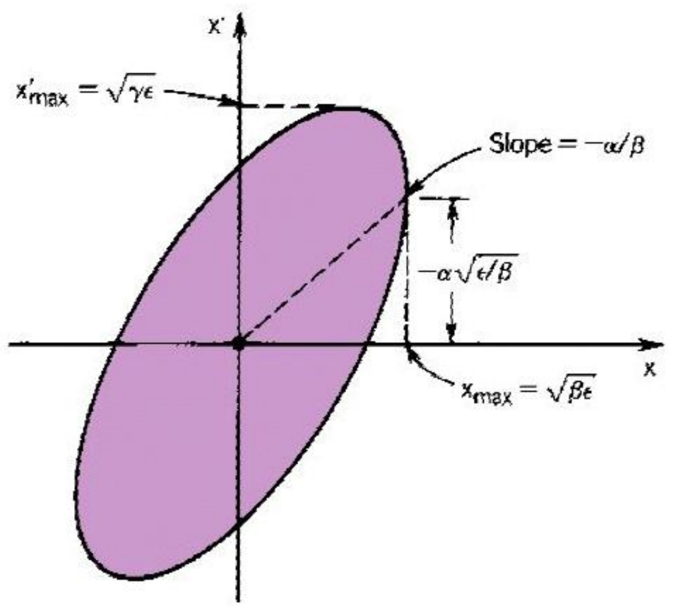
$$\alpha = -(X_o'/X_o)\, m_{12}\, m_{22} - (X_o/X_o')\, m_{11}\, m_{21} ,$$

$$\beta = (X_o'/X_o)\, m_{12}^2 + (X_o/X_o')\, m_{11}^2 .$$

- The quantities α, β and γ are called the transport parameters (or sometimes Twiss parameters). Combined with the emittance, ϵ, they specify the distribution at the output of the optical system.

$$\boxed{\gamma\beta - \alpha^2 = m_{11}m_{22} - m_{12}m_{21} = 1}$$

Courant-Snyder invariant



- The maximum extent in spatial dimension:

$$x_{max} = \sqrt{\beta\epsilon}$$

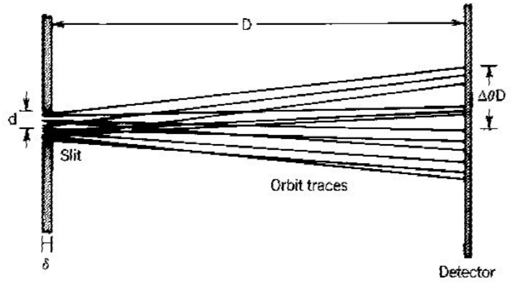
- The maximum extent in angular direction:

$$x'_{max} = \sqrt{\gamma\epsilon}$$

- The quantity α determines the envelope angle of the beam:

$$\frac{dx_{max}}{dz} = x'@x_{max} = -\alpha\sqrt{\epsilon/\beta}$$

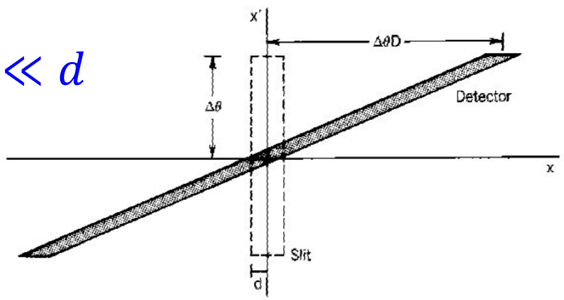
Measurement of emittance: pepperpot diagnostic



(a)

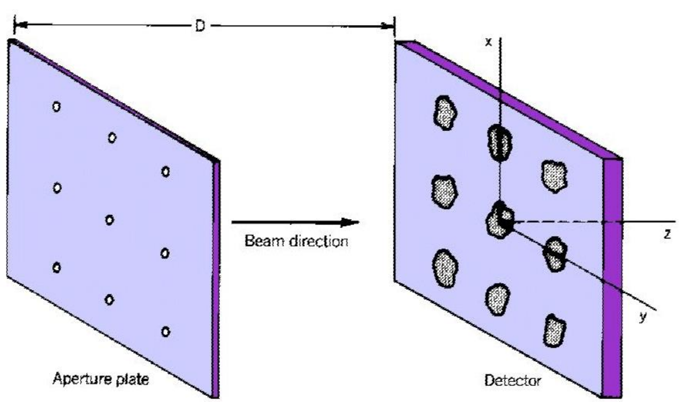
$$\Delta\theta \cdot D \gg d$$

$$\delta \cdot \Delta\theta \ll d$$

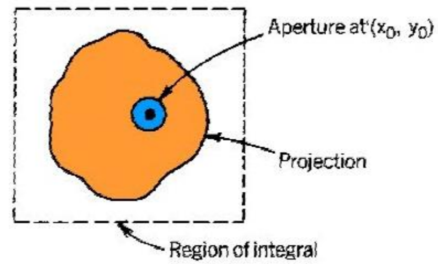


(b)

Pepperpot (imaging detector)

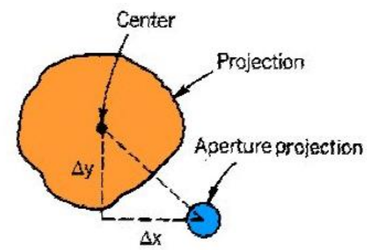


Intensity ~ Current density



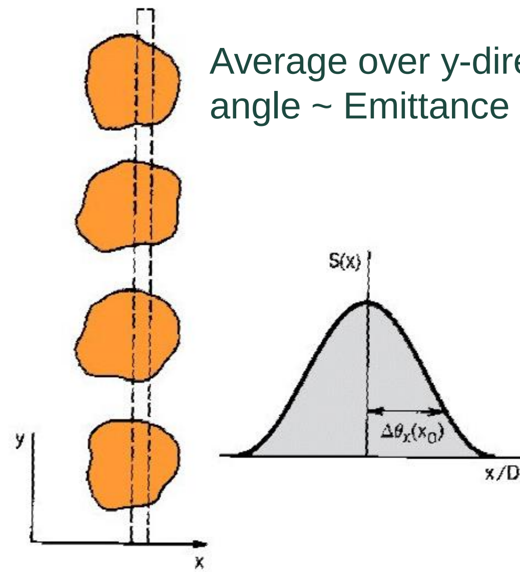
(a)

Displacement ~ Beam direction

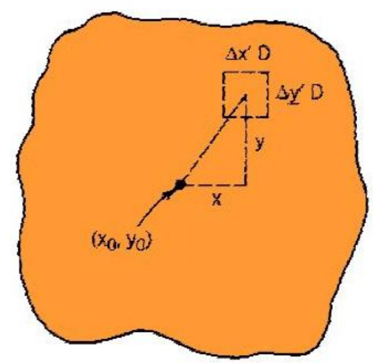


(b)

Average over y-direction ~ Divergence angle ~ Emittance in x-direction



(c)



(d)

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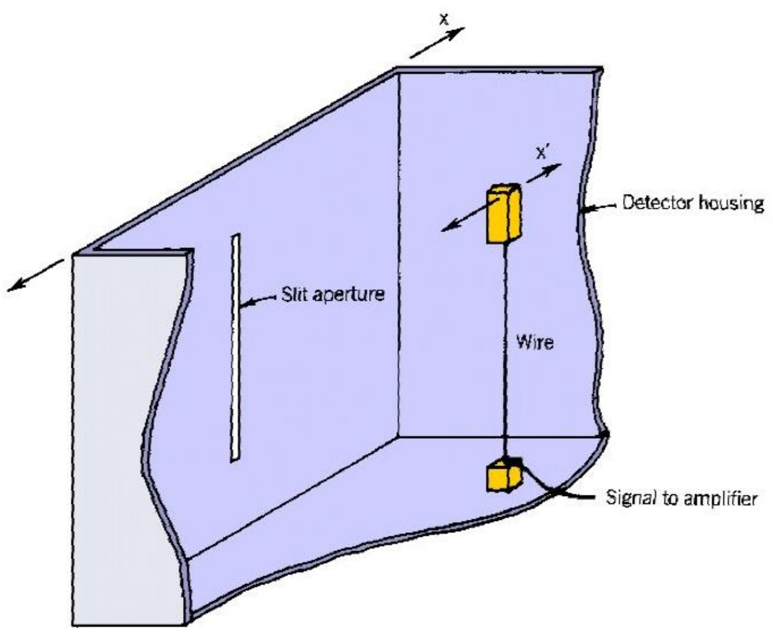
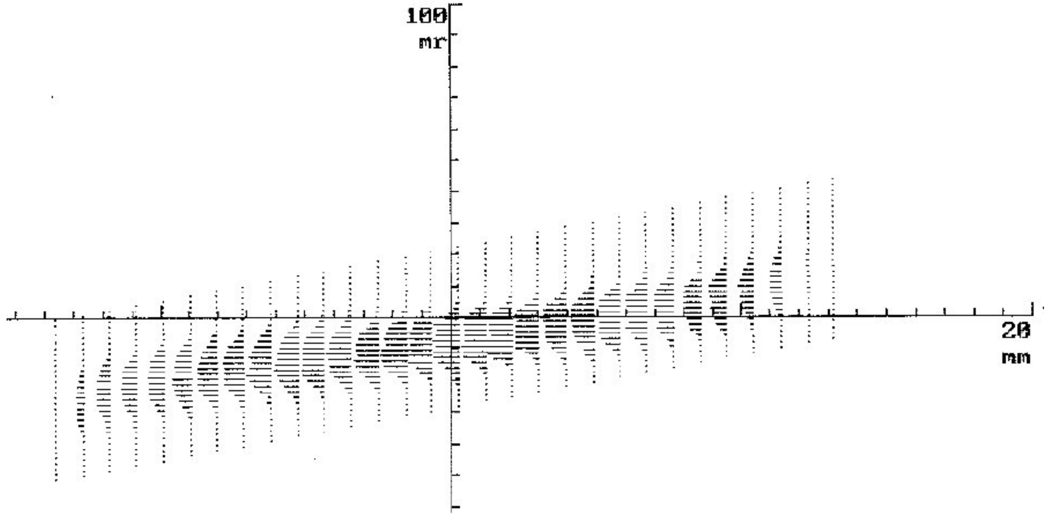
Measurement of emittance: wire scanner diagnostic

- A detector with an analyzing slit moves over the cross section of a beam to sample different values of x. A wire beam collector moves within the detector to sample different values of x'. The extended geometry of the slit and wire gives averaging along y and y'.
- Wire scans are useful only for steady-state or continuously pulsed beams.
- Statistical (rms) emittance:

$$\varepsilon_{x,\mathrm{rms}} = \sqrt{\langle (x-\bar{x})^2 \rangle \langle (x'-\bar{x'})^2 \rangle - \langle (x-\bar{x})(x'-\bar{x'}) \rangle^2}$$



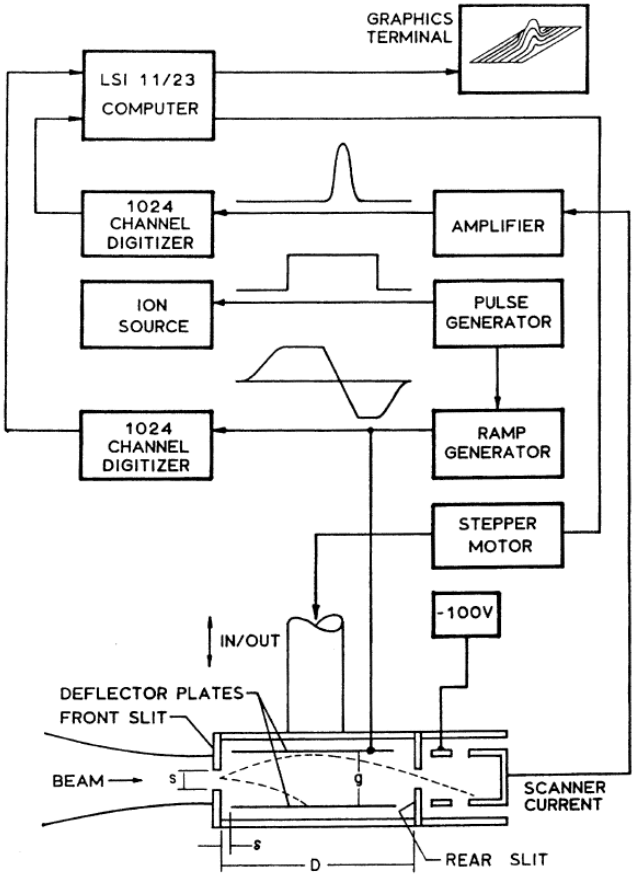


Measurement of emittance: Allison-type scanner

IEEE Transactions on Nuclear Science, Vol. NS-30, No. 4, August 1983

AN EMITTANCE SCANNER FOR INTENSE LOW-ENERGY ION BEAMS*

Paul W. Allison, Joseph D. Sherman, and David B. Holtkamp, AT-2, MS H818
Los Alamos National Laboratory, NM 87545



Using the paraxial ray approximation, we find for ions passing through the rear slit with deflection voltage V applied across the gap g between the plates that

$$x' = \frac{V}{\phi}\frac{(D - 2\delta)}{4g} \quad .$$

The maximum analyzable angle x'_m, limited by ions striking the deflecting plates is

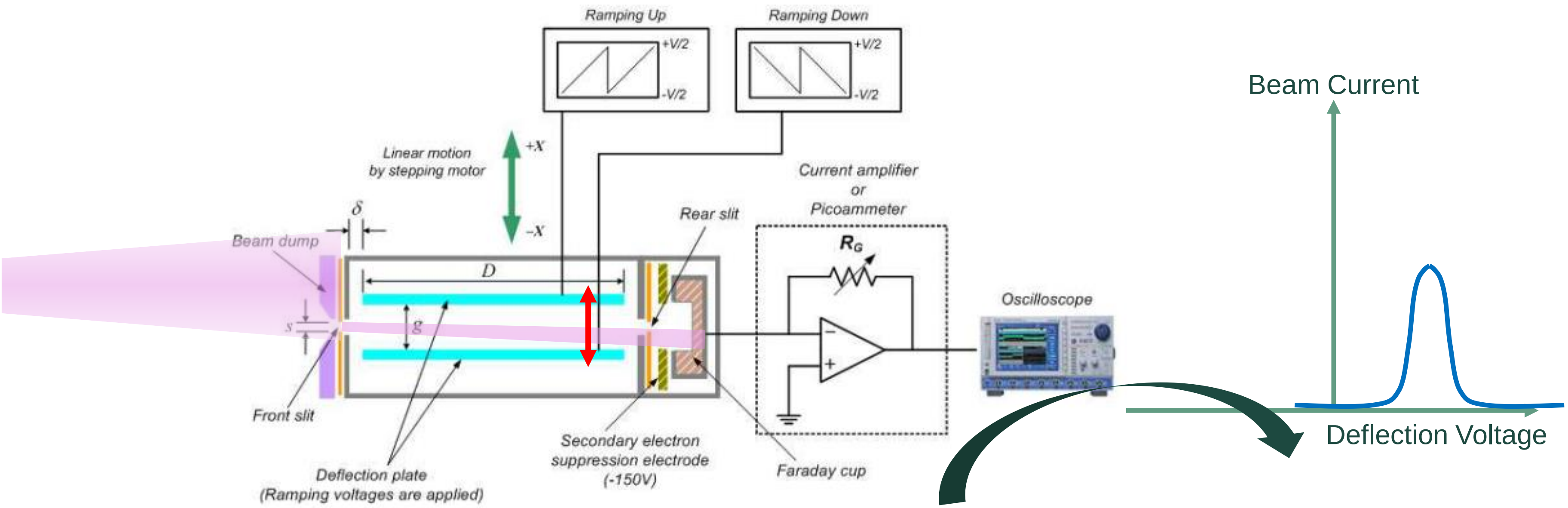
$$x'_m = \frac{\pm 2g}{(D + 2\delta)} \quad ,$$

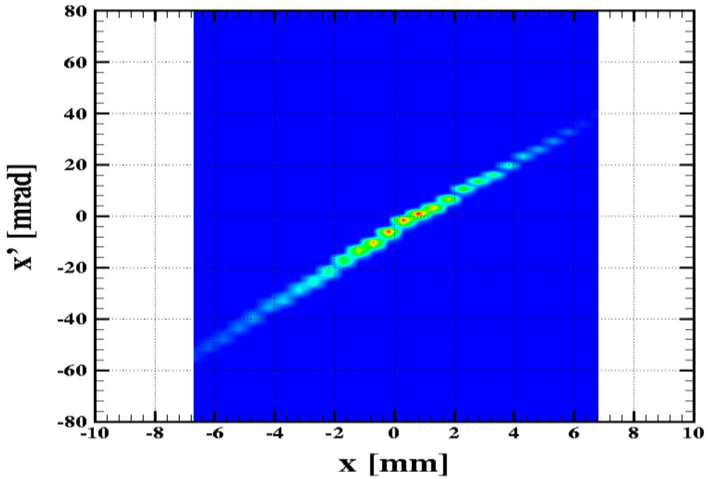
corresponding to a maximum voltage V_m required:

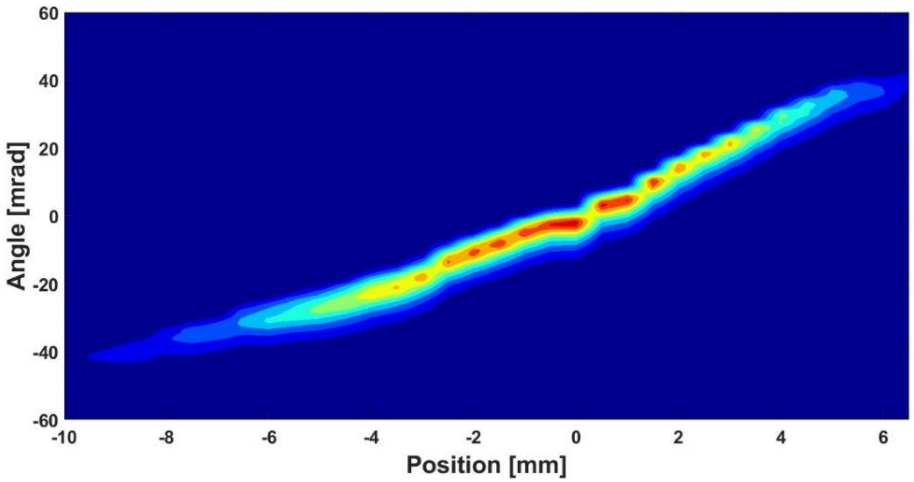
$$V_m = \frac{\pm 8g^2\phi}{D^2 - 4\delta^2}$$

$$= \pm 2(x'_m)^2\,\phi\,\frac{D + 2\delta}{D - 2\delta} \quad .$$

Measurement of emittance: Allison-type scanner







Normalized emittance

- Acceleration generally reduces emittance. The transverse momentum of particles may remain constant while the axial momentum increases, leading to a reduction in x'. We shall find it useful to designate an alternative quantity that remains constant during acceleration, the normalized emittance.
- With the effects of acceleration removed, changes in the normalized emittance indicate a degradation of beam quality resulting from non-linear forces or beam perturbations.
- Although the trace-space volume of a beam decreases during acceleration, we know that the phase-space volume stays constant in a linear focusing system. The transverse momentum is related to the inclination angle by

$$p_x = mv_x = \gamma m_0 x' v_z = \gamma m_0 x' \beta c = x'(\beta\gamma)(m_0 c)$$

- The normalized emittance of a relativistic paraxial beam is: $[x, \beta\gamma x']$

$$\epsilon_{nx} = (\beta\gamma)\epsilon_x = (Area\ in\ x - p_x\ space)/\pi m_0 c \quad (\pi - m - rad)$$

- For non-relativistic beams:

$$\epsilon_{nx} = \beta\epsilon_x = (v_z\epsilon_x)/c = (Area\ in\ x - v_x\ space)/\pi c \quad (\pi - m - rad)$$

Brightness

- The quantity brightness was adopted from conventional optics where it characterizes the quality of light sources.
- In charged particle beam applications, beam brightness is the current density per unit solid angle in the axial direction. Bright beams have high current density and good parallelism.
- The brightness is defined as

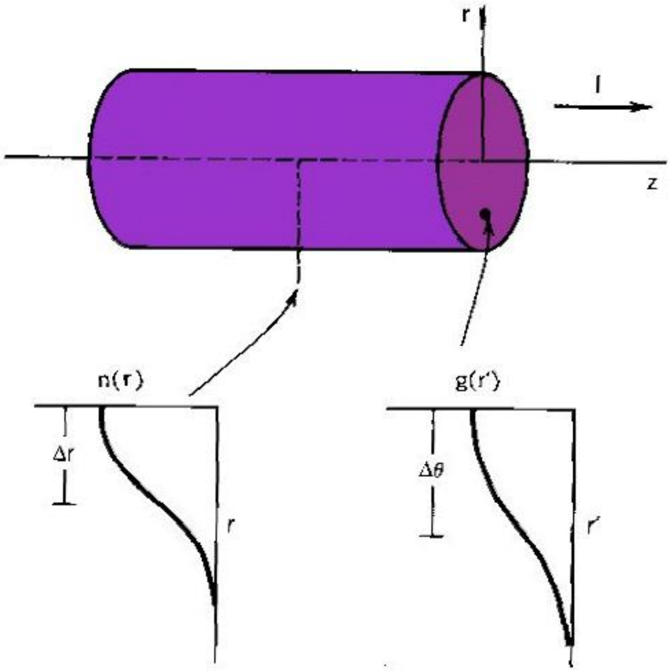
$$B \cong \frac{I}{(\pi \Delta r^2)(\pi \Delta \theta^2)} = \frac{j_b}{\pi \Delta \theta^2} \quad (A/(m-rad)^2)$$

- When beams have Cartesian symmetry in the transverse direction, we can write an expression for brightness in terms of the emittances:

$$B \cong \frac{I}{(\pi x_0 x_0')(\pi y_0 y_0')} = \frac{I}{\pi^2 \epsilon_x \epsilon_y} = \frac{I}{\pi^2 \epsilon^2}$$

- The normalized brightness (relativistic):

$$B_n = \frac{I}{\pi^2 \epsilon_n^2} = \frac{B}{(\beta\gamma)^2}$$



Note that if ϵ is constant, the beam brightness is also a conserved quantity.

Coupled transverse beam distributions

- The focusing system of a high-energy particle accelerator consists mainly of quadrupole lenses and dipole bending magnets. In these optical elements, particle motions in the x and y directions are independent.
- Motion is not separable in a variety of other focusing devices. Some common examples are solenoidal magnetic lenses, liquid metal lenses, or cylindrical electrostatic lenses in acceleration columns.
- Consider a paraxial beam in a focusing system where x and y motions couple but transverse motion is independent of axial motion. Emittances in the x and y directions are no longer separately-conserved quantities. Instead, the total four-dimensional trace-space volume in (x, x', y, y') is constant in the absence of acceleration. The four-dimensional extension of emittance is called hyper-emittance.
- If the distribution fits into the four-dimensional ellipsoid:

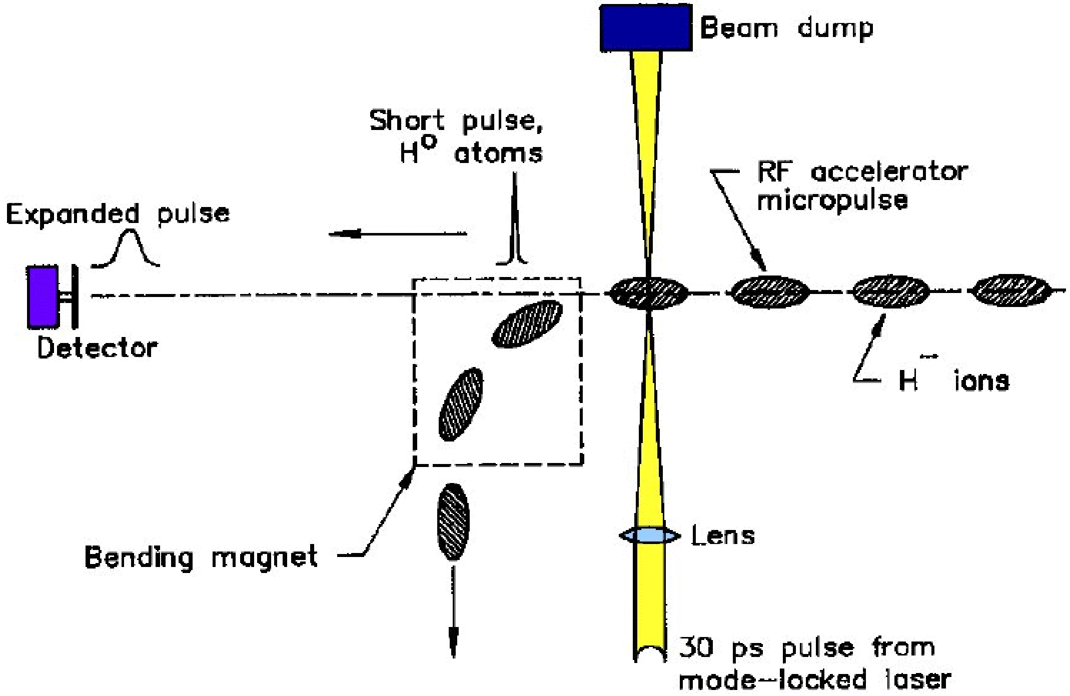
$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{x'}{x_0{}'}\right)^2 + \left(\frac{y}{y_0}\right)^2 + \left(\frac{y'}{y_0{}'}\right)^2 = 1$$

then, the hyper-emittance (ϵ_4) is

$$\epsilon_4 = V_4/\pi^2 = x_0 x_0' y_0 y_0' \ (\pi^2 - m^2 - rad^2)$$

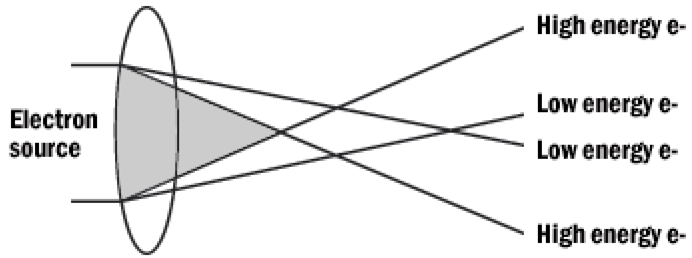
Longitudinal emittance

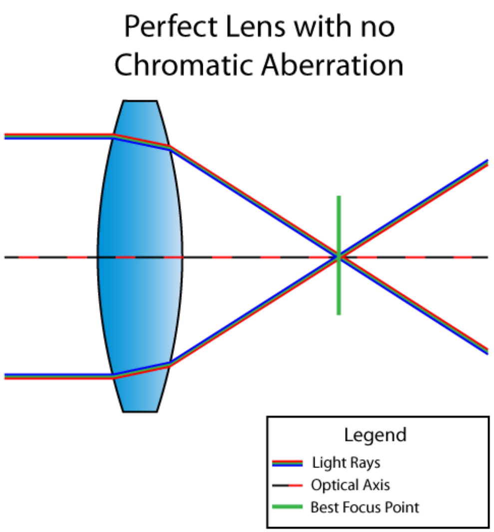
- In extending emittance to the axial direction, we recognize that angles and orbit traces are undefined. Therefore, we must plot orbit vectors to represent distributions directly in $z - p_z$ space.
- In RF accelerators, we measure and plot longitudinal distributions relative to a point of constant phase of the accelerating wave. Here, common distribution coordinates are ϕ, the phase position of the particle relative to the wave, and ΔT, the difference in kinetic energy from the average value.

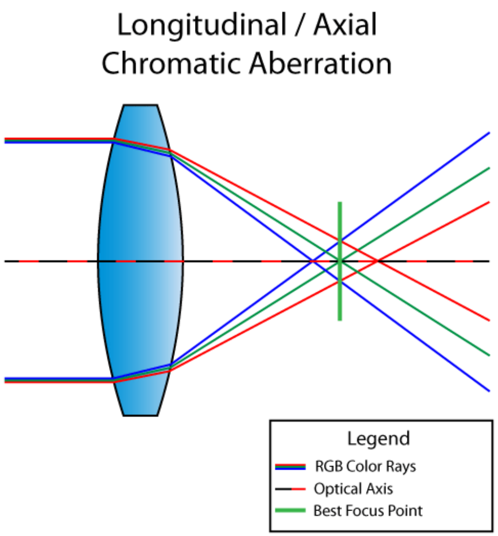


Chromatic aberration

- Sometimes, longitudinal and transverse particle motions couple. For example, the forces in most focusing elements are energy-dependent — transverse deflections depend on the axial velocity. This effect, chromatic aberration, leads to a increase in a focal spot size when beams have a substantial energy spread.



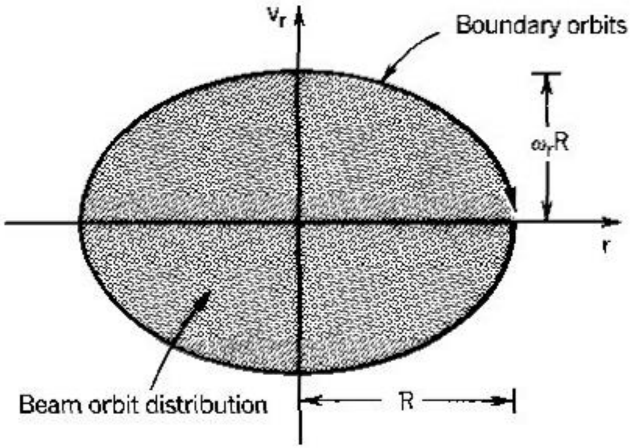




Emittance force

- An ideal laminar beam with parallel orbits propagates indefinitely with no change in radius. In contrast, a beam with non-zero emittance expands — some of the particles are aimed outward.
- To maintain a constant radius for a beam with emittance, focusing forces must be applied to reverse the outwardly directed particles. In a sense, we can view non-zero emittance in terms of an outward force that balances the focusing force to maintain a constant radius beam. We can calculate the effective emittance force by seeking the focusing force that guarantees radial force balance.
- Suppose a linear, axi-centered force confines the beam — the force varies in z over scale lengths long compared with the envelope radius, R. We write the linear focusing force as:

$$F_r(r) = -F_0(r/R_0)$$



- If no other forces act on the beam, the orbit vector points of individual particles follow ellipses in trace-space as the particles perform radial oscillations. The oscillation frequency for all particles is

$$\omega_r = \sqrt{F_0/(\gamma m_0 R_0)}$$

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Emittance force

- By definition, the radial emittance of the beam equals the product of the maximum displacement and angle of the boundary orbit:

$$\epsilon_r = R\frac{v_{ro}}{v_z} = \frac{\omega_r R^2}{v_z} \quad \Longrightarrow \quad \omega_r = \frac{\epsilon_r v_z}{R^2}$$

- By equating two ω_r values at R, we can find the focusing force needed to balance emittance on the beam envelope:

$$F_r(R) = -F_0\frac{R}{R_0} = \epsilon_r^2\frac{\gamma m_0 v_z^2}{R^3}$$

Effective emittance force

- In the quasi-static limit, the following approximate equation describes changes in the envelope radius of a beam with non-zero emittance subject to a linear focusing force:

$$\frac{d^2}{dt^2}(\gamma m_0 R) = -F_0\frac{R}{R_0} + \epsilon_r^2\frac{\gamma m_0 v_z^2}{R^3}$$

- Envelope equation (without acceleration):

$$\frac{d}{dt} = v_z\frac{d}{dz}$$

$$\boxed{R'' = \frac{d^2R}{dz^2} = -\frac{F_0(R/R_0)}{\gamma m_0\beta^2c^2} + \frac{\epsilon_r^2}{R^3}}$$

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Free-space expansion of a cylindrical beam with non-zero emittance

- Without focusing force, the envelope equation is

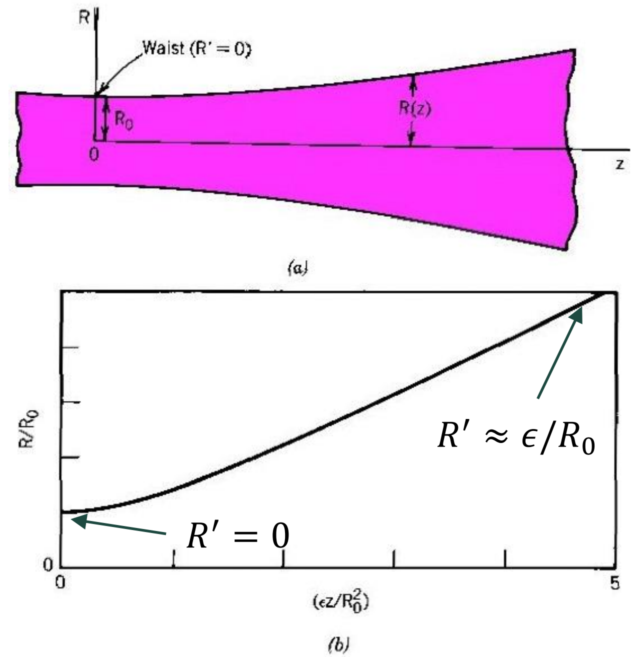
$$R'' = \frac{\epsilon^2}{R^3}$$

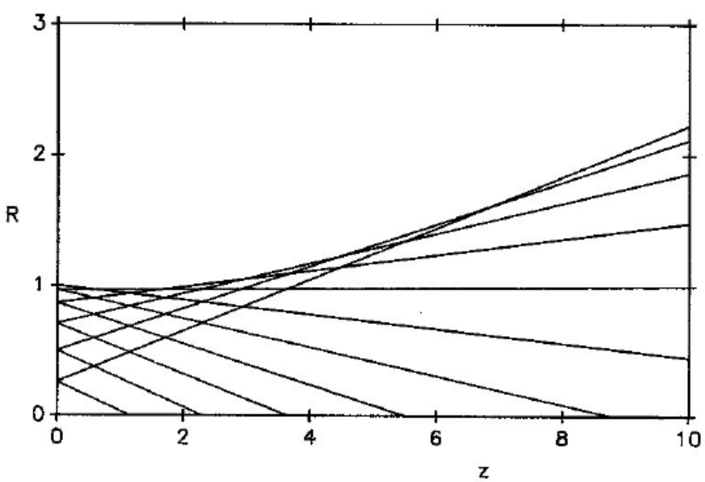
- By multiplying $2R'$ and integrating, we obtain

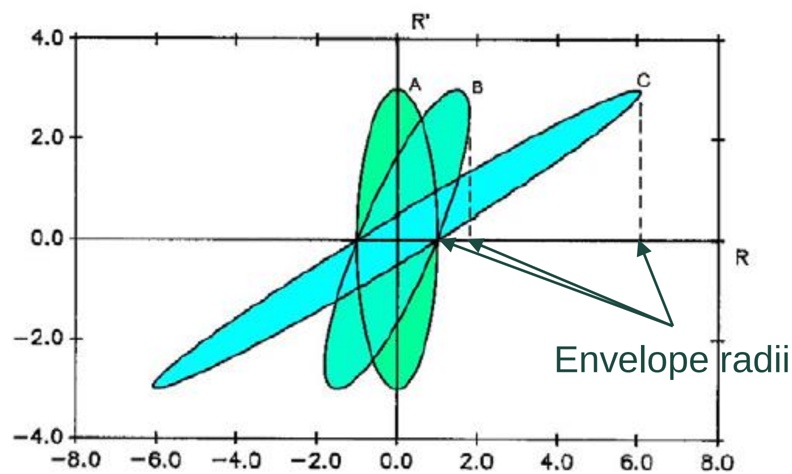
$$(R')^2 = \epsilon^2(1/R_0^2 - 1/R^2)$$

- We finally obtain

$$R(z)^2 = R_0^2 + \epsilon^2 z^2/R_0^2$$





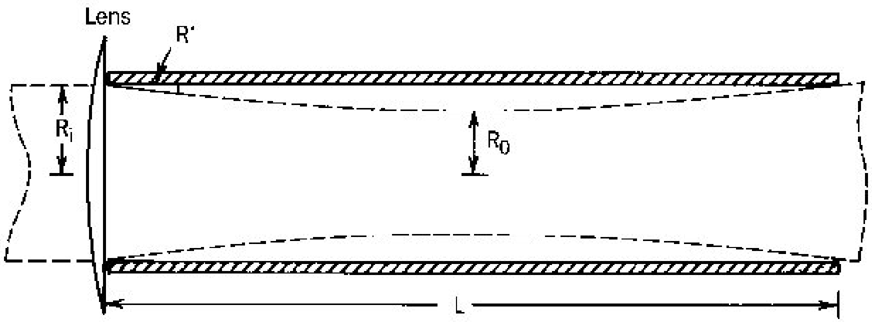


Field-free propagation of a beam through a tube

- The envelope equation for converging or diverging beams that have a waist point at any position z_0:

$$R(z)^2 = R_0^2 + \epsilon^2 (z - L/2)^2 / R_0^2$$

- Find the maximum length of the pipe such that the beam can traverse without losses.



- We assume that a focusing lens at the entrance allows us to adjust the input convergence angle. Different angles give different values of the waist radius, R_0. We seek an maximum tube length by expressing L as a function of R_0 and then setting dL/dR_0 equal to zero.
- At $z = 0$, we obtain $L^2 = (4/\epsilon^2)(R_0^2 R_i^2 - R_0^4)$
- Taking the derivative, the value of $R_0 = R_i/\sqrt{2}$ gives the maximum value of L:

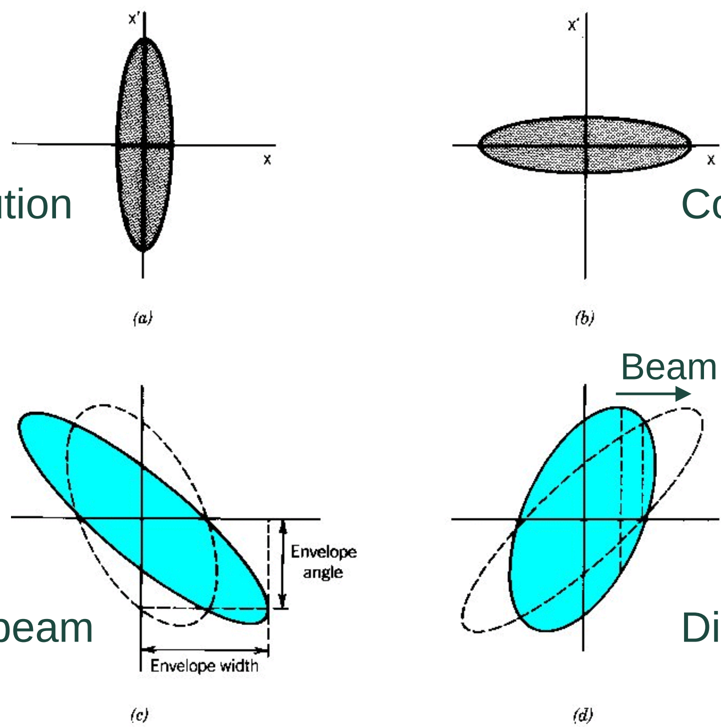
$$L = R_i^2/\epsilon$$

- The envelope angle at the pipe entrance for the optimum solution is

$$R' = -\epsilon\sqrt{1/R_0^2 - 1/R_i^2} = -\epsilon/R_i$$

Non-laminar beams in linear focusing systems

- Accelerator transport systems combine electric or magnetic field lenses, bending elements, and drift spaces to steer beams and to confine them about an axis.
- All transport systems accept particles only within a limited range of displacement and inclination angle from the main axis. We must make certain that all particles in the beam can travel through the system without striking a boundary. We display the allowed particle orbits through a trace-space boundary called the acceptance.



Hot distribution

Cold distribution

Beam cools as it expands

Converging beam

Diverging beam

Effects of linear optical elements on a beam trace-space distribution

- Effect of a drift length

$$x_f = x_i + x_i' d$$

$$x_f' = x_i'$$

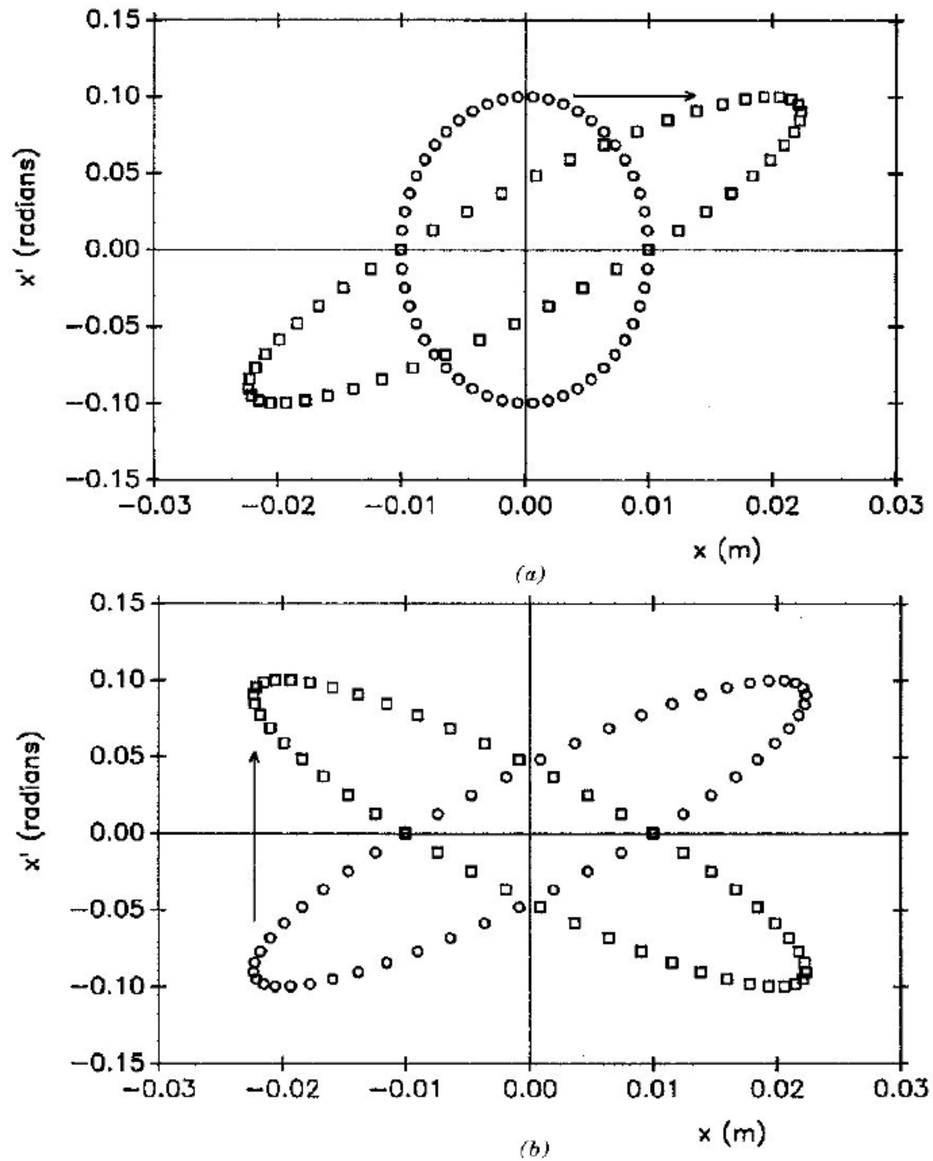
$$\boldsymbol{M} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

- Effect of a lens

$$x_f = x_i$$

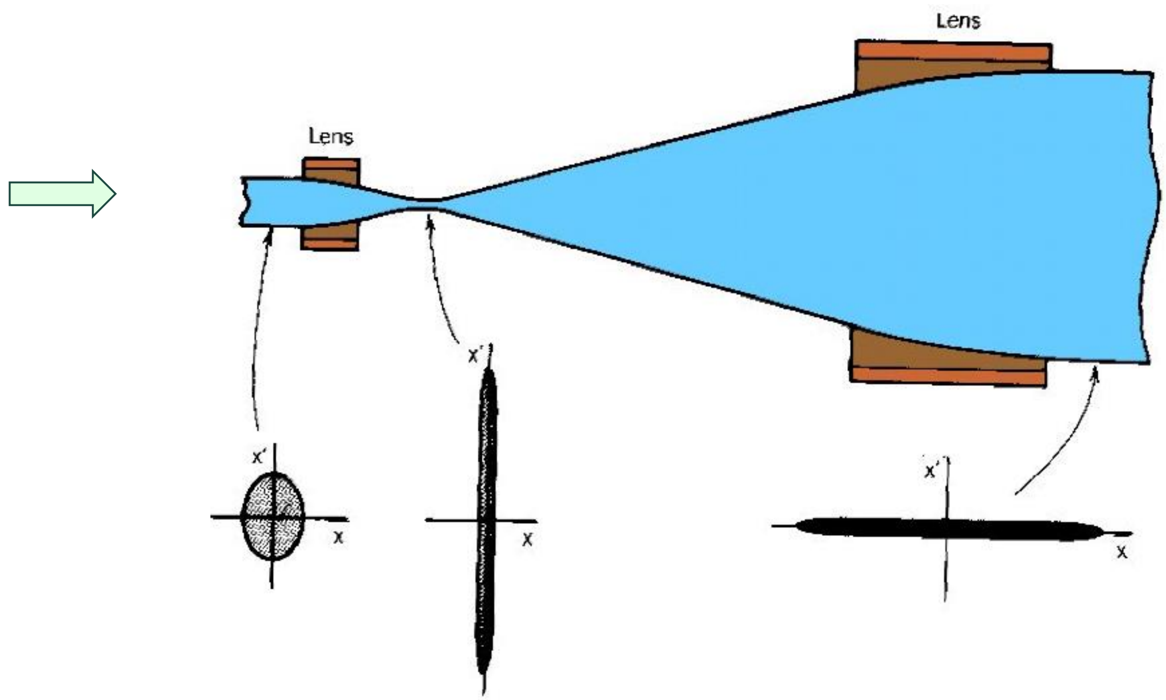
$$x_f' = x_i' - x_i/f$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$



Beam telescope

- The function of the device is to focus a beam from an accelerator to a small spot at long distance from the lens.
- If the output beam from the accelerator were focused directly by a small diameter lens, the focal spot would be large. To reduce the spot size, the telescope first expands the beam and then focuses the cooled beam with a large diameter lens.



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Beam matching

- Assume the force is uniform in z and linear in x:

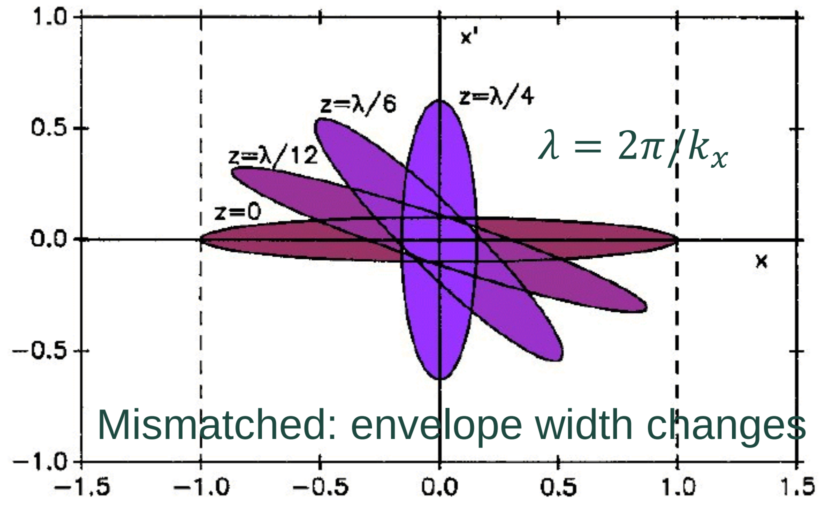
$$F_x = -Ax$$

- In the paraxial limit, individual particles follow orbits described by:

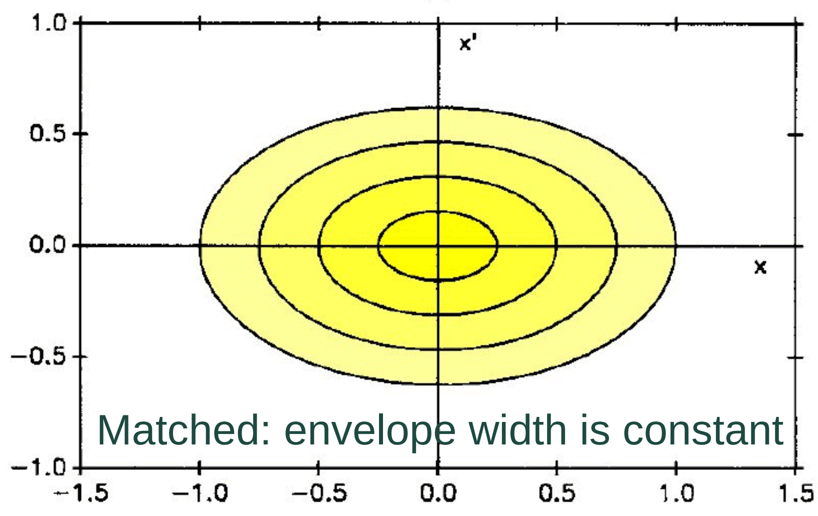
$$x_n(z) = x_{0n}\sin(k_x z + \phi_n)$$

$$x_n'(z) = x_{0n}k_x\cos(k_x z + \phi_n)$$

- Although the amplitude and phase of individual particle orbits vary, all particle orbits rotate in trace-space at the same frequency.
- A matched beam exhibits minimal envelope oscillations. The advantage of matching a beam with a given emittance to a focusing system is that the beam has the smallest possible spatial width as it propagates.



(a)



(b)

Acceptance diagram

- Acceptance refers to the region in trace-space accessible for particle transport.

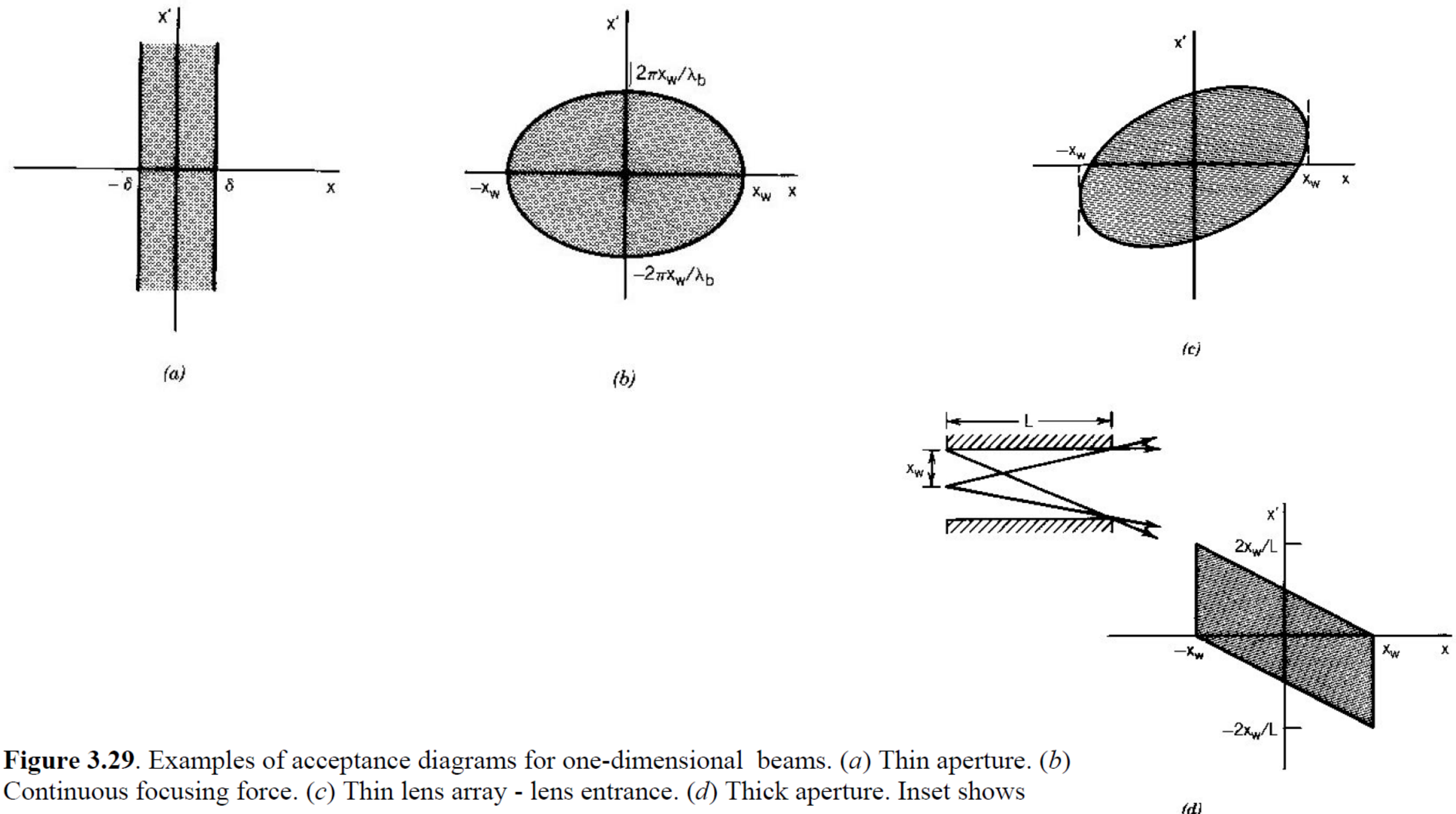


Figure 3.29. Examples of acceptance diagrams for one-dimensional beams. (*a*) Thin aperture. (*b*) Continuous focusing force. (*c*) Thin lens array - lens entrance. (*d*) Thick aperture. Inset shows geometry.

Combined acceptance