Kinetic Energy and Generalized Momenta

Configuration space : $q_i = q_i(x_1(t) \ldots\ldots x_{3N}(t), t)$

$$q_i(t) = \frac{dq_i(t)}{dt} :$$ **using Generalized coordinates !**

Ex) Car-pendulum system : (Fig.2.4)

(q_1) distance + (q_2) angular rotation angle.

- **Ref. Fig.2.11(page 110):**

Elevator model: (Matheu's Equation) : Stability problem.: page 78

Car-pendulum : Vibration problem.

T, V ?

Generalized velocity are <u>not necessarily</u> absolute velocity. Why ?

: $x_i(t) <-> q_i(t)$

<u>For *i-th* particle : $x_i = x_i\,(q_1,,,,,q_n,t)$</u>

|
| **Generalized coordinate**
|
| **Physical coordinate**

Among the *N* particles, choose *i*-th particle, and then dq_i/dt can be expressed in terms of generalized coordinates ! -> <u>Just through a coordinate transformation</u> by using a Chain rule !!

wrt $x -> q$

: Rate of change of physical <u>coordinate</u> x_i depends on the rate of change of the generalized <u>coordinates</u>. It may also depend on time *t* if the change of coordinates

contains *t* explicitly. (* Moving frame)

Single particle : T :

$$\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)\equiv\frac{1}{2}m\dot{x}_i\bullet\dot{x}_j\delta_{ij}\ :$$
$$(\mathrm{x},\mathrm{y},\mathrm{z}) \lhd\rhd\ f(\mathrm{r},\theta,\phi);\mathrm{g}(\mathrm{r},\theta,\mathrm{z})$$
$$;Orthogonality!!$$

(2.14)

Let generalized coordinate q_1, q_2, q_3

Physical coordinate using coordinate transformation, then

$$x_i = x_i(q_1, q_2, q_3, t) \quad (i=1\sim 3)$$

(2.15)

Applying total derivate

$$\frac{dx_i}{dt} = \sum_{j=1}^{3} \frac{\partial x_i}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial x_i}{\partial t} \equiv \frac{\partial x_i}{\partial q_j}\dot{q}_j + \dot{x}_i \equiv x_{i,j}\dot{q}_j + \dot{x}_i \tag{2.16}$$

Thus,

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \equiv \frac{1}{2}m(x_{i,j}\dot{q}_j + \dot{x}_{i,t})(x_{i,\mathrm{k}}\dot{q}_k + \dot{x}_{i,t}) \\
&\equiv \frac{1}{2}m(x_{i,j}x_{i,\mathrm{k}}\dot{q}_j\dot{q}_k + \underline{\dot{x}_{i,t}x_{i,\mathrm{k}}\dot{q}_k + \dot{x}_{i,t}x_{i,\mathrm{j}}\dot{q}_j} + \dot{x}_{i,t}\dot{x}_{i,t}) \\
&\equiv \frac{1}{2}m(x_{i,j}x_{i,\mathrm{k}}\dot{q}_j\dot{q}_k + \underline{2\dot{x}_{i,t}x_{i,\mathrm{j}}\dot{q}_j} + \dot{x}_{i,t}\dot{x}_{i,t}) \\
&\equiv \frac{1}{2}\alpha_{jk}\dot{q}_j\dot{q}_k + \beta_j\dot{q}_j + \frac{1}{2}\gamma
\end{aligned} \tag{2.17}$$

Kinetic energy of system for *N*-particles :

Sum ; $\frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$ ***!! :*** **Using a Chain rule and Applying Einstein summation convention):**

➔ **In terms of generalized coordinate :**

T=T(*q, dq/dt ,t*)

Actually depend on generalized coordinates

~ **Determined by the nature of transformation :**

$$T = T_2 + T_1 + T_0 \qquad (2.21)$$

: **Homogeneous Quadratic + Linear + Constant**

If a coordinate transformation does not dependent on t

→ $T_1 = T_0 = 0$

Generalized momentum = **Rate of change of KE *w.r.t* particular component of generalized velocity** $\dot{q}_i$ **:**

Physical interpretation of a particular component of a generalized momentum p_i

depends on the nature of the corresponding generalized coordinates.

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \tag{2.22}$$

Ex: In 3–dimensional space : m$(v_x^2+v_y^2+v_z^2)/2$ ~ quardratic function !

→ **linear momentum** $p_x = m\dot{x}\ldots$

Generalized coordinates may be actual *x*-,*y*- and *z*-components of position

the definition, $p_x = mv_x$, p_y=.. , p_z=..

Ex: Earth surface : Elliptic ! Coupling of momentum.

Ex: In spherical coordinates, the kinetic energy is

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 + r^2 \cos^2 \phi \, \dot{\theta}^2 \right) \qquad \textbf{(2.24)}$$

;**Generalized coordinates : distance, two angles**

Generalized momenta conjugate to these coordinates :

$$p_r = m\dot{r} \quad (linear..momentum)$$

$$p_\theta = mr^2 \cos^2 \phi \dot{\theta} \; (angular..momentum)$$

$$p_\phi = mr^2 \dot{\phi} \; (angular..momentum)$$

- **We are establishing a formulation based on geometric configurations rather than purely kinematic variables of vector mechanics. The objective of this formalism is to develop the equation of motion of a system in a way that will be independent of the type of generalized coordinates used to formulated the problem.**

Generalized Force

Vector mechanics : Time rate of change of the momenta of a system ~ force, moment

Analytical mechanics : Geometric relationships between generalized coordinates

obscure the distinction between the linear and angular momentum.

Similarly, *Energy* concept ~ Unified concept.

Virtual Work due to the actual forces is defined as

$$\delta W = \vec{F}_i \bullet \delta \vec{r}_i \,(i = 1 \ldots N) \tag{2.28}$$

where $\vec{F}_i = F_i^m \, \vec{\mathrm{i}}_m \,(i = 1 \sim \mathrm{N}, m = 1 \sim 3)$

for $i - th$ **particle**

Applying the chain rule,

$$\delta x_i = \frac{\partial x_i}{\partial q_j} \delta q_j \, (j = 1 \ldots n) \qquad \textbf{(2.29)}$$

Virtual (x *Actual*)displacement is <u>defined</u> as for *time is fixed*

$$\delta t = 0$$

Objective: Instantaneous effect of a force on the coordinate ~ Dynamics

: Virtual displacement of the physical coordinate $\delta \boldsymbol{x_i}$:

~ Simultaneous virtual displacements of the generalized coordinates $\delta \boldsymbol{q}_j$

- **A virtual displacement of a single coordinate, δx_i,**

  ~ produce a simultaneous virtual displacement in

  some or all of the generalized coordinates.

As a result:

Virtual work done by a physical component of force F_{ix} under a virtual displacement

δx_i ~ **Virtual work done under the simultaneous combinations of virtual**

displacements δq_j :

$$F_{(i)x}\delta x_{(i)} = F_{(i)x}\frac{\partial x_{(i)}}{\partial q_j}\delta q_j \qquad \textbf{(2.30)}$$

Individual terms in the summation ~ Contribution of the physical component $F_{(i)x}$ along the direction of the generalized coordinate q_j.

Rearranging the terms and factoring out δq_j, then the Total virtual work :

$$\delta W = \sum_{j=1}^{n}\left[\underline{\sum_{i=1}^{N}(F_{ix}\frac{\partial x_i}{\partial q_j} + F_{iy}\frac{\partial y_i}{\partial q_j} + F_{iz}\frac{\partial z_i}{\partial q_j}}\right]\delta q_j \qquad \textbf{(2.31)}$$

N = ? n = ? : N = n ? (* Constraint ?)

In terms of the simultaneous virtual displacements of the generalized coordinates, it can be written as

$$\delta W = Q_j \delta q_j \quad \textbf{for} \quad j = 1 \ldots n \qquad (2.32)$$

What is Q_j **?** **<u>Generalized force</u>**

~ **Underlined part !!**

- **Generalized force** Q_j **is determined by computing the virtual work done under an <u>infinitesimal change</u>** $\underline{q_j}$ **while leaving the other independent generalized coordinates fixed.**

Imagine the system frozen at an arbitrary instant.

: Consider an arbitrary configuration of the system.

Rewriting the virtual as

$$\delta W = Q_i \delta q_i .(i = 1..n)$$

A generalized force Q_j contributes to δW **only if the corresponding generalized coordinate q_j is given a virtual displacement. (independent !)**

: Virtual work δW **of the actual forces for each individual variation of only the generalized coordinates at a time.**

Since the transformations are <u>invertible</u>, a single variation δq_j will induce a simultaneous variation of one or more of the physical coordinates.

A virtual displacement of a generalized coordinate in physical space ~

A combination of virtual displacements subjected to the constraints of

the system.

Generally, the corresponding virtual work done by the physical components of the forces can be computed and set equal to $Q_{(i)}\,\delta q_{(i)}$.