Beam-generated Forces

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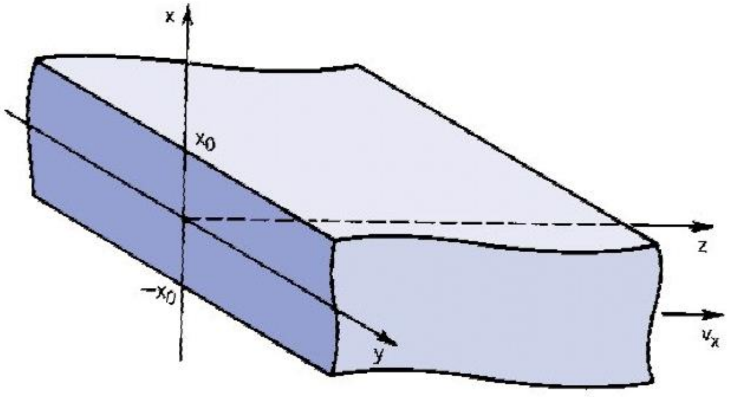
Electric and magnetic fields of a sheet beam

- Sheet beam (infinite in y and z directions, particle motion in parallel to z axis)
- Beam-generated E field

$$\frac{\partial E_x}{\partial x} = \frac{qn(x)}{\epsilon_0} \qquad E_x(x) = \frac{q}{\epsilon_0}\int_0^x n(x')dx'$$



- Beam-generated B field

$$\frac{\partial B_y}{\partial x} = \mu_0 j_z(x) \qquad j_z(x) = qn(x)v_z$$

$$B_y(x) = \mu_0 q v_z \int_0^x n(x')dx' \quad \Rightarrow \quad \boxed{B_y = \epsilon_0\mu_0 v_z E_x = \frac{v_z}{c^2}E_x}$$

- For uniform density distribution, $n(x) = n_0$ $(-x_0 \le x \le x_0)$

$$E_x(x) = \frac{qn_0}{\epsilon_0}x \qquad B_y(x) = \mu_0 q n_0 v_z x = \frac{qn_0 v_z}{\epsilon_0 c^2}x$$

- Electric potential

$$\phi(x) = \phi(0) - \frac{qn_0}{2\epsilon_0}x^2$$

- Paraxial beam approximation holds if

$$\Delta\phi_{max} = \frac{qn_0}{2\epsilon_0}x_0^2 \ll m_0 v_z^2/2q$$

Envelope fields and forces

- Often it is unnecessary or impossible to understand the details of particle motion over the full cross section of a beam. Instead, we use an envelope equation that describes the balance of forces only at the periphery, or envelope, of a beam.
- If v_z is almost constant, the current per unit length of a sheet beam is

$$J = 2qn_0 v_z x_0 = 2qn_0 \beta c x_0$$

- Then, envelope electric and magnetic fields are

$$E_{x0} = \frac{qn_0}{\epsilon_0} x_0 = \frac{J}{2\epsilon_0 \beta c} \qquad B_{y0} = \mu_0 q n_0 v_z x_0 = \frac{\mu_0 J}{2}$$

- The beam-generated electric and magnetic forces acting on the envelope of a sheet beam are

$$F_{x0}(electric) = qE_{x0} = \frac{qJ}{2\epsilon_0 \beta c} \qquad F_{x0}(magnetic) = -qv_z B_{y0} = -\frac{q\beta c \mu_0 J}{2}$$

- The ratio of magnetic to electric force

$$\boxed{\frac{F_{x0}(magnetic)}{F_{x0}(electric)} = -\beta^2}$$

For non-relativistic particles (such as ions) the beam magnetic force is usually negligible. In contrast, magnetic forces are important for relativistic electron beams.

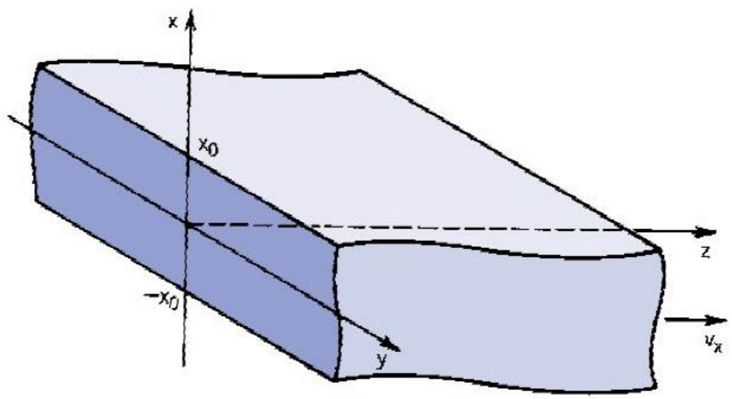
Transverse force on sheet beams by self-generating forces

- The electric and magnetic forces acting on the envelope of a sheet beam carrying a current per unit length (along y) of J (A/m) is:

$$F_{x0}(electric) = qE_{x0} = \frac{qJ}{2\epsilon_0 \beta c} \qquad F_{x0}(magnetic) = -qv_z B_{y0} = -\frac{q\beta c \mu_0 J}{2}$$

- The total beam-generated force on the envelope:

$$\boxed{F_X = \gamma m_0 (\beta c)^2 K_x} \qquad K_x \equiv \frac{qJ}{2\epsilon_0 m_0 \beta\gamma c} \text{ (generalized perveance)}$$



Envelope equation for sheet beams

- The beam envelope follows an equation of motion of the form:

$$\frac{d}{dt}\left[\gamma m_0\left(\frac{dX}{dt}\right)\right] = \frac{d}{dt}[\gamma m_0 \beta c X'] = m_0 \beta c^2[\gamma\beta X'' + \gamma\beta' X' + \gamma'\beta X'] = \sum F_X$$

$$\gamma\beta' + \beta\gamma' = \gamma'/\beta$$

- We obtain the following equation:

$$\gamma m_0(\beta c)^2\left[X'' + \frac{\gamma'}{\gamma\beta^2}X'\right] = -X(m_0 c^2)\gamma'' - \frac{q^2 B_z^2(0,z)}{\gamma m_0}X + \gamma m_0(\beta c)^2 K_x + \epsilon_x^2\frac{\gamma m_0(\beta c)^2}{X^3}$$

- Finally, we obtain the envelope equation for sheet beams:

$$X'' = -\frac{\gamma'}{\gamma\beta^2}X' - \frac{\gamma''}{\gamma\beta^2}X - \left(\frac{qB_z}{\gamma m_0 \beta c}\right)^2 X + K_x + \frac{\epsilon_x^2}{X^3}$$

Emittance force

Focusing by electrostatic lens

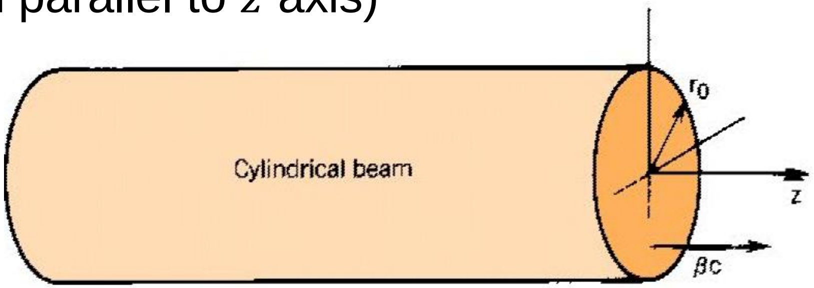
Defocusing by beam-generated forces

Decrease in the envelope angle by beam acceleration

Focusing by magnetic lens

Transverse force on cylindrical beams by self-generating forces

- Cylindrical beam (radius r_0, particle motion in parallel to z axis)



- Beam-generated E field

$$E_r(r) = \frac{q}{2\pi\epsilon_0 r}\int_0^r 2\pi r' n(r')dr'$$

- Beam-generated B field

$$B_\theta(r) = \frac{\mu_0 q v_z}{2\pi r}\int_0^r 2\pi r' n(r')dr'$$

$$\boxed{B_\theta = \frac{v_z}{c^2}E_r}$$

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$$

- For total beam current

$$I = q v_z \int_0^{r_0} 2\pi r' n(r')dr'$$

- The beam-generated electric and magnetic forces acting on the envelope of a cylindrical beam are

$$F_{r0}(electric) = \frac{qI}{2\pi\epsilon_0\beta c r_0}$$

$$F_{r0}(magnetic) = -\frac{q\beta I}{2\pi\epsilon_0 c r_0}$$

- The ratio of magnetic to electric force

$$\boxed{\frac{F_r(magnetic)}{F_r(electric)} = -\beta^2}$$

Space charge expansion (transverse) of a drifting beam

- The beam-generated electric and magnetic forces at the envelope are

$$F_R(electric) = \frac{qI}{2\pi\epsilon_0\beta cR} \qquad F_R(magnetic) = -\frac{qI\beta}{2\pi\epsilon_0 cR}$$

- The motion of envelope particles in the combined fields follows the equation:

$$\frac{d}{dt}\left(\gamma m_0 \frac{dR}{dt}\right) = \frac{qI}{2\pi\epsilon_0\beta c\gamma^2}\frac{1}{R} \qquad \frac{d}{dt} = \beta c\frac{d}{dz}$$

- For the condition of constant energy (β and γ are constant)

$$\boxed{\frac{d^2R}{dz^2} = \frac{qI}{2\pi\epsilon_0 m_0(\beta\gamma c)^3}\frac{1}{R} = \frac{K}{R}} \qquad K \equiv \frac{qI}{2\pi\epsilon_0 m_0(\beta\gamma c)^3}\ \text{(generalized perveance)}$$

- The beam-generated forces cause beam expansion — a converging beam reaches a minimum value of envelope radius (R_m) at $z = 0$ and then expands.

$$\frac{dR}{dz} = \sqrt{2K}\sqrt{\ln(R(z)/R_m)} = \sqrt{2K}\sqrt{\ln(\chi)} \qquad \chi = R(z)/R_m$$

Space charge expansion (transverse) of a drifting beam

- The variation of envelope radius with distance from the neck is given by

$$z = (R_m/\sqrt{2K})\,F(\chi)$$

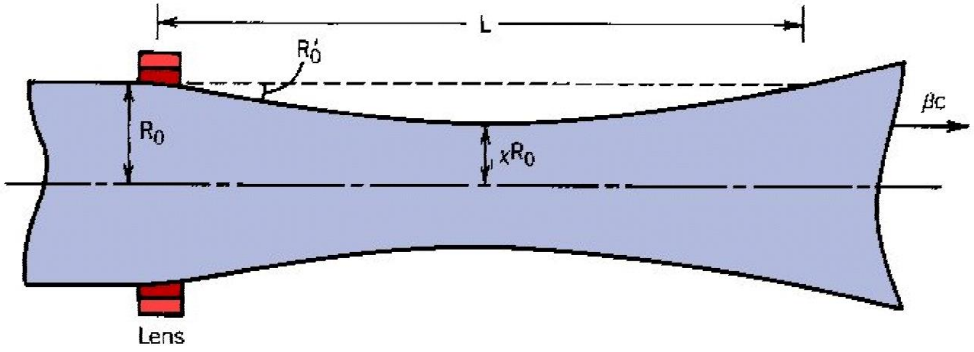
$$R(z) = \sqrt{2K}z[\chi/F(\chi)]$$

$$F(\chi) = \int_1^{\chi} \frac{dy}{\sqrt{\ln y}}$$

- Application: find the maximum distance that a relativistic electron beam can propagate across a vacuum region.

$$L = 2R_0[F(\chi)/\chi]/\sqrt{2K}$$

- The quantity $F(\chi)/\chi$ attains a maximum value of 1.085 at $\chi = 2.35$. The maximum propagation distance is $L_{max} = 1.53R_0/\sqrt{K}$ if we adjust the entrance envelop angle to $R_0' = -1.31\sqrt{K}$.



e.g.) Suppose we have a 100 A, 500 keV electron beam (γ = 1.98, β = 0.863) with an initial radius of 0.02 m.
The generalized perveance is K = 2.3×10^{-3}.
The propagation distance is L_{max} = 0.63 m for an injection angle of -63 mrad (-3.6°).

Value of function $F(\chi)$

χ	$F(\chi)$	$(\ln \chi)^{1/2}$	$F(\chi)/\chi$	χ	$F(\chi)$	$(\ln \chi)^{1/2}$	$F(\chi)/\chi$
1.10	0.6370	0.3087	0.5791	8.00	7.1148	1.4420	0.8894
1.20	0.9082	0.4270	0.7569	8.50	7.4590	1.4629	0.8775
1.30	1.1209	0.5122	0.8622	9.00	7.7985	1.4823	0.8665
1.40	1.3039	0.5801	0.9313	9.50	8.1338	1.5004	0.8562
1.50	1.4681	0.6368	0.9788	10.00	8.4651	1.5174	0.8465
1.60	1.6193	0.6856	1.0121	10.50	8.7929	1.5334	0.8374
1.70	1.7607	0.7284	1.0357	11.00	9.1173	1.5485	0.8289
1.80	1.8945	0.7667	1.0525	11.50	9.4387	1.5628	0.8208
1.90	2.0220	0.8012	1.0642	12.00	9.7573	1.5764	0.8131
2.00	2.1444	0.8326	1.0722	12.50	10.0732	1.5893	0.8059
2.10	2.2625	0.8614	1.0774	13.00	10.3866	1.6016	0.7990
2.20	2.3768	0.8880	1.0804	13.50	10.6976	1.6133	0.7924
2.30	2.4878	0.9126	1.0817	14.00	11.0065	1.6245	0.7862
2.40	2.5960	0.9357	1.0817	14.50	11.3132	1.6353	0.7802
2.50	2.7017	0.9572	1.0807	15.00	11.6180	1.6456	0.7745
2.60	2.8051	0.9775	1.0789	15.50	11.9209	1.6556	0.7691
2.70	2.9064	0.9966	1.0764	16.00	12.2221	1.6651	0.7639
2.80	3.0058	1.0147	1.0735	16.50	12.5215	1.6743	0.7589
2.90	3.1035	1.0319	1.0702	17.00	12.8194	1.6832	0.7541
3.00	3.1997	1.0482	1.0666	17.50	13.1156	1.6918	0.7495
3.10	3.2944	1.0637	1.0627	18.00	13.4105	1.7001	0.7450
3.20	3.3877	1.0785	1.0587	18.50	13.7039	1.7082	0.7408
3.30	3.4798	1.0927	1.0545	19.00	13.9959	1.7159	0.7366
3.40	3.5708	1.1062	1.0502	19.50	14.2867	1.7235	0.7327
3.50	3.6607	1.1193	1.0459	20.00	14.5761	1.7308	0.7288
3.60	3.7495	1.1318	1.0415	20.50	14.8644	1.7379	0.7251
3.70	3.8374	1.1438	1.0371	21.00	15.1516	1.7449	0.7215
3.80	3.9244	1.1554	1.0327	21.50	15.4376	1.7516	0.7180
3.90	4.0105	1.1666	1.0283	22.00	15.7225	1.7581	0.7147
4.00	4.0958	1.1774	1.0240	22.50	16.0063	1.7645	0.7114
4.10	4.1804	1.1879	1.0196	23.00	16.2892	1.7707	0.7082
4.20	4.2642	1.1980	1.0153	23.50	16.5711	1.7768	0.7052
4.30	4.3473	1.2077	1.0110	24.00	16.8520	1.7827	0.7022
4.40	4.4298	1.2172	1.0068	24.50	17.1321	1.7885	0.6993
4.50	4.5117	1.2264	1.0026	25.00	17.4112	1.7941	0.6965
4.60	4.5929	1.2353	0.9985	25.50	17.6894	1.7996	0.6937
4.70	4.6736	1.2440	0.9944	26.00	17.9669	1.8050	0.6910
4.80	4.7537	1.2524	0.9904	26.50	18.2435	1.8103	0.6884
4.90	4.8333	1.2607	0.9864	27.00	18.5193	1.8154	0.6859
5.00	4.9123	1.2686	0.9825	27.50	18.7943	1.8205	0.6834
5.50	5.3007	1.3057	0.9638	28.00	19.0686	1.8254	0.6810
6.00	5.6788	1.3386	0.9465	28.50	19.3421	1.8303	0.6787
6.50	6.0482	1.3681	0.9305	29.00	19.6149	1.8350	0.6764
7.00	6.4101	1.3950	0.9157	29.50	19.8871	1.8397	0.6741
7.50	6.7654	1.4195	0.9021	30.00	20.1585	1.8442	0.6720

$$\frac{F\,(\text{emittance})}{F\,(\text{self}-\text{field})} = \frac{\epsilon^2}{KR^2} \approx \frac{\Delta R'^2}{K}$$

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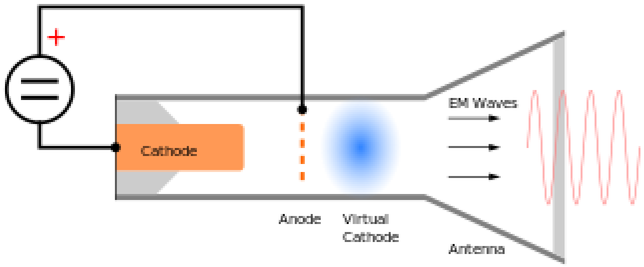
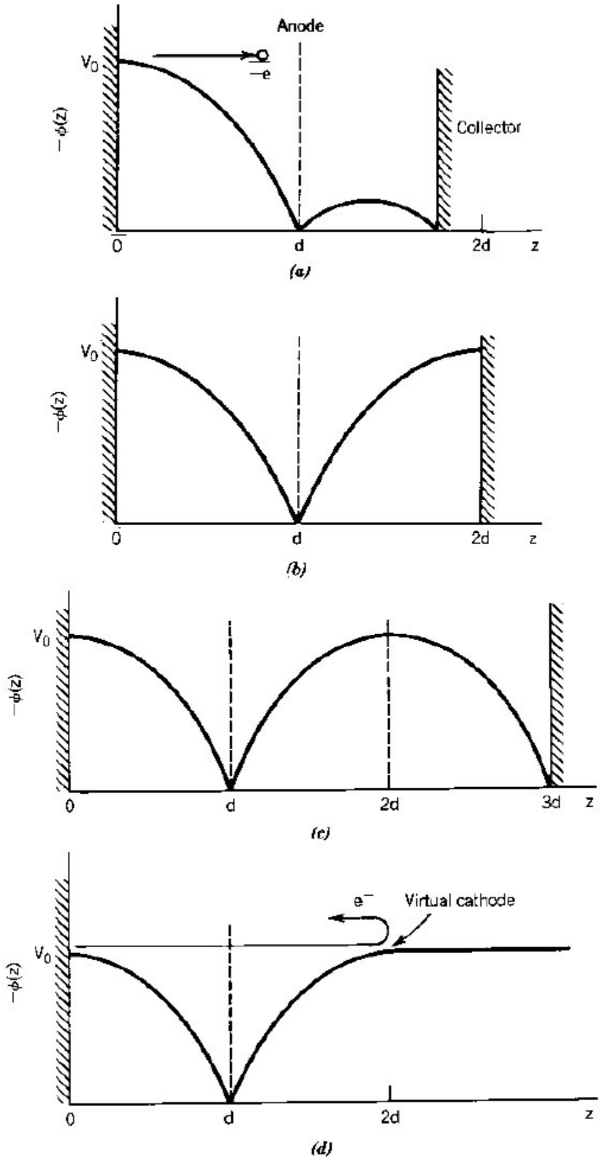
Beam-generated electrostatic potential in downstream

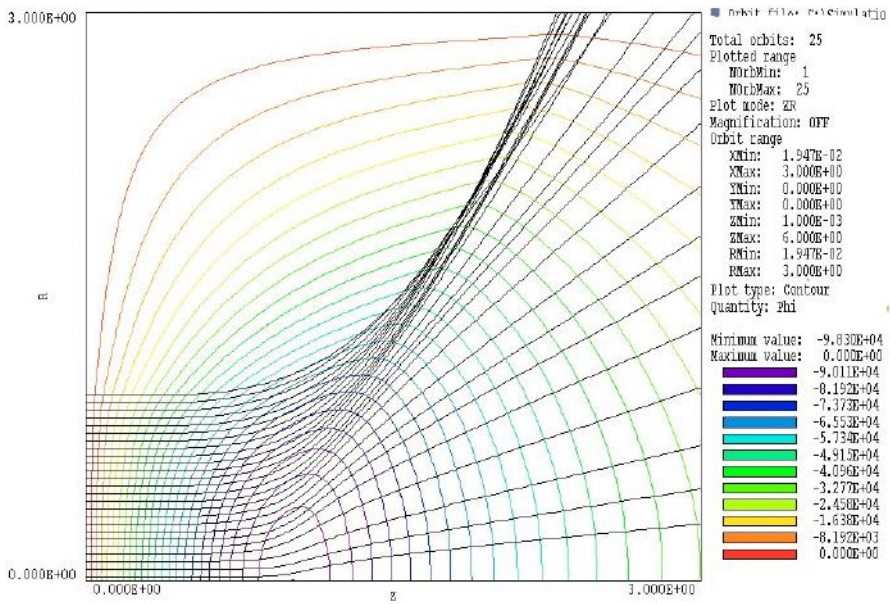
- Beams extracted from a one-dimensional, space-charge-limited extractor cannot propagate an indefinite distance in vacuum. The space-charge of the beam creates electric fields — depending on the geometry of the propagation region, the fields may be strong enough to reverse the direction of the beam.
- If the spacing between the anode and collector exceeds $2d$, the electrostatic potential causes electron reflection. The reflection plane, where the potential reaches $-V_0$, is called a virtual cathode.
- Hence, beams generated by a one-dimensional space-charge-limited extractor cannot propagate a distance greater than twice the extraction gap width.

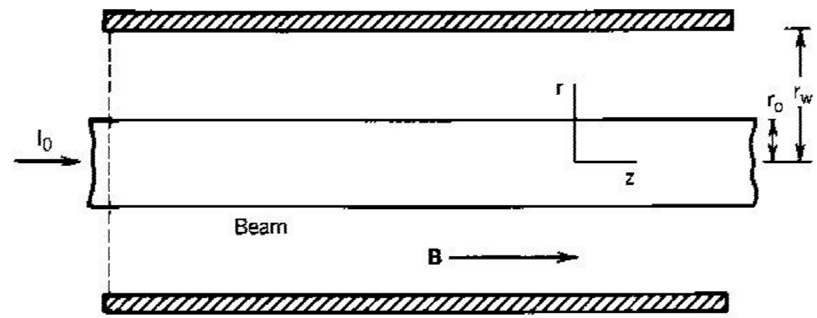




Space-charge effects can strongly influence the transport of a high-current beam injected into a vacuum region

- The theoretical description of high-current beam propagation can become complex when we address the full three-dimensional problem.
- Figure illustrates some of the interactive processes that can take place when a cylindrical beam enters a transport tube through an anode mesh. Some of the particles travel forward while others are reflected at a virtual cathode — all particles are subject to transverse space-charge deflections.





Incident laminar beam properties: 0.01 m radius, 100 keV energy, 700 A current, uniform current density

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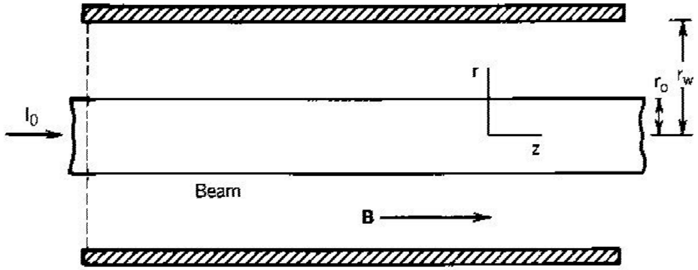
Longitudinal transport limits for magnetically-confined electron beams

- Assume a cylindrical relativistic electron beam in a transport pipe with a strong axial magnetic field. The magnetic field allows electrons to move only in the axial direction— we say that electrons are tied to the field lines.

Monoenergetic electrons with $\gamma = 1 + eV_0/m_e c^2$

Uniform current density $j_z(r, z) = j_0$, so that $I_0 = j_0(\pi r_0^2)$



- In the downstream region far from the entrance mesh ($z \gg r_w$), we assume that the cylindrical beam is axially uniform; therefore, axial variations of electric and magnetic field are small. In this region, the space-charge potential is a function of radius only, $\phi(r)$. The conservation of total energy gives

$$\gamma(r) = 1 + \frac{eV_0}{m_e c^2} + \frac{e\phi(r)}{m_e c^2}$$

- For uniform density inside the downstream beam and $-e\phi \ll eV_0$,

$$E_r = (\rho_0/(2\epsilon_0))r, \qquad (0 \le r \le r_0)$$

$$E_r = (\rho_0/(2\epsilon_0))(r_0^2/r), \qquad (r_0 \le r \le r_w)$$

Longitudinal transport limits for magnetically-confined electron beams: the maximum potential

- The electrostatic potential on the axis (maximum value)

$$e\phi(0) = e\int_0^{r_w} E_r dr = \left[\frac{e\rho_0 \pi r_0^2}{2\pi\epsilon_0}\right]\left[\int_0^{r_0}\frac{r}{r_0^2}dr + \int_{r_0}^{r_w}\frac{1}{r}dr\right]$$

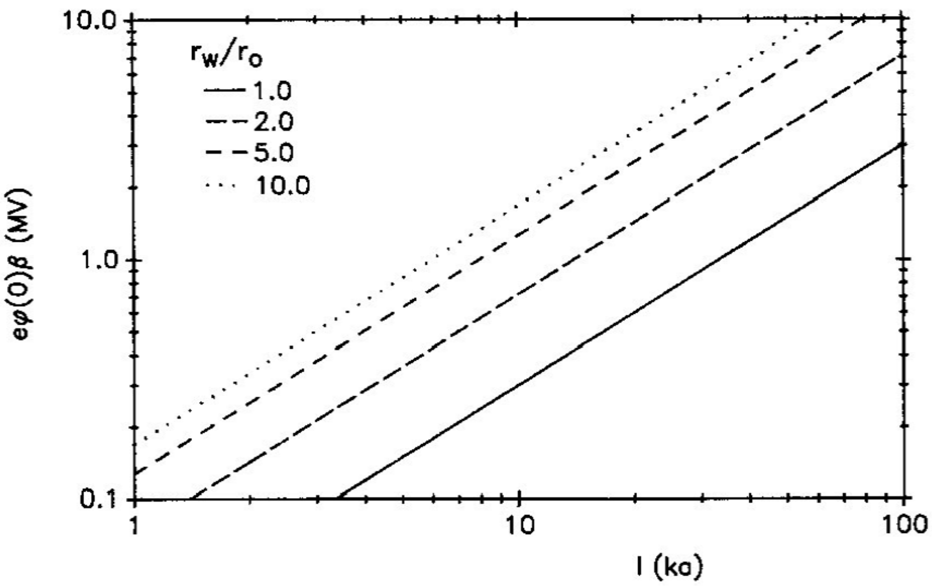
- Integrating and utilizing $I_0 = \rho_0 \pi r_0^2 \beta c$

$$e\phi(0) = \frac{-eI_0}{4\pi\epsilon_0\beta c}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]$$

- The maximum potential in volts

$$\phi(0) = \frac{-30I_0[A]}{\beta}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]$$

A current of 10 kA is typical of high power linear induction accelerators. It is shown that the corresponding space-charge potential is high, nearly 1 MV.
For this reason, induction linac injectors operate at high voltage (V_0 > 2 MV).



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Longitudinal transport limits for magnetically-confined electron beams: the maximum current

- To find space-charge limit, we must find the value of peak potential in the range $0 \leq \phi \leq V_0$ that gives the highest value of I_0 for a given γ.
- For a narrow beam ($r_0 \ll r_w$), the conservation of energy implies that

$$e\Delta\phi \cong \underbrace{m_e c^2(\gamma_0 - \gamma)}_{\text{Injection energy of electrons}} \cong \underbrace{\frac{eI_0}{4\pi\epsilon_0\beta c}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]}_{\text{Maximum potential difference by space charge}}$$

- We obtain

$$\frac{eI_0}{4\pi\epsilon_0 m_e c^3}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right] = \frac{(\gamma_0 - \gamma)\sqrt{\gamma^2 - 1}}{\gamma}$$

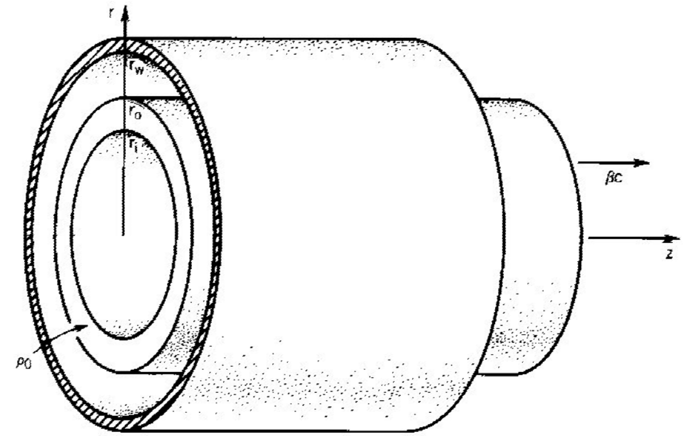
- The current becomes maximum at $\gamma = \gamma_0^{1/3}$

$$I_{max} = \frac{4\pi\epsilon_0 m_e c^3/e}{1 + 2\ln(r_w/r_0)}\left(\gamma_0^{2/3} - 1\right)^{3/2} \approx \frac{17.1\ [kA]}{1 + 2\ln(r_w/r_0)}\left(\gamma_0^{2/3} - 1\right)^{3/2}$$

e.g.) Consider the propagation of a 0.01 m radius beam in a 0.03 m radius pipe. The injection energy of 1.5 MeV corresponds to $\gamma_0 = 3.94$. The space-charge-limiting current is 9.2 kA. At this value, the space-charge potential of the beam is 1.2 MV — the beam propagates with an average kinetic energy of 0.3 MeV ($\gamma = 1.58$).

Annular beam can carry more current than solid beam

- Longitudinal space-charge effects can be reduced, in principle, by using beams with nonuniform current density. An annular beam can carry more current in equilibrium than a solid beam.



- The electric field

$$E_r = (\rho_0/(2\epsilon_0))((r^2 - r_i^2)/r), (r_i \le r \le r_o)$$

$$E_r = (\rho_0/(2\epsilon_0))((r_o^2 - r_i^2)/r), (r_o \le r \le r_w)$$

- The maximum space charge potential

$$e\phi_{max} = \frac{eI_0}{4\pi\epsilon_0\beta c}\left[1 - 2r_i^2 \ln\left(\frac{r_o/r_i}{r_o^2 - r_i^2}\right) + 2\ln\left(\frac{r_w}{r_o}\right)\right]$$

$$I_0 = \pi(r_o^2 - r_i^2)\rho_0\beta c$$

- The maximum current for a thin annular beam ($r_o/r_i \to 1$)

$$I_{max} = \frac{4\pi\epsilon_0 m_e c^3/e}{2\ln(r_w/r_o)}\left(\gamma_0^{2/3} - 1\right)^{3/2} \approx \frac{17.1\ [kA]}{2\ln(r_w/r_o)}\left(\gamma_0^{2/3} - 1\right)^{3/2}$$

e.g.) Consider the propagation of a 0.09 m outer radius beam in a 0.1 m radius pipe. The injection energy of 1.5 MeV corresponds to γ_0 = 3.94. The space-charge-limiting current is 148 kA (note that 9.2 kA for the narrow solid beam).

Paraxial ray equation

- In a cylindrical system, symmetry permits only certain components of electric and magnetic field:

    1. axial and radial components of the applied electric field,
    2. radial electric field resulting from space-charge,
    3. axial and radial magnetic field components generated by axi-centered circular coils, and
    4. beam-generated toroidal magnetic field.

- In the paraxial limit, we can relate the radial components of applied fields to the axial field by:

$$E_r(r,z) \approx -\frac{r}{2}\left(\frac{\partial E_z}{\partial z}\right)_{r=0} \qquad B_r(r,z) \approx -\frac{r}{2}\left(\frac{\partial B_z}{\partial z}\right)_{r=0}$$

- Particles gain azimuthal velocity when they move through the radial magnetic fields of a solenoidal lens. For forces with cylindrical symmetry, the canonical angular momentum is a constant of particle motion:

$$\gamma m_0 r v_\theta + q r A_\theta = P_\theta = \text{constant}$$

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Paraxial ray equation

- We can derive the following equation for axial variation of the envelope of a cylindrical beam:

$$\frac{d}{dt}\left(\gamma m_0 \frac{dR}{dt}\right) - \gamma m_0 \frac{v_\theta^2}{R} = q(E_r + v_\theta B_z) + \epsilon_r^2 \frac{\gamma m_0 v_z^2}{R^3} + \frac{qI}{2\pi\epsilon_0 \beta c \gamma^2} \frac{1}{R}$$

$$\Downarrow$$

$$\psi_0 = \int_0^{R_s} 2\pi R B_z(R, Z_s) dR$$

$$R'' = -\frac{\gamma'}{\gamma\beta^2} R' - \frac{\gamma''}{2\gamma\beta^2} R - \left(\frac{qB_z}{2\gamma m_0 \beta c}\right)^2 R + \frac{\epsilon^2}{R^3} + \left(\frac{q\psi_0}{2\pi\gamma m_0 \beta c}\right)^2 \frac{1}{R^3} + \frac{K}{R}$$

Decrease in the envelope angle by beam acceleration

Electrostatic focusing from radial components of applied electric fields

Magnetic focusing from applied solenoidal fields

Emittance force

Non-zero angular momentum

Defocusing by beam-generated forces

Example: limiting current for paraxial beams with a uniform solenoid field

- Radial force balance for a cylindrical, paraxial electron beam in a uniform solenoid field B_0.

$$R'' = -\left(\frac{qB_0}{2\gamma m_0 \beta c}\right)^2 R + \frac{\epsilon^2}{R^3} + \frac{K}{R} = 0$$

- The acceptance α is defined as the allowed beam emittance for a given envelope radius when there are no beam-generated forces, i.e. $K = 0$:

$$\alpha^2 = \left(\frac{qB_0}{2\gamma m_0 \beta c}\right)^2 R^4$$

- Using the expression for the generalized perveance, we obtain the matched beam current:

$$K = \frac{\alpha^2}{R^2} - \frac{\epsilon^2}{R^2} = \frac{eI}{2\pi\epsilon_0 m_0 (\beta\gamma c)^3}$$

$$I = \left[\frac{\pi\epsilon_0 ec}{2m_0}\right](\beta\gamma)(B_0 R)^2 \left[1 - \frac{\epsilon^2}{\alpha^2}\right]$$

- If there is no emittance, the beam-generated forces exactly balance the focusing force of the axial magnetic field. Here, particle flow is laminar and the allowed current has a maximum value.